MSc Introductory Material - Block A Introduction to Probability and Statistics Solutions to Exercises

1.

(a) $P(4$ cards of same suit $)= \frac{\binom{4}{1}\times\binom{13}{4}}{\binom{52}{4}} = 0.0106$

(b) P (at least two aces)

$$\begin{aligned}
&= 1 - P(\text{ no aces }) - P(1 \text{ ace })\\
&= 1 - \frac{\binom{4}{0}\binom{48}{4}}{\binom{52}{4}} - \frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}}\\
&= 1 - 0.7187 - 0.2556 = 0.0257
\end{aligned}$$

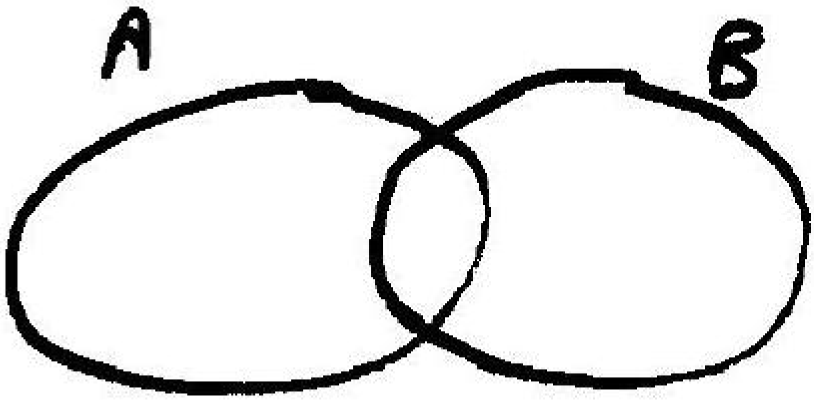
(c) P (same number ‘clubs’ as ‘spades’) $= P(0C, 0S) + P(1C, 1S) + P(2C, 2S)$

$$\begin{aligned}
&= \frac{\binom{13}{0}\binom{13}{0}\binom{26}{4}}{\binom{52}{4}} + \frac{\binom{13}{1}\binom{13}{1}\binom{26}{2}}{\binom{52}{4}} + \frac{\binom{13}{2}\binom{13}{2}\binom{26}{0}}{\binom{52}{4}}\\
&= 0.0552 + 0.2029 + 0.0225\\
&= 0.2806
\end{aligned}$$

2.

(a) $A \cup B = A \cup (B \backslash A)$

Therefore $P(A \cup B) = P(A) + P(B \backslash A)$ (mutually exclusive). Also $(B \backslash A) \cup (B \cap A) = B$.



Therefore $P(B \backslash A) + P(B \cap A) = P(B)$, from Axiom 3, which implies that $P(A \cap B) = P(A) + P(B) - P(A \cap B)$

(b)

$$
\begin{aligned}
P(A \cup B \cup C) &= P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C] \text{ from (a)} \\
&= P(A) + P(B) - P(A \cap B) + P(C) - \{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)\} \\
&\Rightarrow \text{ result}
\end{aligned}
$$

Generally $P\left[\bigcup_{i=1}^n A_i\right] = \sum_i^n P\left(A_i\right) - \sum\sum_{i<j} P\left(A_i \bigcap A_j\right)$

$$
+\sum\sum\sum_{i<j<k} P\left(A_i \bigcap A_j \bigcap A_k\right) \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right).
$$

(c) $P(A \cap B) \geq 0$ therefore $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.

General: not immediately obvious because of alternating signs. Use induction: True for $n = 1, 2$. Assume true for $n = k$ Therefore $P\left(A_1 \bigcup \ldots \bigcup A_k\right) \leq P\left(A_1\right) + \ldots + P\left(A_k\right)$ Now

$$
\begin{aligned}
P\left(A_1 \cup \ldots \cup A_{k+1}\right) &= P\left[\left(A_1 \bigcup \cdots \bigcup A_k\right) \bigcup A_{k+1}\right] \\
&\leq P\left(A_1 \bigcup \cdots \bigcup A_k\right) + P\left(A_{k+1}\right) \quad [\text{ result for } n = 2] \\
&\leq P\left(A_1\right) + P\left(A_2\right) + \ldots + P\left(A_{k+1}\right) \text{ as required}
\end{aligned}
$$

3.

(a) $P\left[\bigcap_1^n A_i\right] = 1 - P\left[\left(\bigcap_1^n A_i\right)^c\right] = 1 - P\left[\bigcup_1^n A_i^c\right] \geq 1 - \sum_1^n P\left(A_i^c\right)$ from 2(c) (b) Require $P\left[\bigcap_1^n A_i\right] \geq 1 - \alpha$ and know $P\left[\bigcap_1^n A_i\right] \geq 1 - \sum_1^n P\left(A_i^c\right)$

Take $\sum_1^n P\left(A_i^c\right) = \alpha$, i.e. $P\left(A_i^c\right) = \frac{\alpha}{n}$ common. Therefore $P\left(A_i\right) = 1 - \frac{\alpha}{n}$. 4.

