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Abstract

We consider network formation. A set of locations can be connected in various network configurations. Every network has a cost and every agent has an individual value of every network. A planner aims at implementing a welfare maximizing network and allocating the resulting cost, but information is asymmetric: agents are fully informed and the planner is ignorant. Full implementation in Nash and strong Nash equilibria is studied. We show the correspondence consisting of welfare maximizing networks and individually rational cost allocations is implementable. We construct a minimal Nash implementable, welfare maximizing, and individually rational solution in the set of upper hemi-continuous and Nash implementable solutions. It is not possible to have full implementation single valued solutions such as the Shapley value.

Keywords: Networks; Welfare maximization; Nash Implementation; Strong Nash Implementation.

JEL Classification: C70, C72, D71, D85.

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1 Introduction

Overview of the paper: We consider network formation in a model a la Jackson and Wolinski (1996). A set of locations can be connected in various network configurations. Every network has a cost and every agent has an individual value of every network. The problem is to implement and allocate costs of a welfare maximizing network. If agents are left to themselves to establish and allocate the cost of the network, the outcome will typically not be a welfare maximizing network. Indeed, it is well-known that the core of the induced cooperative game can be empty (Megiddo, 1978; Tamir, 1991; Hougaard and Tvede, 2022). Consequently, decentralized organization of networks can result in inefficient networks or no network at all pointing to the need for an external agent such as a benevolent social planner.

We therefore take a mechanism design approach, where agents have complete information and study classic Nash implementation (see, e.g., the survey in Maskin and Sjöström, 2002): a benevolent social planner wants all equilibrium outcomes to be *desirable* in that the chosen network is *welfare maximizing* and payments are *individually rational* for all possible costs of networks and all possible values of networks for agents. Agents are fully informed, but the planner is ignorant about the true costs and values.

Specifically, we examine the possibility of full implementation in Nash and strong Nash equilibria. All Nash implementable solutions can be implemented by the canonical (unbounded) mechanism described in Maskin (1977, 1999). Modified and more informationally efficient versions can be found in Saijo (1988), Lombardi and Yoshihara (2013), and Tatamitani (2001). Alternatively, our solutions can also be implemented by a bounded mechanism as in Jackson et al. (1994).

We first show that it is impossible to implement budget-balanced cost sharing rules for which there is a unique distribution of the welfare generated by the network (Theorem 1). These cost sharing rules include the celebrated Shapley value. Therefore, we focus on the correspondence from states, where states are costs and values, to all desirable outcomes and show that it is Nash and strong Nash implementable (Theorems 2 and 5). Since the correspondence of desirable outcomes is rather large, it is natural to examine whether minimally implementable correspondences exist. A simple example demonstrates that they do not (Theorem 3). Adding continuity would be appealing in terms of robustness. But welfare maximizing networks vary discontinuously with costs and individual values of networks and payments vary discontinuously with networks. Hence, we consider upper hemi-continuity instead of continuity and construct a minimal correspondence in the set of upper hemi-continuous and Nash implementable correspondences from costs and values to desirable outcomes (Theorem 4). Finally, we discuss how the informational requirements of Maskin's canonical mechanism can be reduced in our case, though our modified mechanism remains unbounded.

Summing up, the planner on the one hand can implement welfare maximizing networks with individually rational payments, and on the other hand has to be flexible in assigning cost shares and not use a specific cost sharing rule such as the Shapley value. Welfare gains need not be equally distributed: specifically, there is no way to ensure that all agents get a positive share. Consequently, centralized organization of networks can result in welfare maximizing networks but not necessarily equitable distributions of welfare.

Related literature: Our paper relates to several strands of literature.

There is a large literature on cost sharing in networks. For the minimum cost spanning tree model various forms of implementation have been considered. For instance, Bergantinos and Lorenzo (2004, 2005) provide an empirical example of a decentralized network formation process, where agents connect sequentially to a source. Bergantinos and Vidal-Puga (2010) consider implementation of minimum cost spanning trees via a decentralized bargaining process inspired by the bidding mechanism of Perez-Castrillo and Wettstein (2001). Hougaard and Tvede (2012) consider central implementation and suggest a specific game form, where agents report connection costs to a planner. The game form fully Nash implements minimum cost spanning trees using a broad class of cost allocation rules like, for instance, the Folk-solution (Bergantinos and Vidal-Puga, 2007).

Non-cooperative behavior in the more general connection network model was initially studied in Anshelevich et al., (2008) and Chen et al., (2010). Both papers focus on equilibrium performance measured by the "Price of Anarchy", respectively "Price of Stability", i.e., the ratio between maximum welfare and the minimal, respectively maximal, welfare obtainable in Nash equilibrium. In a context where the planner is fully informed but unable to enforce a centralized network solution, Juarez and Kumar (2013) use a game form inspired by the model in Anshelevich et al. (2008). Loosely speaking, they show that a cost allocation rule implements an efficient network (in the sense that an efficient network is a Nash equilibrium outcome, and it Pareto dominates the outcome of all other equilibria) if and only if the allocation rule is a function of total network cost only: adding equal treatment of equals, in effect, leaves the equal split rule as the only possibility.

Generalizing the game form in Hougaard and Tvede (2012) to connection networks, Hougaard and Tvede (2015) obtain similar results in a centralized setting. Full Nash implementation of an efficient network is only possible provided the planner knows the connection demand of every agent, and only under very strong assumptions on the cost allocation rule, in effect violating individual rationality. In case the planner does *not* know connection demands, truthful reporting is a Nash equilibrium that implements a cost minimizing connection network, but other equilibria can induce highly inefficient networks. Indeed, the "Price of Anarchy" is unbounded even if the planner has full knowledge of connection costs. It is therefore somewhat striking that full Nash implementation of desirable outcomes is possible in the more general version of the model, where agents have limited willingness to pay for connectivity, albeit not when using a specific cost sharing rule.

