# SOME DISCRETE DISTRIBUTIONS

**SOME DISCRETE DISTRIBUTIONS**

| **Name** | **Genesis** | **Notation** | **p.f.** | **E(X)** | **V(X)** | **Applications** | **Comments** |
|---|---|---|---|---|---|---|---|
| Uniform (discrete) | Set of $k$ equally likely outcomes (usually, not necessarily, the integers) | $\mathrm{U}(1,...,k)$ (not standard) | $p(x)=1/k$ <br> $x=1,...,k$ | $\frac{k+1}{2}$ | $\frac{k^2-1}{12}$ | Dice | |
| Bernoulli trial | Expt. with two outcomes: 'success' w.p. $\theta$ and 'failure' w.p. $1-\theta$ <br> $X\equiv$ no. successes | $\mathrm{Ber}(\theta)$ | $p(x)=\theta^x(1-\theta)^{1-x}$ <br> $x=0,1$ <br> $\theta\in[0,1]$ | $\theta$ | $\theta(1-\theta)$ | Coins, constituent of more complex distributions | |
| Binomial | $X\equiv$ no. successes in $n$ ind. $Ber(\theta)$ trials | $\mathrm{Bi}(n,\theta)$ | $p(x)=\binom{n}{x}\theta^x(1-\theta)^{n-x}$ <br> $x=0,1,2,...,n$ <br> $\theta\in[0,1]$ | $n\theta$ | $n\theta(1-\theta)$ | Sampling with replacement | $\mathrm{Bi}(1,\theta)\equiv\mathrm{Ber}(\theta)$ |
| Geometric | $X\equiv$ no. failures until 1st success in sequence of ind. $\mathrm{Ber}(\theta)$ trials | $\mathrm{Ge}(\theta)$ | $p(x)=\theta(1-\theta)^x$ <br> $x=0,1,2,...$ <br> $\theta\in[0,1]$ | $\frac{1-\theta}{\theta}$ | $\frac{1-\theta}{\theta^2}$ | Waiting times (for single events) | Alternative formulation in terms of $Y\equiv$ no. of trials to 1st success ($Y=X+1$) |
| Negative binomial (or Pascal) | $X\equiv$ no. failures to $m$th success in sequence of ind. $\mathrm{Ber}(\theta)$ trials. Generalization of Geometric | Neg $\mathrm{Bi}(m,\theta)$ (not standard) | $p(x)=\binom{m+x-1}{x}\theta^m(1-\theta)^x$ <br> $x=0,1,2,...$ <br> $\theta\in[0,1]$ | $\frac{m(1-\theta)}{\theta}$ | $\frac{m(1-\theta)}{\theta^2}$ | Waiting times (for compound events) | Neg $\mathrm{Bi}(1,\theta)\equiv\mathrm{Ge}(\theta)$ <br> Remains valid for any $k>0$ <br> (not necessarily integer). <br> Alternative formulation as above. |
| Hypergeometric | $X\equiv$ no. of defectives in sample of size $n$ taken without replacement from population of size $N$ of which $d$ are defective | $\mathrm{Hypergeom}(N,d,n)$ (not standard, esp. order of arguments) | $p(x)=\frac{\binom{d}{x}\binom{N-d}{n-x}}{\binom{N}{n}}$ <br> $x=\max(0,n+d-N),...,$ <br> $...,\min(n,d)$ | $\frac{nd}{N}$ | $\frac{N-n}{N-1}n\frac{d}{N}\left(1-\frac{d}{N}\right)$ | Sampling without replacement | Sampling with replacement leads to the $\mathrm{Bi}(n,\frac{d}{N})$ - a suitable approx if $\frac{n}{N}<0.1$ |
| Poisson | Arises empirically or via Poisson Process (PP) for counting events. For PP rate $\nu$ the no. of events in time $t\sim\mathrm{Po}(\nu t)$. Also as an approx. to the Binomial | $\mathrm{Po}(\lambda)$ | $p(x)=\frac{e^{-\lambda}\lambda^x}{x!}$ <br> $x=0,1,2,...$ <br> $\lambda>0$ | $\lambda$ | $\lambda$ | Counting events occurring 'at random' in space or time | $\mathrm{Bi}(n,\theta)\equiv\mathrm{Po}(n\theta)$ if $n$ large, $\theta$ small |