$$
\begin{aligned}
P\left(B \mid A^c\right) = P\frac{\left(B \cap A^c\right)}{P\left(A^c\right)} &= \frac{P(B) - P(A \cap B)}{1 - P(A)} (\text{ as in } Q.2) \\
&= \frac{P(B) - P(A) P(B \mid A)}{1 - P(A)} \\
&< \frac{P(B) - P(A) P(B)}{1 - P(A)} = P(B)
\end{aligned}
$$

5.

$$
\begin{aligned}
P\left(A \cap B^c\right) &= P(A) - P(A \cap B) \quad (\text{ as in Q.2}) \\
&= P(A) - P(A) P(B) \quad \text{ since A, B independent} \\
&= P(A). P\left(B^c\right) \\
&\text{ie } A, B^c \text{ independent}
\end{aligned}
$$

Replace A by B^c, B by $A \Rightarrow$ same argument $\Rightarrow A^c, B^c$ independent. Generalization: if we take a collection of independent events, and replace any or all of them by their complements, then we still have a collection of independent events. 6. $P(N = 2) = P(RR) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}$
$P(N = 3) = P(RWR) + P(WRR) = \frac{2}{7} \cdot \frac{5}{6} \cdot \frac{1}{5} + \frac{5}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{21}$ Similarly
$P(N = 4) = \frac{3}{21}, P(N = 5) = \frac{4}{21}, P(N = 6) = \frac{5}{21}, P(N = 7) = \frac{6}{21}$. Distribution function
$P(N \leq 2) = \frac{1}{21}$ $P(N \leq 3) = \frac{3}{21}$ $P(N \leq 4) = \frac{6}{21}$ $P(N \leq 5) = \frac{10}{21}$

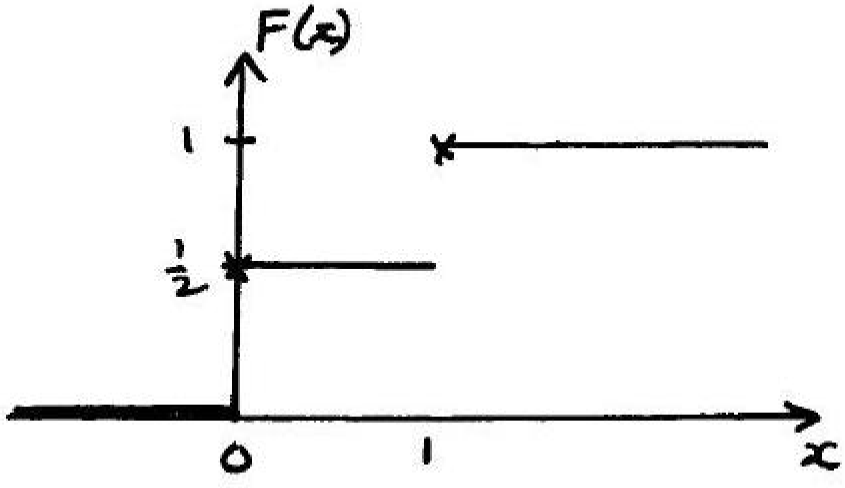
$$
\begin{aligned}
P(N \leq 6) &= \frac{15}{21} \\
P(N \leq 7) &= \frac{21}{21}
\end{aligned}
$$

7.

i.

$$f(x) = \begin{cases} \frac{1}{2} & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



(ii)

$$f(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

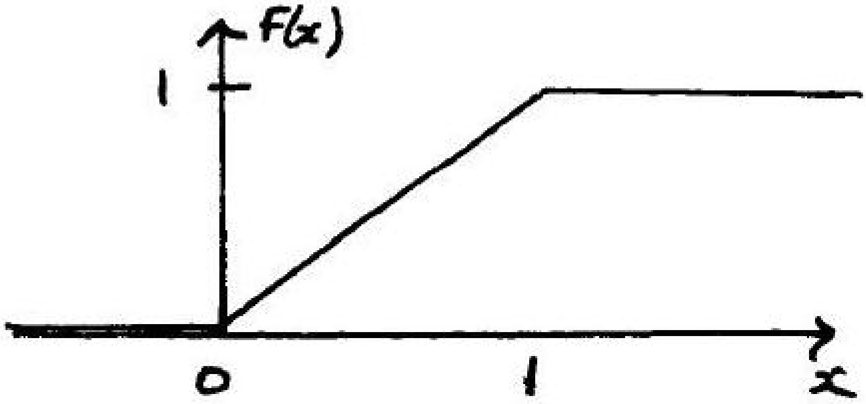
$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

$$\xrightarrow[0]{1/4.\ \int_2^{F(x)} \times}$$

iii.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



8.