Considering a network model a la Jackson and Wolinsky (1996) as in our case, Mutuswami and Winter (2002) show that a specific solution, namely the Shapley value, can be implemented in subgame perfect Nash equilibrium. A difference from our setting is that the planner knows the connection costs. Generally, as shown by Jackson et al. (1994) and Sjöström (1994), any social choice function is boundedly implementable provided it satisfies certain separability requirements. We could apply a mechanism similar to theirs and implement efficient networks where, for instance, agents pay in proportion to their value, provided that it would be possible to exclude agents from getting access to the network (making the network an excludable public good). Another option would be to allow for unbalanced payments. Along those lines, Young (1998) presents a simple auction mechanism to implement a welfare maximizing network in strong Nash equilibrium accepting that the mechanism can produce a surplus to the planner. Recently, Mackenzie and Trudeau (2023) provide an example of a Groves-like mechanism in a general exclusion model where surpluses are accepted as well.

Finally, our results have a parallel interpretation concerning the provision of multiple public goods (Mutuswami and Winter, 2004; Hougaard and Moulin, 2014).

2 The Model

In the present section, we introduce our framework and discuss the model.

Set Up

Let $\mathscr{L}$ be a finite set of n locations (nodes). Let g^n denote the complete graph. The set of possible undirected networks (graphs) is $\mathscr{G} = \{ g \mid g \subset g^n \}$.

Every network has a cost. *Costs* are a function from the set of networks to non-negative real numbers $C : \mathscr{G} \to \mathbb{R}_+$. We write C^g for the cost of network g and assume $C^{\emptyset} = 0$, and $C^g > 0$ for every $g \neq \emptyset$. As such, the networks are congestion-free. The set of costs is denoted $\mathscr{C}$. Costs are *additive* provided for every pair of locations a and b there is connection cost c_{ab} such that $C^g = \sum_{ab \in g} c_{ab}$ for every network g.

Let $\mathscr{M} = \{1, \ldots, m\}$ be a set of finitely many agents with $m \geq 3$. Every agent has a value associated with each network. *Values* are a function from the set of networks to vectors of individual values, $V : \mathscr{G} \to \mathbb{R}^m$. We write V_i^g for agent i's value of, or willingness to pay for, network g, and assume $V_i^{\emptyset} = 0$ for every $i \in \mathscr{M}$. The set of values is denoted $\mathscr{V}$.

The *social welfare* of costs, values and network, (C,V,g), is $\sum_i V_i^g - C^g$. For costs and values (C,V), a *Welfare Maximizing Network* (WMN) g is a graph such that for every other graph h,

$$\sum_i V_i^g - C^g \;\geq\; \sum_i V_i^h - C^h.$$

The set of welfare maximizing networks is non-empty and finite because the set of graphs is non-empty and finite.

Costs are *strictly monotonic* provided for every pair of networks g and h, $g \subset h$ and $g \neq h$ implies $C^g < C^h$. Values are *non-cyclic* provided for every pair of networks g and h, there is path from a to b in g if and only if there is a path from a to b in h imply $V^g = V^h$. Recall that a graph is a tree provided there is a unique path between every pair of locations in the graph and a forest provided it is a union of disjoint trees. Clearly, if costs are strictly monotonic and values are non-cyclic, then g being a WMN implies g is a tree or a forest.

An *outcome* is a network and a list of cost shares, (g, π^g), where $g \in \mathscr{G}$ and $\pi^g = (\pi_i^g)_{i \in \mathscr{M}}$ with $\sum_i \pi_i^g = 1$. Cost shares can be positive or negative corresponding to agents paying or being paid. The outcome (g, π^g) results in the network g and costs $(\pi_i^g C^g)_i$ for the agents. For costs and values (C,V) and an outcome (g,π^g), the utility of agent i is $u_i^g(C,V,\pi^g) = V_i^g - \pi_i^g C^g$. Let $\mathscr{O}$ be the set of outcomes.

A *desirable outcome* is an outcome (g,π^g) for which: g is a WMN; and, nobody pays more than their willingness to pay, i.e., $u_i^g = V_i^g - \pi_i^g C^g \geq 0$ for each $i \in \mathscr{M}$. A *minimal-subsidy (MS) desirable outcome* is a desirable outcome (g,π^g) for which subsidies are minimized: for an agent i with $V_i^g \leq 0$ the cost share is determined by $\pi_i^g = V_i^g / C^g$ for $C^g > 0$ so $u_i^g = V_i^g - \pi_i^g C^g = 0$. For costs and values (C,V), let $\mathscr{O}^d(C,V) \subset \mathscr{O}$ be the set of desirable outcomes, and let $\mathscr{O}_0^d(C,V) \subset \mathscr{O}^d(C,V)$ the set of MS-desirable outcomes. Note that $\mathscr{O}_0^d(C,V)$ is non-empty since all agents have non-negative utility of any network and the outcome is welfare maximizing.

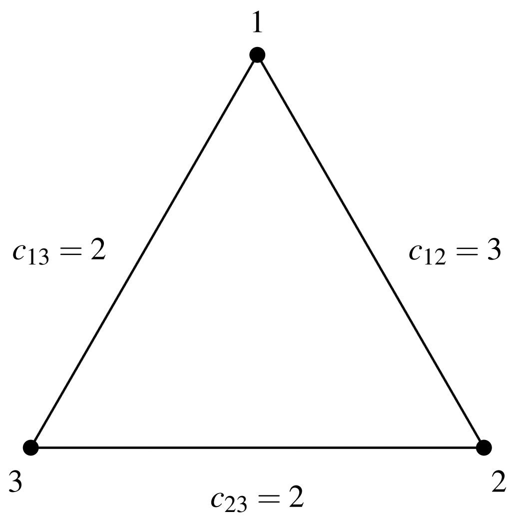
A *solution* $\Gamma : \mathscr{C} \times \mathscr{V} \to \mathscr{O}$ is a correspondence from costs and values to outcomes. We consider two solutions: the desirable solution Γ^d mapping costs and values to sets of all desirable outcomes $\Gamma^d(C,V) = \mathscr{O}^d(C,V)$; and, the MS-desirable solution Γ_0^d mapping costs and values to sets of all MS-desirable outcomes $\Gamma_0^d(C,V) = \mathscr{O}_0^d(C,V)$.