# SOME CONTINUOUS DISTRIBUTIONS

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| **Name** | **Notation** | **p.d.f.** | **E(X)** | **V(X)** | **Applications** | **Comments** |
|---|---|---|---|---|---|---|
| Uniform (continuous) (or Rectangular) | $\mathrm{Un}(\alpha,\beta)$ | $f(x)=\frac{1}{\beta-\alpha}$ <br> $x\in[\alpha,\beta]$ <br> $\alpha<\beta$ | $\frac{\alpha+\beta}{2}$ | $\frac{(\beta-\alpha)^2}{12}$ | Rounding errors $Un(-\frac{1}{2},\frac{1}{2})$. Simulating other distributions from $Un(0,1)$. | |
| Exponential | $\mathrm{Ex}(\lambda)$ | $f(x)=\lambda e^{-\lambda x}$ <br> $x>0$ <br> $\lambda>0$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | Inter-event times for Poisson Process. Models lifetimes of non-ageing items. | Alternative parameterization in terms of $1/\lambda$ <br> $\mathrm{Ga}(1,\lambda)\equiv\mathrm{Ex}(\lambda)$ |
| Gamma | $\mathrm{Ga}(\alpha,\beta)$ | $f(x)=\frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$ <br> $x\geq 0$ <br> $\alpha,\beta>0$ | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ | Times between $k$ events for Poisson Process. Lifetimes of ageing items. | Alternative parameterization in terms of $1/\beta$ <br> $\mathrm{Ga}(1,\lambda)\equiv\mathrm{Ex}(\lambda)$, <br> $\mathrm{Ga}(\nu/2,1/2)\equiv X^2_\nu$, |
| Beta | $\mathrm{Be}(\alpha,\beta)$ | $f(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ <br> $x\in[0,1]$ <br> $\alpha,\beta>0$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ | Useful model for variables with finite range. Also as a Bayesian conjugate prior. | $\mathrm{Be}(1,1)\equiv\mathrm{Un}(0,1)$ <br> $\mathrm{Be}(\alpha,\beta)$ is reflection about $\frac{1}{2}$ of $\mathrm{Be}(\beta,\alpha)$. <br> Can transform $\mathrm{Be}(\alpha,\beta)$ on $[0,1]$ to any finite range $[a,b]$ by $Y=(b-a)X+a$ |
| Normal | $\mathrm{N}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ <br> $x\in(-\infty,\infty)$ | $\mu$ | $\sigma^2$ | Empirically and theoretically (via CLT etc.) a good model in many situations. Often easy to handle mathematically. | $X\sim\mathrm{N}(\mu,\sigma^2)\Longrightarrow$ <br> $aX+b\sim\mathrm{N}(a\mu+b,a^2\sigma^2)$ <br> $\Longrightarrow Z=\frac{X-\mu}{\sigma}\sim\mathrm{N}(0,1)$ <br> So <br> $P[X\in(u,v)]=P[Z\in(\frac{u-\mu}{\sigma},\frac{v-\mu}{\sigma})]$ <br> $\mathrm{N}(0,1)$ special case has p.d.f. denoted $\phi$, c.d.f. $\Phi$ (tabulated). <br> Note $\Phi(-z)=1-\Phi(z)$. |
| Chi-square | $\chi^2_\nu$ | $f(x)=2^{-\nu/2}\Gamma(\alpha)^{-1}x^{\nu/2-1}e^{-x/2}$ <br> $x>0$ <br> $\nu>0$ | $\nu$ | $2\nu$ | Sum of squares of $\nu$ standard normals | $X^2_\nu\equiv\mathrm{Ga}(\nu/2,1/2)$ <br> If $X_1,X_2,\ldots,X_n\sim\mathrm{N}(0,1)$ independent, then $\sum_{i=1}^n X_i^2\sim\chi^2_n$ |
| Student $t$ | $\mathrm{t}_\nu$ | $f(x)=$ <br> $\nu^{-1/2}B\left(\frac{1}{2},\frac{\nu}{2}\right)^{-1}\left(1+x^2/\nu\right)^{-(\nu+1)/2}$ <br> $x\in(-\infty,\infty)$ <br> $\nu>0$ | 0 <br> (if $\nu>1$) | $\frac{\nu}{\nu-2}$ <br> (if $\nu>2$) | Useful alternative to Normal for variables with heavy tails. | If $X\sim\mathrm{N}(0,1)$ and $Y\sim\chi^2_\nu$ independent then $\frac{X}{\sqrt{Y/\nu}}\sim\mathrm{t}_\nu$. <br> $\mathrm{t}_1\equiv$ Cauchy. $\mathrm{t}^2_\nu\equiv\mathrm{F}_{1,\nu}$. |
| F | $\mathrm{F}_{\nu,\delta}$ | $f(x)=\frac{\nu^{\nu/2}\delta^{\delta/2}x^{\nu/2-1}}{B(\nu/2,\delta/2)(\nu x+\delta)^{(\nu+\delta)/2}}$ <br> $x>0$ <br> $\nu,\delta>0$ | $\frac{\delta}{\delta-2}$ <br> (if $\delta>2$) | $\frac{2\delta^2(\nu+\delta-2)}{\nu(\delta-2)^2(\delta-4)}$ <br> (if $\delta>4$) | Scaled ratio of chi-squares. Used in tests to compare variances | If $X\sim\chi^2_\nu$ and $Y\sim\chi^2_\delta$ independent then $\frac{X/\nu}{Y/\delta}\sim\mathrm{F}_{\nu,\delta}$. <br> If $T\sim\mathrm{t}_\nu$ then $T^2\sim\mathrm{F}_{1,\nu}$. <br> If $Z\sim\mathrm{Be}(\alpha,\beta)$ then $\frac{\beta Z}{\alpha(1-Z)}\sim\mathrm{F}_{2\alpha,2\beta}$. |