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x(1-\theta)^{x-1}\theta = \theta \frac{1}{[1-(1-\theta)]^2} = \frac{1}{\theta} \\
E\{X(X-1)\} &= \sum_{1}^{\infty} x(x-1)(1-\theta)^{x-1}\theta = \theta(1-\theta)\frac{2}{[1-(1-\theta)]^3} = \frac{2(1-\theta)}{\theta^2} \\
\text{Therefore } \operatorname{var}(X) &= E\{X(X-1)\} + E(X) - [E(X)]^2 \\
&= \frac{2(1-\theta)}{\theta^2} + \frac{1}{\theta} - \frac{1}{\theta^2} = \frac{1-\theta}{\theta^2}
\end{aligned}$$

9.

$$\begin{aligned}
P(a \le X \le b) = P(X \le b) - P(X < a) \\
&= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \mathrm{dx} - \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \mathrm{dx} \\
&= \int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\infty}^{\frac{a-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \left[\text{Put } z = \frac{x-\mu}{\sigma}; dz = \frac{dx}{\sigma}\right] \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \quad \text{since integrand is } \phi(z) \text{ the } N(0,1) \text{ density}
\end{aligned}$$

10.

$$E(X) = \int_0^{\infty} x \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x} dx}{\Gamma(\alpha)} = \frac{\alpha}{\lambda} \int_0^{\infty} \frac{\lambda^{\alpha+1} x^{\alpha+1-1} e^{-\lambda x}}{\Gamma(\alpha+1)} dx = \frac{\alpha}{\lambda}$$

$$\text{Similarly } E\left(X^2\right) = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\text{Therefore } \operatorname{var}(X) = E\left(X^2\right) - [E(X)]^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

11. $P(X > a+b \mid X > a) = \frac{P((X>a+b) \bigcap (X>a))}{P(X>a)} = \frac{P(X>a+b)}{P(X>a)}$.

Geometric: $P(X > a) = \sum_{a+1}^{\infty} (1-\theta)^i \theta = \theta(1-\theta)^{a+1}/[1-(1-\theta)] = (1-\theta)^{a+1}$. Thus $\frac{P(X>a+b)}{P(X>a)} = \frac{(1-\theta)^{a+b+1}}{(1-\theta)^{a+1}} = (1-\theta)^b = P(X \ge b)$. If you take the Geometric p.f. in the form $\theta(1-\theta)^{x-1}$ (for $X =$ number of trials, not number of failures, before first success) you obtain the result

$$P(X > a+b \mid X > a) = P(X > b)$$

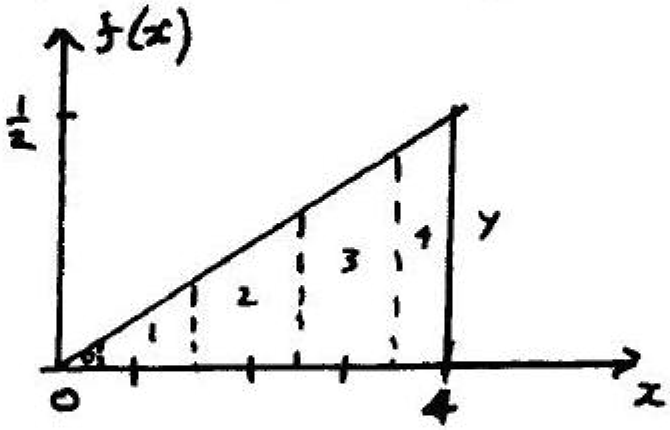
In either case, the key property, that $P(X > a+b \mid X > a)$ is independent of a, is established.

Exponential: $P(X > a) = \int_a^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda a}$. Thus $\frac{P(X>a+b)}{P(X>a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b} = P(X \ge b)$. 12.

$$
\begin{aligned}
\operatorname{var}(X) &= \mu \text{ for } \operatorname{Po}(\mu) \\
\text{so } \operatorname{var}[g(X)] &\simeq [g'(\mu)]^2 \mu \\
&= k, \text{ say} \\
\text{so need } g'(\mu) &= \sqrt{\frac{k}{\mu}} \\
\text{ie } g(\mu) &= 2\sqrt{k\mu}(+\text{ constant}) \\
\text{ie take } g(X) &= 2\sqrt{k}\sqrt{X} \text{ to obtain approximately constant variance} \\
& [\text{ any multiple of } \sqrt{X}]
\end{aligned}
$$

13. $P(Y=0) = P(X \in [0, 0.5)) = \frac{1}{64}$ $P(Y=1) = P(X \in [0.5, 1.5)) = \frac{8}{64}$
$P(Y=2) = P(X \in [1.5, 2.5)) = \frac{16}{64}$ $P(Y=3) = P(X \in [2.5, 3.5)) = \frac{24}{64}$

$P(Y=4) = P(X \in [3.5, 4]) = \frac{15}{64}$



$E(X) = \int_0^4 x \cdot \frac{x}{8} dx = 2.67$ $E\left(X^2\right) = 8 \Rightarrow \operatorname{var}(X) = 0.889$
$E(Y) = \frac{1}{64}[0 \times 1 + 1 \times 8 + \ldots + 4 \times 15] = 2.6878$ $E\left(Y^2\right) = 8.25 \Rightarrow \operatorname{var}(Y) = 1.027$

Grouping has increased both mean and variance. 14. (a)

		Y		
	$P(x,y)$	1	2	3
	1	0	$\frac{1}{6}$	$\frac{1}{6}$
X	2	$\frac{1}{6}$	0	$\frac{1}{6}$
	3	$\frac{1}{6}$	$\frac{1}{6}$	0

Cannot both take same value. Otherwise all possible outcomes equally probable.