Desirable outcomes are appealing because they are efficient and individually rational. The networks are WMNs, their costs are exactly covered, and nobody pays more than their value (willingness to pay). In terms of fairness, individual rationality can be seen as a minimum requirement, at least nobody gets punished by realizing a desirable outcome. Adding MS to desirable outcomes ensures that every agent with positive value weakly contributes and every agent with $V_i^g \leq 0$ is compensated. Yet, this does not ensure that everybody strictly benefits from desirable outcomes.

Comments

A particular instance of our model, where agents are characterized by connection demands in the form of two locations they want connected and a willingness to pay for that connection, relates to the standard cost allocation model in connection networks (see e.g., Bergantinos and Vidal-Puga, 2007; Anschelevich et al., 2008; Bogomolnaia et al., 2010; Bogomolnaia and Moulin, 2010; Trudeau, 2012; Moulin, 2014; Hougaard and Tvede, 2015). As shown in Panova (2023) adding willingnesses to pay, might seem like a minor variation, but it fundamentally changes the standard model. In particular, it introduces the basic design question of which connection demands to satisfy. In contrast to the standard framework, where all connection demands have to be satisfied, we can now compare the cost of satisfying an agent's demand with their willingness to pay. Thus, we are able to address the optimal size of the network by aiming at social welfare maximization, in contrast to the cost minimization of the standard models. Consider the example below.

Example: Consider three locations $\mathscr{L} = \{1,2,3\}$ with additive connection costs, where $c_{12} = 3$ and $c_{13} = c_{23} = 2$ as illustrated below. Suppose one agent, Titika, has value V_T



if locations 1 and 2 are connected and zero otherwise. Another agent, Yi, has value V_Y if locations 1 and 3 are connected and zero otherwise. If both agents have to have their pairs of locations connected, then the cost minimizing network is clearly $g = \{13, 23\}$ with a total cost of 4. The remaining problem is to determine how the total cost should be allocated between the two agents. However, if the valuations of the agents are taken into account, then the problem becomes radically different in that both the locations to be connected and the allocation of total cost have to be determined. Indeed: if $V_T \geq 3$ and $V_Y \leq 1$, then it is welfare maximizing to not satisfy Yi and build $g_T = \{12\}$; if $V_T \leq 2$ and $V_Y \geq 2$, then it is welfare maximizing to not satisfy Titika and build $g_Y = \{13\}$; and, if $V_T \leq 3$, $V_Y \leq 2$ and $V_T + V_Y \leq 4$, then it is welfare maximizing to not satisfy any of the two agents and build no network.

The model induces a cooperative game $(\mathscr{M}, v)$, where the value $v(S)$ of every coalition $S \subset \mathscr{M}$, is naturally defined as the maximum total welfare obtainable by agents in S. As demonstrated by examples in Tamir (1991) and Hougaard and Tvede (2022) the core of such games can be empty. Consequently, decentralized mechanisms cannot be expected to work.

3 Implementation

A planner aims at implementing desirable or MS-desirable outcomes and designs a game that agents play. The equilibria of the game have to be the desirable or MS-desirable outcomes that the planner aims to implement. We assume that the planner is uninformed, but the agents know the costs and values (C,V). By restricting outcomes to be desirable, we restrict outcomes to be welfare maximizing and individually rational in that no agent pays more than her value of the network.

A mechanism $F = ((S_i)_i, f)$ consists of a strategy set for every agent, S_i, and a map from lists of individual strategies to outcomes, $f : \times_i S_i \to \mathscr{O}$. A list of individual strategies $(\bar{s}_i)_i$ is a *Nash equilibrium* provided there is no agent j and strategy s_j, such that $u_j(f(s_j, (\bar{s}_i)_{i \neq j})) > u_j(f((\bar{s}_i)_i))$. A list of individual strategies $(\bar{s}_i)_i$ is a *strong Nash equilibrium* provided there is no group of agents $T \subset \mathscr{M}$, and list of individual strategies for agents in T, $(s_j)_{j \in T}$, such that $u_j(f((s_j)_{j \in T}, (\bar{s}_k)_{k \in T^{\complement}})) > u_j(f((\bar{s}_i)_i))$ for every $j \in T$. A solution $\Gamma : \mathscr{C} \times \mathscr{V} \to \mathscr{O}$ is *implementable* in (strong) Nash equilibrium provided there exists a mechanism F such that for all costs and values (C,V), the set of (strong) Nash equilibria for F is $\Gamma(C,V)$.

Let $L_i^g(C,V,\pi^g) = \{(h,\pi^h) \in \mathscr{O}\} \mid u_i^h(C,V,\pi^h) \leq u_i^g(C,V,\pi^g)\}$ be the set of outcomes (h,π^h) that are weakly worse than (g,π^g) for agent i. A solution Γ is *monotonic* provided that for all outcomes $(g,\pi^g) \in \mathscr{O}$ and all pairs of costs and values $(C,V),(C',V') \in \mathscr{C} \times \mathscr{V}$, $(g,\pi^g) \in \Gamma(C,V)$ and $L_i^g(C,V,\pi^g) \subset L_i^g(C',V',\pi^g)$ for every i imply $(g,\pi^g) \in \Gamma(C',V')$.

Implementation in Nash equilibrium

The desirable and the MS-desirable solutions are appealing in that they maximize welfare and respect individual rationality. Two less appealing features of these solutions are that they are "big" and that they can be perceived as unfair. Indeed the MS-desirable solution maps problems to all pairs of WMNs and cost allocations, where individual cost shares are bounded from below by zero and from above by values. Specifically, in case two agents have identical values it is possible that one agent pays their value and gets utility zero and the other agent pays zero and gets utility equal to their value. Solutions that for all pairs of problems and WMNs have a unique cost allocation are much "smaller" and can be more fair in the sense that costs are allocated using a specific allocation rule, which may possess certain desirable fairness properties. However, these solutions are not Nash implementable, even if the planner knows the costs. In particular, single-valued solutions based on cooperative games, such as the Shapley value, are not Nash implementable.

Theorem 1 *Assume the planner knows the costs C. Suppose a solution $\Gamma : \mathscr{V} \to \mathscr{O}$ has the following properties:*

- *For all V and every g, there is either a unique or no π^g such that $(g, \pi^g) \in \Gamma(V)$.*
- *For all V, $\Gamma(V) \subset \Gamma^d(V)$.*

Then Γ is not Nash implementable.

Proof: To show there is no solution with the imposed properties a simple counterexample is presented. Suppose there are three locations $\mathscr{L} = \{1,2,3\}$ and costs C are additive with $c_{13} > c_{12}, c_{23}$ and $c_{12}+c_{23} = 1$. Furthermore, suppose the m agents can be split into two groups T and $T^{\complement}$, where agents in T have total value $V_T > 0$ of every graph in which locations 1 and 2 are connected and value zero otherwise and agents in $T^{\complement}$ have total value $V_{T^{\complement}} > 0$ of every graph in which locations 1 and 3 are connected and zero otherwise.

The network $g = (12,23)$ is the unique WMN provided

$$\begin{cases} V_T + V_{T^{\complement}} > 1 & (g \text{ is strictly better than no network}) \\ V_{T^{\complement}} > c_{23} & (g \text{ is strictly better than } g' = (12)) \\ V_T > 1 - c_{13} & (g \text{ is strictly better than } g'' = (13)) \end{cases}$$

The cost allocation for the two groups is $\pi = (\pi_T, \pi_{T^{\complement}})$ with $\pi_T + \pi_{T^{\complement}} = 1$. Furthermore, $(g,\pi) \in \Gamma^d(V)$ implies $\pi_T \in [1-V_{T^{\complement}}, V_T]$ and $\pi_{T^{\complement}} \in [1-V_T, V_{T^{\complement}}]$.

Let $V_T = c_{12}+\delta_T$ and $V_{T^\complement} = c_{23}+\delta_{T^\complement}$. Then the inequalities ensuring g is the unique WMN are satisfied if and only if

$$\begin{cases} \delta_T + \delta_{T^\complement} & > & 0 \\ \delta_{T^\complement} & > & 0 \\ \delta_T & > & c_{23} - c_{13} \end{cases}$$

so δ_T can be negative because $c_{23} < c_{13}$. Since $(g, \pi^g) \in \Gamma^d(V)$, the cost allocation has to satisfy

$$\begin{cases} \pi^T & \in & [c_{12}-\delta_{T^\complement}, c_{12}+\delta_T] \\ \pi^{T^\complement} & \in & [c_{23}-\delta_T, c_{23}+\delta_{T^\complement}]. \end{cases}$$

Suppose $(\delta_T, \delta_{T^\complement})$ and $(\delta'_T, \delta'_{T^\complement})$ satisfy the three inequalities with $\delta_T < -\delta'_{T^\complement}$. If $(g, \pi) \in \Gamma(\delta_T, \delta_{T^\complement})$ and $(g, \pi') \in \Gamma(\delta'_T, \delta'_{T^\complement})$, then $\pi_T < \pi'_T$ and $\pi_{T^\complement} > \pi'_{T^\complement}$.

Suppose Γ is a Nash implementable solution. Then Γ is monotonic according to Theorem 1 in Maskin and Sjöström (2002). Therefore, for all $(\delta''_T, \delta''_{T^\complement})$ satisfying the three inequalities as well as $\delta''_T \geq \max\{\delta_T, \delta'_T\}$ and $\delta''_{T^\complement} \geq \max\{\delta_{T^\complement}, \delta'_{T^\complement}\}$, $(g, \pi), (g, \pi') \in \Gamma(\delta''_T, \delta''_{T^\complement})$ contradicting there is either a unique or no π^g such that $(g, \pi^g) \in \Gamma(\delta''_T, \delta''_{T^\complement})$. □

Fortunately, it turns out both desirable solutions are Nash implementable. According to Theorem 2 in Maskin (1999), if a solution satisfies monotonicity and no veto power, then the solution is Nash implementable. A solution satisfies no veto power provided that if an outcome is top ranked by $m-1$, then it is in the solution. However, no outcome is top ranked by any agent. Indeed, since there is h with $C^h > 0$, for all (g, π^g) if $\pi_i^h < (V_i^g - \pi_i^g C^g - V_i^h)/C^h$, then $V_i^h - \pi_i^h C^h > V_i^g - \pi_i^g C^g$. Hence, from Theorems 2 and 3 in Maskin (1999) it follows that solutions in our setting are implementable if and only if they are monotonic. Consequently, we simply show that both desirable solutions are monotonic. Moreover, by Proposition 1 in Jackson et al. (1994), solutions in our setting are implementable by bounded mechanisms.

Theorem 2 *The desirable and the MS-desirable solutions are Nash implementable.*

Proof: Our setting fits the setting in Maskin and Sjöström (2002) with costs and values being states. Therefore, Theorem 2 in Maskin and Sjöström can be applied to show that Γ^d and Γ_0^d are Nash implementable.

To show that Γ^d or Γ_0^d is monotonic, suppose there are a pair of costs and values (C, V) and (C', V') and an outcome (g, π^g) with $(g, \pi^g) \in \Gamma^d(C, V)$ and $(g, \pi^g) \notin \Gamma^d(C', V')$ or $(g, \pi^g) \in \Gamma_0^d(C, V)$ and $(g, \pi^g) \notin \Gamma_0^d(C', V')$. Then for (C', V') either (g, π^g) is maximizing social welfare or (g, π^g) is not maximizing social welfare. If (g, π^g) is maximizing social welfare for (C', V'), then there is an agent i such that $u_i^g(C', V', \pi^g) < 0$.