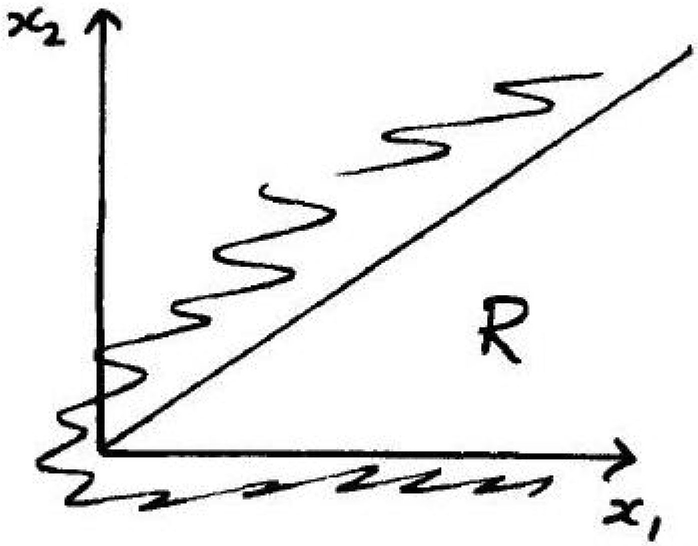
$$P(X < Y) = P\{(1,2), (1,3), (2,3)\} = \frac{1}{2}$$

(b)

x	$p(x)$	y	$p(y)$
1	$\frac{1}{3}$	1	$\frac{1}{3}$
2	$\frac{1}{3}$	2	$\frac{1}{3}$
3	$\frac{1}{3}$	3	$\frac{1}{3}$

$$
\begin{aligned}
P(X = x \mid Y = y) &= \frac{p(x,y)}{p(Y=y)} \\
&= \begin{cases} \frac{1}{2} & x \neq y \\ 0 & x = y \end{cases}
\end{aligned}
$$

(c) No, since, e.g. $P(X=1, Y=1)=0 \neq P(X=1)P(Y=1)=\frac{1}{9}$ 15. Need $\iint_R ke^{-x_1}dx_1dx_2 = 1$ i.e. $\int_{x_2=0}^{\infty} k \int_{x_1=x_2}^{\infty} e^{-x_1}dx_1dx_2 = 1$



i.e. $\int_{x_2=0}^{\infty} ke^{-x_2}dx_2 = 1$ i.e. $k=1$

$$f_{X_1}(x_1) = \int_{x_2=0}^{x_1} e^{-x_1}dx_2 = x_1e^{-x_1} \; (0 < x_1 < \infty)$$

16.

$$\begin{aligned}
\operatorname{var}(X+Y) &= \operatorname{cov}(X+Y, X+Y) \\
&= \operatorname{cov}(X,X) + \operatorname{cov}(Y,X) + \operatorname{cov}(X,Y) + \operatorname{cov}(Y,Y) \\
&= \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y)
\end{aligned}$$

17. Not uniquely invertible - so proceed from first principles

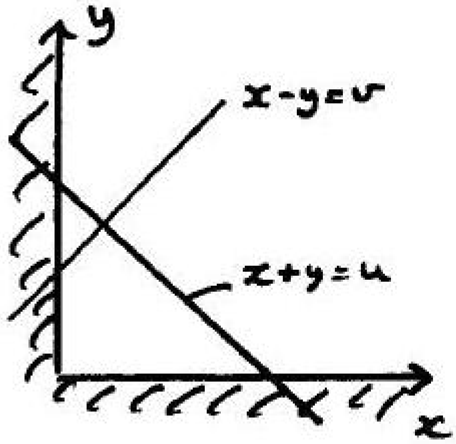
Let $Y = X^2 (y > 0)$

$$\begin{aligned}
F_Y(y) = P(Y \leq y) = P\left(X^2 \leq y\right) &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \text{ by symmetry}
\end{aligned}$$

$$\text{Put } u^2 = v \text{ ie } 2udu = dv \Rightarrow \int_0^y \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}v}}{v^{\frac{1}{2}}} dv$$

Therefore $f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} (y > 0)$ ie $Ga\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi_1^2$. 18. X, Y independent $Ex(1) = Ga(1,1) \quad \Rightarrow X + Y \sim Ga(2,1)$ $f_{X,Y}(x,y) = e^{-(x+y)} (x > 0, y > 0)$. Put

$u = x + y, \quad v = x - y$



therefore $x = \frac{u+v}{2}, y = \frac{u-v}{2}$

$J(u,v) = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$ therefore
$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \left|-\frac{1}{2}\right| = \frac{1}{2} e^{-u} (u > 0, -u < v < u)$ therefore
$f_U(u) = \int_{-u}^{u} dv \cdot \frac{1}{2} e^{-u} = u e^{-u}$ ie $Ga(2,1)$. 19.