Therefore $(\emptyset,(0,\dots,0)) \in L_i^g(C,V,\pi^g)$ and $(\emptyset,(0,\dots,0)) \notin L_i^g(C',V',\pi^g)$, so $L_i^g(C,V,\pi^g) \not\subset L_i^g(C',V',\pi^g)$. If (g,π_i^g) is not maximizing social welfare for (C',V'), then there is an outcome (h,π^h) such that $\sum_i u_i^h(C',V',\pi^h) > \sum_i u_i^g(C',V',\pi^g)$. Hence, there is an outcome (h,π'^h) such that $u_i^h(C',V',\pi'^h) > u_i^g(C',V',\pi^g)$ for every i. Since $\sum_i u_i^h(C,V,\pi'^h) \leq \sum_i u_i^g(C,V,\pi^g)$, there is an agent i such that $u_i^h(C,V,\pi'^h) \leq u_i^g(C,V,\pi^g)$. Hence $(h,\pi'^h) \in L_i^g(C,V,\pi^g)$ and $(h,\pi'^h) \notin L_i^g(C',V',\pi^g)$, so $L_i^g(C,V,\pi^g) \not\subset L_i^g(C',V',\pi^g)$. To sum up, Γ^d and Γ_0^d are monotonic and consequently Nash implementable. □

Both solutions Γ^d and Γ_0^d can be implemented by a bounded mechanism designed as in the proof of Theorem 1 in Jackson et al., (1994) (see also Proposition 1).

Minimal Nash implementable solutions

The desirable and the MS-desirable solutions are correspondences and they are "big". Indeed, they map costs and values (C,V) to sets containing all outcomes where individual cost shares are bounded from below by zero and from above by the individual values of the network. Solutions that map all (C,V) to a single cost allocation for every WMN are "small", but not Nash implementable as shown in Theorem 1. Therefore, obvious questions are whether there are minimal Nash implementable solutions and if so, what they look like.

A Nash implementable solution Γ is minimal provided there is no other Nash implementable solution Φ such that $\Phi(C,V) \subset \Gamma(C,V)$ for all (C,V) and $\Phi(C,V) \neq \Gamma(C,V)$ for some (C,V). The following Theorem shows that there is no minimal solution in the full set of Nash implementable solutions.

Theorem 3 *Assume that the planner knows the costs C, but not the values V. Then there is no minimal Nash implementable solution with $\Gamma(V) \subset \Gamma^d(V)$ for all V.*

Proof: To show there is no minimal Nash implementable solution with $\Gamma(V) \subset \Gamma^d(V)$ for all V, a simple counterexample is presented. There are two locations $\mathscr{L} = \{1,2\}$ with cost $c_{12} = 1$ and m agents that want to have locations 1 and 2 connected. Then $g = \emptyset$ is an WMN provided $\sum_i V_i \leq 1$ and $g = \{12\}$ is an WMN provided $\sum_i V_i \geq 1$. Suppose $\Gamma : \mathscr{V} \to \mathscr{O}$ is a Nash implementable solution. Then Γ is monotonic according to Theorem 1 in Maskin and Sjöström (2002).

There is $\tilde{V}$ with $\tilde{V}_i < 1$ for every i and $\sum_i \tilde{V}_i > 1$. For $(\tilde{g},\tilde{\pi}^g) \in \Gamma(\tilde{V})$ let another correspondence $\Phi : \mathscr{V} \to \mathscr{O}$ be defined by $\Phi(\tilde{V}) = \Gamma(\tilde{V}) \backslash \{(\tilde{g},\tilde{\pi}^{\tilde{g}})\}$ for all V so $\Phi(V) \subset \Gamma(V)$ for all V and $\Phi(\tilde{V}) \neq \Gamma(\tilde{V})$. Furthermore, Φ is monotonic because Γ is monotonic. There is V with $V_i < V_i'$ for every i, $\sum_i V_i > 1$ and $V_i < \tilde{\pi}_i^{\tilde{g}}$ for some i. Since $V_i < V_i'$ for every i, and $\sum_i V_i^{\{12\}} > 1$, $\Gamma(V) \subset \Gamma(\tilde{V})$ because Γ is monotonic. Since $V_i < \tilde{\pi}_i^{\tilde{g}}$ for some

i, $(\tilde{g}, \tilde{\pi}^{\tilde{g}}) \notin \Gamma(V)$. Hence, $\Phi(V) \neq \emptyset$ for all V with $\sum_i V_i > 1$. For all V with $\sum_i V_i < 1$, $\Gamma(V) = (\emptyset, (0, \ldots, 0))$ so $(\emptyset, (0, \ldots, 0)) \in \Gamma(V')$ for all V' with $\sum_i V'_i \leq 1$ because Γ is monotonic. Hence $\Phi(V) \neq \emptyset$ for all V so Φ is a solution implying Γ is not minimal. □

The set of Nash implementable solutions contains less appealing solutions such as the one constructed in the proof of Theorem 3. Thus, it seems natural to require additional properties of solutions. In terms of robustness, continuity is an appealing property of solutions. However, solutions mapping problems to WMNs and cost allocations are not continuous. The set of WMNs varies discontinuously with problems and cost allocations vary discontinuously with WMNs. The second best in terms of robustness is upper hemi-continuity. Recall that a solution Γ is *upper hemi-continuous* if at all (C,V), all $(g, \pi^g) \in \Gamma(C,V)$ and all sequences $(C_n, V_n)_n$ converging to (C,V) there is a sequence $(g_n, \pi^g_n)_n$ with $(g_n, \pi^g_n) \in \Gamma(C_n, V_n)$ for every n converging to (g, π^g). Trivially, the desirable and the MS-desirable solutions are upper hemi-continuous. Moreover, there are minimal solutions in sets of upper hemi-continuous and Nash implementable solutions with their graphs being contained in the graph of the MS-desirable solution.

To formalize the notion of sizes of solutions, let Ω be a set of solutions. A solution $\Gamma \in \Omega$ is Ω-*minimal* provided that all solutions $\Phi \in \Omega$, $\Phi(C,V) \subset \Gamma(C,V)$ for all (C,V) implies $\Phi(C,V) = \Gamma(C,V)$ for all (C,V).

Theorem 4 *There are minimal upper hemi-continuous solutions* Γ *with* $\Gamma(C,V) \subset \Gamma^d_0(C,V)$ *for all* (C,V).

Proof: First, an upper hemi-continuous and Nash implementable solution Γ is constructed. Second, it is shown that Γ is minimal in the set of upper hemi-continuous and Nash implementable solutions.

For every graph g let the set $A^g \subset \mathscr{C} \times \mathscr{V}$ be the set of costs and values for which g is a WMN. Then the set A^g is convex, closed and non-empty. Trivially, $(A^g)_g$ is a cover of $\mathscr{C} \times \mathscr{V}$. Let $(A^g)_{g \in \mathscr{H}}$ be a minimal cover of $\mathscr{C} \times \mathscr{V}$: $(A^g)_{g \in \mathscr{H}}$ is a cover of $\mathscr{C} \times \mathscr{V}$; and, for every $h \in \mathscr{H}$, $(A^g)_{g \in \mathscr{H} \setminus \{h\}}$ is not a cover of $\mathscr{C} \times \mathscr{V}$.

Fix a graph g and costs C. For $A^g(C) \subset \mathscr{V}$ being the set of valuations V such that $(C,V) \in A^g$, $A^g(C) = \{V \in \mathscr{V} \mid (C,V) \in A^g\}$, let $L^g(C) \subset A^g(C)$ be the set of minimal valuations in $A^g(C)$,

$$L^g(C) \;=\; \{V \in A^g(C) \mid \forall V' \in A^g(C) : \max_i V^g_i - V'^g_i > 0 \;\Rightarrow\; \min_k V^g_k - V'^g_k < 0\}.$$

First, for $V^{g+}_i = \max\{V^g_i, 0\}$ and $V^{g-}_i = \min\{V^g_i, 0\}$ let the function $\lambda^g(C, \cdot) : L^g(C) \to \mathbb{R}^m$

be defined by: in case $g=\emptyset$ so $C^g=0$, $\lambda_i^g(C,V)=1/m$; and, in case $g\neq\emptyset$ so $C^g>0$,

$$\lambda_i^g(C,V) \;=\; \begin{cases} \dfrac{V_i^{g-}}{C^g} & \text{for } V_i^g\leq 0 \\ \dfrac{V_i^{g+}}{\sum_k V_k^{g+}}\,\dfrac{C^g-\sum_k V_k^{g-}}{C^g} & \text{for} V_i^g\geq 0. \end{cases}$$

Then $V_i^g\leq 0$ implies $u_i^g(C,V,\lambda^g(C,V))=V_i^g-\lambda_i^g(C,V)C^g=0$ and $V_i^g\geq 0$ implies

$$u_i^g(C,V,\lambda^g(C,V)) \;=\; V_i^g-\lambda_i^g(C,V)C^g \;=\; \frac{V_i^g}{\sum_k V_k^{g+}}\,(\sum_k V_k^g-C^g) \;\geq\; 0.$$

Moreover,

$$\sum_i \lambda_i^g(C,V) \;=\; \sum_i \frac{V_i^{g-}}{C^g}+\sum_i \frac{V_i^{g+}}{\sum_k V_k^{g+}}\,\frac{C^g-\sum_k V_k^{g-}}{C^g} \;=\; \frac{\sum_i V_i^{g-}+C^g-\sum_i V_i^{g-}}{C^g} \;=\; 1.$$

Second, let the function $\Gamma^g(C,\cdot):L^g(C)\to\mathscr{O}$ be defined by $\Gamma^g(C,V)=(g,\lambda^g(C,V))$. Third, extend the function $\Gamma(C,\cdot):L^g(C)\to\mathscr{O}$ to a correspondence $\Gamma^g(C,\cdot):A^g(C)\to\mathscr{O}$ defined by

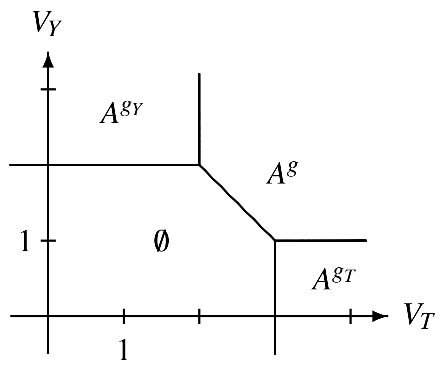
$$\Gamma^g(C,V) \;=\; \{\,(g,\pi^g)\mid \exists\tilde{V}\in L^g(C):\tilde{V}^g-V^g\in-\mathbb{R}_+^m \text{ and } \pi^g=\lambda^g(C,\tilde{V})\,\}.$$

Then by construction Γ^g is a non-empty, continuous and monotonic correspondence on $A^g(C)$. Finally, let the solution $\Gamma:\mathscr{C}\times\mathscr{V}\to\mathscr{O}$ be defined by $\Gamma(C,V)=\cup_{g\in\mathscr{H}}\Gamma^g(C,V)$. Then $\Gamma(C,\cdot):\mathscr{V}\to\mathscr{O}$ is upper hemi-continuous and monotonic.

To show Γ is a minimal upper hemi-continuous solution, consider another upper hemi-continuous and monotonic correspondence $\Phi:\mathscr{C}\times\mathscr{V}\to\mathscr{O}$ with $\Phi(C,V)\subset\Gamma(C,V)$ for all (C,V). Assume $(g,\pi)\in\Gamma(C,\tilde{V})$. By construction of Γ there is $(C,V)\in L^g(C)$ such that $\lambda(C,V)=\pi^g$ and $\tilde{V}^g-V^g\in-R_+^m$. Consider a sequence $(C_n,V_n)_{n\in\mathbb{N}}$ converging to $(C,\tilde{V})$ with g being the unique WMN for every n. If $(g,\pi_n^g)\in\Gamma(C_n,V_n)$ for every n, then $\lim_{n\to\infty}\pi_n^g=\pi^g$ because Γ is upper hemi-continuous. For all sequences $(h_n,\pi_n^h)_{n\in\mathbb{N}}$ with $(h_n,\pi_n^h)\in\Phi(C_n,V_n)$ for every n, $h_n=g$ for every n, because g is the unique WMN for every n, and $\lim_{n\to\infty}\pi_n^g=\pi^g$, because $\Phi(C_n,V_n)\subset\Gamma(C_n,V_n)$ for every n. Therefore, $(g,\pi^g)\in\Phi(C,V)$. Since Φ is monotonic, $(g,\pi^g)\in\Phi(C,\tilde{V})$ so $\Gamma(C,\tilde{V})\subset\Phi(C,\tilde{V})$. □

Remark: The solution constructed in the proof of Theorem 4 is not necessarily unique. Indeed, non-uniqueness could be caused by multiplicity of minimal covers of $\mathscr{C}\times\mathscr{V}$ and multiplicity of possible cost shares at $(C,V)\in L^g$.