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(\mu, \mathbf{\Sigma}) = N\left(\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}\right)$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \mathbf{B} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\text{Thus } \mathbf{B}\mu = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, B\mathbf{\Sigma}B' = \begin{pmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\text{So } \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right).$$

20. $X_1, \ldots, X_n \sim \mathrm{Ex}(\lambda) \equiv Ga(1, \lambda)$ therefore $\sum_1^n X_i \sim Ga(n, \lambda)$ therefore

$$P(\bar{X} \le x) = P\left(\sum X_i \le nx\right) = \int_0^{nx} \frac{\lambda^n u^{n-1} e^{-\lambda u}}{\Gamma(n)} du$$
$$= \int_0^{x} \frac{(n\lambda)^n v^{n-1} e^{-n\lambda v}}{\Gamma(n)} dv$$

ie $\bar{X} \sim Ga(n, n\lambda)$. Therefore $E\left(\frac{1}{\bar{X}}\right) = \int_0^\infty \frac{1}{v} \frac{(n\lambda)^n v^{n-1} e^{-n\lambda v}}{\Gamma(n)} dv = \frac{n\lambda}{n-1} \int_0^\infty \frac{(n\lambda)^{n-1} v^{(n-1)-1} e^{-n\lambda v}}{\Gamma(n-1)} dv$

$$= \frac{n\lambda}{n-1}, n \ge 2.$$

21. $X_1 \sim Po(a\lambda), X_2 \sim Po(b\lambda)$

(i) $T_1 = \frac{X_1 + X_2}{a+b}$: $E(T_1) = \frac{1}{a+b}\{E(X_1) + E(X_2)\} = \frac{1}{a+b}(a\lambda + b\lambda) = \lambda$ unbiased
$\mathrm{var}(T_1) = \frac{1}{(a+b)^2}\{\mathrm{var}(X_1) + \mathrm{var}(X_2)\} = \frac{1}{(a+b)^2}(a\lambda + b\lambda) = \frac{\lambda}{a+b}$

(ii) $T_2 = \frac{X_1 - X_2}{a-b}$: Not very sensible, since could take $-ve$ values
$E(T_2) = \frac{1}{(a-b)}[a\lambda - b\lambda] = \lambda$ unbiased

$\operatorname{var}(T_2) = \frac{1}{(a-b)^2}[\operatorname{var}(X_1)+\operatorname{var}(X_2)] = \frac{a+b}{(a-b)^2}\lambda$

(iii) $T_3 = \frac{1}{2}\left(\frac{X_1}{a}+\frac{X_2}{b}\right) : E\left(T_3\right) = \frac{1}{2a}a\lambda + \frac{1}{2b}b\lambda = \lambda$ unbiased

$$\operatorname{var}(T_3) = \frac{1}{4a^2}a\lambda + \frac{1}{4b^2}b\lambda = \frac{a+b}{4ab}\lambda$$

Dismiss estimator T_2; T_1, T_3 both unbiased, but $\operatorname{var}(T_1) \leq \operatorname{var}(T_3) \Rightarrow T_1$ is best
[check: $\frac{1}{a+b} < \frac{a+b}{4ab} \Leftrightarrow (a+b)^2 > 4ab \Leftrightarrow (a-b)^2 > 0$ which is true]
[In fact, T_1 is the minimum variance unbiased estimator of λ].

22. $L(\mathbf{X} \mid \theta) = \prod_{i=1}^n \theta X_i^{\theta-1} = \theta^n \prod_1^n X_i^{\theta-1}$
therefore $\ell(\theta) = \ln L(\mathbf{X} \mid \theta) = n\ln\theta + (\theta-1)\sum_1^n \ln X_i$
therefore $\frac{d\ell}{d\theta} = \frac{n}{\theta} + \sum \ln X_i = 0$ if $\hat{\theta} = \frac{-n}{\sum_1^n \ln X_i}$
$\left[\frac{d^2\ell}{d\theta^2} = -\frac{n}{\theta^2} < 0 \text{ so maximum}\right]$.

23. $L(X \mid \theta) = \binom{n}{X}\theta^X(1-\theta)^{n-X}$
therefore $\ell(\theta) = \ln\binom{n}{X} + X\ln\theta + (n-X)\ln(1-\theta)$
therefore $\frac{d\ell}{d\theta} = \frac{X}{\theta} - \frac{n-X}{1-\theta} = 0 \Rightarrow$ ML Estimator $\hat{\theta} = \frac{X}{n}$
(check $\frac{d^2\ell}{\partial\theta^2} < 0$)
$E(\hat{\theta}) = \frac{1}{n}E(X) = \frac{1}{n}.n\theta = \theta$ ie unbiased.

24. $f_{X_i}\left(x_i\right) = \lambda_i e^{-\lambda_i x_i} = \beta z_i e^{-\beta z_i x_i}$
therefore $L(\mathbf{X} \mid \beta) = \Pi_{i=1}^n \beta z_i e^{-\beta z_i X_i}$
therefore $\ell(\beta) = n\ell n\beta + \sum_1^n \ell n z_i - \beta\sum_1^n z_i X_i$
therefore $\frac{d\ell}{d\beta} = 0 \Rightarrow \frac{n}{\hat{\beta}} - \sum z_i X_i = 0 \Rightarrow \text{MLE}\, \hat{\beta} = \frac{n}{\sum_{i=1}^n z_i X_i}$