Example (continued): To illustrate Theorems 3 and 4, we use the example with three locations and two agents. Costs are $c_{12}=3$ and $c_{13}=c_{23}=2$ and for the two agents, Titika



has valuation V_T if and only if locations 1 and 2 are connected and Yi has valuation V_Y if and only if locations 1 and 3 are connected. The possible WMNs depending on the valuations are $\emptyset$, $g_T = \{12\}$, $g_Y = \{13\}$ and $g = \{13, 23\}$. The graph $g' = \{12, 13\}$ is not an WMN for any valuations because $c_{12} > c_{23}$. The combinations of valuations and WMNs are illustrated in the figure.

In the proof of Theorem 4, valuations on the line between $(2,2)$ and $(3,1)$ are used to construct cost shares for all valuations in A^g. Indeed, for any valuation $(V_T, V_Y) \in A^g$, valuations smaller than those and on the line are used to construct cost shares. Thereby, for all valuations in A^g the relevant cost shares are subsets of the line between $(1/2, 1/2)$ and $(3/4, 1/4)$ so more extreme cost shares are excluded even though they can be individually rational.

In the proof of Theorem 3, for all valuations in the interior of A^g the graph g is the unique WMN. The solution correspondence being monotonic implies that for a pair valuations (V_T, V_Y) and (V_Y', V_Y') in the interior of A^g, if, $V_T \geq V_T'$ and $V_Y \geq V_Y'$, then cost shares for (V_T', V_Y') must be cost shares for (V_T, V_Y) too because of monotonicity. Therefore, consider a dense set of points on the line between $(2,2)$ and $(3,1)$ and use them to construct cost shares for valuations in the interior of A^g. Next, consider the dense set of points on the line except one point and use it to construct cost shares in the interior of A^g. Then the graph of correspondence constructed without the point is a strict subset of the graph of the correspondence constructed with the point. There is no smallest dense set of points on the line, so the process can continue.

Implementation in strong Nash equilibrium

Agents may be able to coordinate their actions. Therefore, implementation of solutions in strong Nash equilibrium is considered. Using a modified, and informationally more efficient, version of the mechanism in the proof of Theorem 3 in Maskin (1978) we show that the desirable solution Γ^d is strong Nash implementable.

Theorem 5 *The desirable solution Γ^d is strong Nash implementable.*

Proof: For the mechanism implementing Γ^d, let the strategy set of every agent be the set of outcomes $S_i = \mathscr{O}$ and the map from lists of individual strategies to outcomes $f^d : S^m \to \mathscr{O}$ be

$$f^d(s) = \begin{cases} (g, \pi^g) & \text{if } s_1 = \ldots = s_m = (g, \pi^g) \\ (\emptyset, (0, \ldots, 0)) & \text{otherwise.} \end{cases}$$

Suppose every agent uses the strategy (g, π^g). If (g, π^g) is a desirable outcome for the true state, then no coalition of agents has an incentive to change its strategy. If (g, π^g) is not a desirable outcome for the true state, then there is another strategy $(h, \pi^h) \in \mathscr{O}^d(C,V)$ such that $u_i^h(C,V,\pi^h) > u_i^g(C,V,\pi^g)$ for every i or there is some i such that $u_i^{\emptyset}(C,V,(0,\ldots,0)) > u_i^g(C,V,\pi^g)$. Therefore, the mechanism implements Γ^d. □

We note that the MS-desirable solution is only partially strong Nash implementable. This will be further discussed in the next section.

4 Discussion

We have used that monotonicity of solutions is necessary and sufficient for Nash implementation by Maskin's canonical mechanism (Maskin, 1977, 1999) or the bounded mechanism in Jackson et al. (1994). Both mechanisms have large strategy sets. For applications, it can be desirable to reduce strategy sets in order to make the mechanism more informationally efficient.

A mechanism for Nash implementation

The strategy sets of the canonical mechanism can be reduced as already shown in Saijo (1988). In both Saijo (1988) and Jackson et al. (1994), fundamentals are outcomes and preferences instead of states. In Saijo (1988), every agent submits preferences for themselves and another agent, an outcome and a natural number. Since preferences in the present setting depend on costs and values, every agent would have to submit costs, values for themselves and another agent, an outcome, and a natural number. In Jackson et al. (1994), every agent submits an alternative, two sets of preferences, and an integer between $-(m+3)$ and m. In the present setting, every agent would have to submit costs and values for themselves and another agent, an outcome and an integer between $-(m+3)$ and m.

We show that in the present setting all Nash implementable solutions can be implemented by use of mechanisms in which every agent submits a part of the costs, values for themselves and another agent and an outcome. This is less information than in both Saijo (1988) and Jackson et al. (1994), but compared to Jackson et al. the mechanism is not bounded.

Before the result can be stated, the part of costs every agent must submit has to be specified. There are $\eta = \sum_{i=1}^{n-1}(n-i)$ connections between different pair of locations so there are $|\mathscr{G}| = 2^\eta$ possible graphs. For $q \in \mathbb{N}$ defined by

$$\frac{2|\mathscr{G}|}{m} \leq q < \frac{2|\mathscr{G}|}{m} + 1,$$

Let $(\mathscr{Q}_i)_i$ be a cover of the set of graphs $\mathscr{G}$ with $|\mathscr{Q}_i| = q$ as well as $\mathscr{Q}_i \subset \mathscr{Q}_{i-1} \cup \mathscr{Q}_{i+1}$ and $\mathscr{Q}_i \cap \mathscr{Q}_{i+1} \neq \emptyset$ for every i where $i+1 = 1$ for $i = m$.

Theorem 6 *All Nash implementable solutions Γ can be implemented by a mechanism $((S_i)_i, F)$ with $S_i = \mathbb{R}^q_+ \times (\mathbb{R}^{|\mathscr{G}|})^2 \times \mathscr{O}$ for every i.*

Proof: Consider an outcome (g, π^g).