25. $L\left(\mathbf{X} \mid \alpha, \beta, \sigma^2\right) = \frac{1}{(2\pi)^{n/2}}\frac{1}{\left(\sigma^2\right)^{n/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_1^n\left(X_i-\alpha-\beta t_i\right)^2\right\}$
therefore $\ell\left(\mathbf{X} \mid \alpha, \beta, \sigma^2\right) = -\frac{n}{2}\ell n(2\pi) - \frac{n}{2}\ell n\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_1^n\left(X_i-\alpha-\beta t_i\right)^2$

Put $\sigma^2 = v \Rightarrow \ell(\mathbf{X};\alpha,\beta,v) = -\frac{n}{2}\ell n(2\pi) - \frac{n}{2}\ell n v - \frac{1}{2v}\sum\left(X_i-\alpha-\beta t_i\right)^2$

$$\frac{\partial\ell}{\partial\alpha} = 0 \Rightarrow \frac{1}{v}\sum\left(X_i-\alpha-\beta t_i\right) = 0 \tag{1}$$
$$\frac{\partial\ell}{\partial\beta} = 0 \Rightarrow \frac{1}{v}\sum t_i\left(X_i-\alpha-\beta t_i\right) = 0 \tag{2}$$
$$\frac{\partial\ell}{\partial\nu} = 0 \Rightarrow -\frac{n}{2\nu} + \frac{1}{2\nu^2}\sum\left(X_i-\alpha-\beta t_i\right)^2 = 0 \tag{3}$$

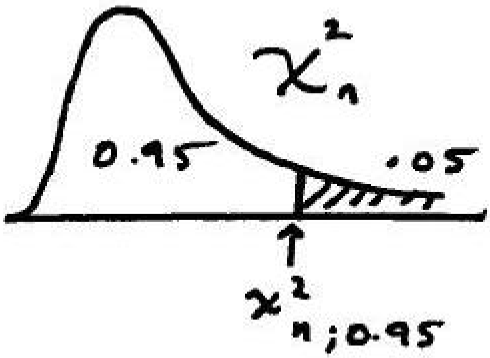
$$(1)\&(2) \Rightarrow \left.\begin{array}{l}\hat{\alpha}+\hat{\beta}\bar{t}_i \\ \hat{\alpha}n\bar{t}_i + \hat{\beta}\sum t_i^2 = \sum t_i\bar{X}\end{array}\right\}$$
$$\Rightarrow \hat{\beta} = \frac{\sum t_i\left(X_i-\bar{X}\right)}{\sum\left(t_i-\bar{t}\right)^2}$$
$$\hat{\alpha} = \bar{X} - \hat{\beta}\bar{t}$$
$$(3) \Rightarrow \hat{v} = \frac{1}{n}\sum_1^n\left(X_i-\hat{\alpha}-\hat{\beta}t_i\right)^2$$

See Block B section 4 for more on this. 26.

$$X_1, X_2, \ldots, X_n \sim N\left(0, \sigma^2\right)$$

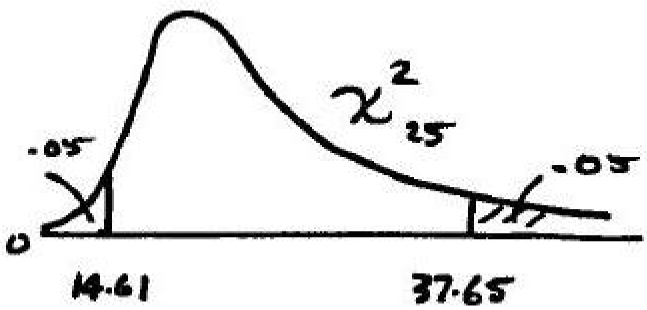
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 = \sigma_1^2\left(>\sigma_0^2\right)$ $\frac{1}{n}\sum X_i^2$ is MLE of σ^2 Form of test : critical region $\chi_1 = \left\{\mathbf{X} : \sum X_i^2 > c\right\}$ Now $\frac{X_i}{\sigma} \sim N(0,1)$, so $\frac{X_i^2}{\sigma^2} \sim \chi_1^2$ therefore $\frac{\sum X_i^2}{\sigma^2} \sim \chi_n^2$ (additivity of χ^2)

Thus to evaluate c,



$$
\begin{aligned}
&\text{set} \quad P\left(\sum X_i^2 > c \mid \sigma^2 = \sigma_0^2\right) &&= \alpha = 0.05,\ \text{say}\\
&\text{ie} \quad P\left(\frac{\sum X_i^2}{\sigma_0^2} > \frac{c}{\sigma_0^2}\right) &&= 0.05\\
&\text{ie} \quad P\left(\chi_n^2 > \frac{c}{\sigma_0^2}\right) = 0.05\\
&\text{ie} \quad c &&= \sigma_0^2 \chi_{n;0.95}^2
\end{aligned}
$$

If $n = 25$



$$
\begin{aligned}
\text{power } &= P\left(\sum X_i^2 > c \mid \sigma^2 = \sigma_1^2\right)\\
&= P\left(\chi_{25}^2 > \frac{c}{\sigma_1^2}\right) \text{ as above}
\end{aligned}
$$

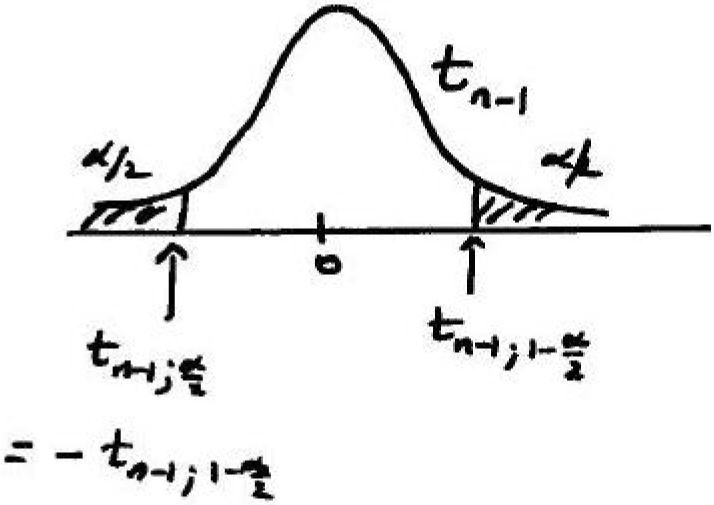
So for power $= 0.95$, we require

$$
\begin{aligned}
\frac{c}{{\sigma_1}^2} &= \chi_{25;0.05}^2 = 14.61\\
\text{Also } \frac{c}{{\sigma_0}^2} &= \chi_{25;0.95}^2 = 37.65\\
\Rightarrow \frac{{\sigma_1}^2}{{\sigma_0}^2} &= \frac{37.65}{14.61} = 2.577
\end{aligned}
$$

27. From Section 6.4.3. if $X_1, X_2, \ldots X_n \sim N\left(\mu, \sigma^2\right)$ then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$ where $\bar{X} = \frac{1}{n}\sum X_i, S^2 = \frac{1}{n-1}\sum\left(X_i - \bar{X}\right)^2$. So

$$
P\left(-t_{n-1;1-\frac{\alpha}{2}} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{n-1;1-\frac{\alpha}{2}}\right) = 1-\alpha
$$

$\Longrightarrow 100(1-\alpha)\%$ CI of form (as in normal case) $\left(\bar{x}-t_{n-1;1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}, \bar{x}+t_{n-1;1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right)$.



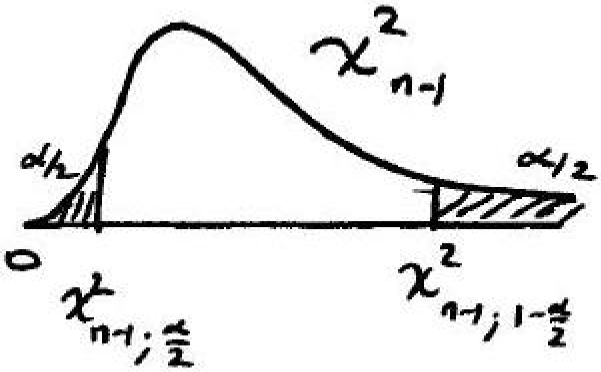
For $n=16, \bar{x}=95.8, s^2=30.25, \alpha=0.1$
$\left(95.8-1.753 \times \frac{\sqrt{30.25}}{4}, 95.8+1.753 \times \frac{\sqrt{30.25}}{4}\right) \Rightarrow (93.39, 98.21)$ 28.

$$X_1, X_2, \ldots, X_n \sim N\left(\mu, \sigma^2\right) \Rightarrow \frac{(n-1) S^2}{\sigma^2} \sim \chi_{n-1}^2$$

Therefore

$$P\left(\chi_{n-1;\frac{\alpha}{2}}^2 \leq \frac{(n-1) S^2}{\sigma^2} \leq \chi_{n-1;1-\frac{\alpha}{2}}^2\right)=1-\alpha$$

$100(1-\alpha)\%\mathrm{CI}$ of form



$$\left(\frac{(n-1) s^2}{\chi_{n-1;1-\frac{\alpha}{2}}^2}, \frac{(n-1) s^2}{\chi_{n-1;\frac{\alpha}{2}}^2}\right)$$

Here

$$s^2=25, n=40 \Rightarrow\left(\frac{39 \times 25}{58.12}, \frac{39 \times 25}{23.65}\right) \Rightarrow(16.78, 41.23)$$

29. $X \sim Bi(2, \theta)$ $H_0: \theta=\frac{1}{2}$ $H_1: \theta=\frac{3}{4}$

$p(x\|\theta)$	$x=0$	1	2
$H_0 : \theta = \frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$H_1 : \theta = \frac{3}{4}$	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{9}{16}$

Test	χ_1	χ_0	α	β
1	$\{0\}$	$\{1,2\}$	$\frac{1}{4}$	$\frac{15}{16}$
2	$\{1\}$	$\{0,2\}$	$\frac{1}{2}$	$\frac{5}{8}$
3	$\{2\}$	$\{0,1\}$	$\frac{1}{4}$	$\frac{7}{16}$
4	$\{0,1\}$	$\{2\}$	$\frac{3}{4}$	$\frac{9}{16}$
5	$\{0,2\}$	$\{1\}$	$\frac{1}{2}$	$\frac{3}{8}$
6	$\{1,2\}$	$\{0\}$	$\frac{3}{4}$	$\frac{1}{16}$
7	$\{0,1,2\}$	ϕ	1	0
8	ϕ	$\{0,1,2\}$	0	1

Test 3 probably best in terms of simultaneous reduction of α and β.