For a Nash implementable solution Γ let a mechanism $((S_i)_i, F)$ be described by $S_i = \mathbb{R}^q_+ \times (\mathbb{R}^{|\mathscr{G}|})^2 \times \mathscr{O}$ for every i and $F : S \to \mathscr{O}$ defined as follows:

- In case there is $(C, V, g, \pi^g) \in \mathscr{C} \times \mathscr{V} \times \mathscr{O}$ with $(g, \pi^g) \in \Gamma(C, V)$ such that for every i, $s_i = ((C^g)_{g \in \mathscr{Q}_i}, V_i, V_{i+1}, g, \pi^g)$, $F(s) = (g, \pi^g)$.

- In case there are j and $(C, V, g, \pi^g) \in \mathscr{C} \times \mathscr{V} \times \mathscr{O}$ with $(g, \pi^g) \in \Gamma(C, V)$ such that $s_i = ((C^g)_{g \in \mathscr{Q}_i}, C_i, V_i, V_{i+1}, g, \pi^g)$ for every $i \neq j$,

$$F(s) = \begin{cases} \mathrm{pr}_{\mathscr{O}}\, s_j & \text{for } \mathrm{pr}_{\mathscr{O}}\, s_j \in L^g_j(C, V, \pi^g) \\ (g, \pi^g) & \text{for } \mathrm{pr}_{\mathscr{O}}\, s_j \notin L^g_j(C, V, \pi^g) \end{cases}$$

  with $\mathrm{pr}_{\mathscr{O}}\, s_j$ being the outcome of agent j's strategy.

- In all other cases, for $\tilde{C}$ defined by $\tilde{C}^g = \max_i\{C^g_i\}$ and $\tilde{V}$ by $\tilde{V}_i = V^i_{i-1}$ for every i (where V^i_{i-1} is what $i-1$ reports is i's valuation), let $F(s) = (g_i, \pi^{g_i})$ for i chosen at random from the set

$$\{\, i \mid \forall j : \textstyle\sum_{k \neq i} \tilde{V}^{g_i}_k - \tilde{C}^{g_i} \geq \sum_{\ell \neq j} \tilde{V}^{g_j}_\ell - \tilde{C}^{g_j} \,\}$$

  endowed with the uniform distribution.

Let $NE : \mathscr{C} \times \mathscr{O} \to S$ be the Nash equilibrium correspondence.

First, it is shown that $\Gamma(C, V) \subset F \circ NE(C, V)$. Suppose $s_i = ((C^g)_{g \in \mathscr{Q}_i}, V_i, V_{i+1}, g, \pi^g)$ for every i and some $(g, \pi^g) \in \Gamma(C, V)$. Then

$$F(S_i, s_{-i}) = L^g_i(C, V, \pi^g).$$

Therefore $s \in NE(C,V)$. Second it is shown that $F \circ NE(C,V) \subset \Gamma(C,V)$. In the first case, where there is $(\bar{C},\bar{V},g,\pi^g) \in \mathscr{C} \times \mathscr{V} \times \mathscr{O}$ with $(g,\pi^g) \in \Gamma(\bar{C},\bar{V})$ such that $s_i = ((\bar{C}^g)_{g \in \mathscr{Q}_i}, \bar{V}_i, \bar{V}_{i+1}, g, \pi^g)$ for every i, a deviating agent j is able to move into the second case. Therefore

$$F(S_i, s_{-i}) \;=\; L_i^g(\bar{C},\bar{V},\pi^g).$$

If $L_i^g(\bar{C},\bar{V},\pi^g) \not\subset L_i^g(C,V,\pi^g)$ for some i, then s is not a Nash equilibrium. If $L_i^g(\bar{C},\bar{V},\pi^g) \subset L_i^g(C,V,\pi^g)$ for every i, then s is a Nash equilibrium. Since Γ is Nash implementable, it is monotonic, so $(g,\pi^g) \in \Gamma(C,V)$. In the second and the third cases, there is a deviating agent i, who is able to move into the third case, so

$$F(S_i, s_{-i}) = \mathscr{O}.$$

Hence, s is not a Nash equilibrium. □

As can be seen in the proof of Theorem 6, if agents submit different strategies, then the values submitted by agent i are used to evaluate the outcome submitted by agent i. Thereby, agent i is able to have any outcome selected. Since every agent can have all outcomes selected, there will be no Nash equilibrium, where agents submit different strategies.

Partial Implementation

Partial implementation is a weaker notion of implementation where the set of Nash equilibria includes the solution correspondence. Both the desirable and the MS-desirable solutions are partially Nash and strong Nash implementable using the modified mechanism in the proof of Theorem 5. The mechanism has strategy set $S_i = \mathscr{O}$ for every agent and payoff function $f^d : S^m \to \mathscr{O}$ defined by

$$f^d((s_i)_i) \;=\; \begin{cases} (g,\pi^g) & \text{for } s_1 = \ldots = s_m = (g,\pi^g) \\ (\emptyset,(0,\ldots,0)) & \text{otherwise.} \end{cases}$$

Indeed, the mechanism fully strong Nash implements the desirable solution as shown in Theorem 5. It is straightforward to check that it partially implements the three other combinations of solutions and forms of implementation. Therefore, the best equilibrium is welfare maximizing making the price of stability equal to one for all four combinations.

However, since $s_i = (\emptyset,(0,\ldots,0))$, for every i, is a Nash equilibrium, the price of anarchy is unbounded for Nash implementation. If the utility is non-positive for some agent i, then $s_i = (\emptyset,(0,\ldots,0))$, for every i, is a strong Nash equilibrium for the MS-desirable solution. Indeed, agent i has to increase her utility by changing her strategy. However, to increase the utility of agent i, her cost share must be negative. Consequently, the price of anarchy is unbounded. Obviously, the problem with the MS-desirable solution is that it may not be possible to transfer welfare between agents.

To sum up: welfare maximization can be obtained in the best case Nash equilibrium for all four combinations of desirable and MS-desirable solution versus partial and strong Nash, while the worst case efficiency is unbounded, except for the combination covered in Theorem 5; and, both the desirable and the MS-desirable solutions can be implemented in Nash.

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