30. $X_1, X_2, \ldots, X_n$ from $\mathrm{Po}(\lambda)$ so $L(\mathbf{X} \mid \lambda) = \frac{\lambda^{\sum X_i} e^{-n\lambda}}{\Pi_1^n X_i!}$ $H_0 : \lambda = \lambda_0$ $H_1 : \lambda = \lambda_1\ (> \lambda_0)$ N-P lemma suggests critical region of form

$$\left\{\mathbf{X} : \frac{L\left(\mathbf{X} \mid \lambda_0\right)}{L\left(\mathbf{X} \mid \lambda_1\right)} < c\right\}$$

ie reject H_0 if $\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum X_i} e^{-n(\lambda_0 - \lambda_1)} < c$
ie if $\sum_1^n X_i \ln\left(\frac{\lambda_0}{\lambda_1}\right) - n\left(\lambda_0 - \lambda_1\right) < c'$
ie if $\sum_1^n X_i \ln\left(\frac{\lambda_0}{\lambda_1}\right) < c''$
ie if $\sum X_i > c''' = k$, say since $\ln\left(\frac{\lambda_0}{\lambda_1}\right) < 0$]
Now $\sum X_i \sim \mathrm{Po}(n\lambda)$ by additivity of Poissons

$$\sim \text{ approximately } N(n\lambda, n\lambda) \text{ if } n\lambda \text{ large.}$$

Therefore set $P\left(\sum X_i > k \mid \lambda = \lambda_0\right) = 1 - \Phi\left\{\frac{k - n\lambda_0}{\sqrt{n\lambda_0}}\right\} = \alpha$
ie $k = n\lambda_0 + \sqrt{n\lambda_0}\Phi^{-1}(1-\alpha)$.

31. $X_1, X_2, \ldots, X_n$ from $Ex(\lambda)$
so $L(\mathbf{X} \mid \lambda) = \lambda^n e^{-\lambda \sum X_i}$
$H_0 : \lambda = \lambda_0$
$H_1 : \lambda \neq \lambda_0$
Use likelihood ratio method:

$$\text{MLE } \hat{\lambda} = 1/\bar{X}$$

so $\Lambda = \frac{L(\mathbf{X}|\lambda_0)}{L(\mathbf{X}|1/\bar{X})} = \frac{\lambda_0{}^n e^{-n\lambda_0 \bar{X}}}{(\bar{X})^{-n} e^{-n}} = \left(\lambda_0 \bar{X}\right)^n e^{-n\left(\lambda_0 \bar{X} - 1\right)}$
so test has critical region

$$\chi_1 = \left\{\mathbf{X} : \left(\lambda_0 \bar{X}\right)^n e^{-n\left(\lambda_0 \bar{X} - 1\right)} < c\right\}$$
$$\Leftrightarrow \ln\left(\lambda_0 \bar{X}\right) - \lambda_0 \bar{X} < c'$$

Consider

$$f(y) = \ln y - y$$
$$f'(y) = \frac{1}{y} - 1 = 0 \text{ if } y = 1$$
$$f''(y) = -\frac{1}{y^2}$$

Thus, $f(y)$ increases from $-\infty$ as y increases from zero, until it reaches a maximum of -1 when $y = 1$, and then it decreases steadily to $-\infty$ as y increases further. Thus the curve is above c (which must be < -1) provided y is between the two roots of $f(y) = c'$. Hence

$$\chi_1 = \{\mathbf{X} : \bar{X} < a \text{ or } b < \bar{X}\}$$

where a, b are roots of $f(y) = c'$.

32. $X_1, X_2 \ldots, X_n \sim N\left(\mu, \sigma^2\right)$ $\left(\sigma^2 \text{ known }\right)$
$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$
$L(\mathbf{X} \mid \mu) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$ MLE $\hat{\mu} = \bar{X}$
Thus $\Lambda = \frac{L(\mathbf{X}|\mu_0)}{L(\mathbf{X}|\bar{X})} = \exp\left\{-\frac{1}{2\sigma^2}\left[\sum (X_i - \mu_0)^2 - \sum \left(X_i - \bar{X}\right)^2\right]\right\} = \exp\left\{-\frac{n}{2\sigma^2}\left(\bar{X} - \mu_0\right)^2\right\}$
Therefore

$$-2 \ln \Lambda = \frac{n}{\sigma^2}\left(\bar{X} - \mu_0\right)^2 = \left(\frac{\sqrt{n}\left(\bar{X} - \mu_0\right)}{\sigma}\right)^2$$

Now

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right) \text{ if } H_0 \text{ is true } \Rightarrow \frac{\sqrt{n}\left(\bar{X} - \mu_0\right)}{\sigma} \sim N(0,1) \text{ under } H_0$$

Therefore

$$\text{under } H_0, -2 \ln \Lambda \sim \chi_1^2$$