MSc Introductory Material
Mathematical Methods
Exercises

These exercises are on a range of mathematical methods which are useful in different parts of the MSc, especially calculus and linear algebra. They are largely taken from the mathematical methods course in the Graduate Certificate in Statistics.

There are a large number of questions here; we do not necessarily expect you to do all of them but, as many of the techniques involved come up in the MSc, you should use the questions to see whether you need help with any of them. A help session will be provided in Intro Week.

Functions and sets; permutations and combinations

1. Let h be the length of a chord of a circle of radius 5 whose distance from the centre is x.
   (a) Use Pythagoras' theorem to express h in terms of x.
   (b) What is h when $x = 4$?
   (c) What is the domain of this function h ?
2. What are the ranges of the following functions?
   (a) $y = \sqrt{1 - x^2}$, domain $[-1, 1]$;
   (b) $y = x^3$, domain $\mathbb{R}$;
   (c) $y = \frac{1}{(1+x^2)}$, domain $\mathbb{R}$.
3. Write the following sets in list form, i.e. as $\{a_1, a_2, \dots a_n\}$.
   (a) $\{x \in \mathbb{R} : x^2 + x - 6 = 0\}$;
   (b) $\{x \in \mathbb{Q} : (x - 3)(x^2 - 3) = 0\}$
4. Find $A \cap B$ in each case.
   (a) $A = \{1, 2, 3, 4, 5\}, B = \{0, 3, 5, 7, 8, 9\}$;
   (b) $A = \mathbb{N}, B = \{x \in \mathbb{R} : x^2$ lies between 10 and 30$\}$;
   (c) $A = \emptyset, B = \mathbb{R}$
   (d) $A = \mathbb{Q}, B = \mathbb{R}$
   (e) $A = \mathbb{Q}, B = \mathbb{Q}$
   (f) $A = \{$ English cities with more than one Premier League football team $\}, B = \{$ English cities with more than one University }
5. (a) A bag contains 11 balls, of which a combination of 3 balls is to be drawn. In how many possible ways can this be done?
   (b) Suppose that of these 11 balls, 5 are red, 4 are blue and 2 are white. In how many of the above possible combinations are there balls of two colours with the third colour absent?

Inequalities

6. Solve the following inequalities:
   (a) $5 - 2x < 9 + 3x$;
   (b) $|t + 5| < 6$;
   (c) $|3y - 9| < 4$;
   (d) $x^2 - x - 6 < 0$;
   (e) $(x + 3)^2 < 2$;
   (f) $|3 - 2/t| < 1/2$.

7. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}$$

Confirm your findings algebraically.

Logarithms

8. Let

$$f(x) = \left\{ \frac{\left(1+x^2\right)^5 \cdot \sin^7 5x}{x^{2x}} \right\}^{\frac{1}{3}}$$

Express $\log_e f(x)$ as simply as possible. 9. When working with likelihoods you will encounter expressions like

$$L = \prod_{j=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(x_j-\mu\right)^2}{2\sigma^2}}$$

Find the $\log$ (to base e) of L and simplify it as much as you can.

Summation of series

10. Find the value of k such that

$$\sum_{j=1}^{\infty} kp^j = 1$$

11. (a) Find the value of the sum of the first 257 odd numbers.
    (b) Also find the sum of all these numbers excluding those which are divisible by 3 .
12. (a) Find the value of

$$1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \frac{81}{256}$$

expressed as a fraction in its lowest terms. [Do not do the calculation directly, but use the general theory.] (b) Express the recurring decimal $0.45454545\ldots$ as a fraction in its lowest terms.
(c) Evaluate

$$\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{3n-2} x^n$$

stating for what values of x the series converges.

13. Let

$$a_n = \frac{n+4}{n(n+1)(n+2)}$$

Use partial fractions and telescoping series techniques to find $\sum_{j=1}^{n} a_j$ and hence $\sum_{j=1}^{\infty} a_j$.

Limits

14. Let $f(x) = \frac{x^4-x^3+x^2-1}{x-1}$ for $x \neq 1$. Do a table of values of $f(x)$ for some values of x very close to 1 . Prove algebraically that $\lim_{x\to 1} f(x)$ is what it appears to be.

15. Do you think that $\lim_{x \to \infty} (1 + 1/x)^x$ exists? Plug in larger and larger values of x to the formula $(1 + 1/x)^x$ and do a table. Now reconsider your opinion.

Differentiation

16. For each of the functions $y = x^2$ and $y = x^3$, find $\delta y/\delta x$ from basics, and hence find dy/dx.
17. Let $f(x) = x^3$. Calculate $\frac{f(x)-f(2)}{x-2}$ for a range of values of x approaching 2 from above and repeat for a range of values of x approaching 2 from below. Do the values appear to be converging to a limit as $x \to 2$? Compare your answers with the derivative of $f(x)$ evaluated at $x = 2$.
18. Evaluate:
    (a) $\frac{d}{dx}(2x^n + 4x^m + 122)$ (where n and m are both > 0);
    (b) $\frac{d}{dx}\left(23x^5 + 24x^3 + 12x + 29\right)$
19. Differentiate:
    (a) $(x-1)\left(x^2 - x + 1\right)$ and
    (b) $5/\left(x^2 - 1\right)$.
20. Differentiate:
    (a) $y = \left(x^2 + 1\right)^7$;
    (b) $y = \left(x^3 + 1\right)^5$;
    (c) $y = \sqrt{x^2 + 1}$.
21. Differentiate $y = \left(\frac{1+x}{1-x}\right)^3$
22. Differentiate:
    (a) $\sin\left(x^2\right)$;
    (b) $\sin^2 x$;
    (c) $\cos 3x$;
    (d) $\tan x$.
23. Establish the formulae:

$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}(\text{ for } a > 0)$$

and

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$$

24. Find $\frac{dy}{dx}$ where:
    (a) $y = e^{\cos x^2}$ (note $\cos x^2$ is not to be confused with $\cos^2 x$);
    (b) $y = xe^{2x}$;
    (c) $y = e^{2x}\sin 4x$
25. Find $\frac{dy}{dx}$ where:
    (a) $y = \log\left(x^2 + 2x + 3\right)$;
    (b) $y = \log(\cos 4x)$;
    (c) $y = \log(\log x)$

Integration

26. Use the formula $\frac{1}{2}\times$ base $\times$ height for the area of a triangle to calculate $\int_0^3 (2 - x)dx$.

27. Let $f(x) = \frac{1}{x^2}$. Show that the indefinite integral $\int f(x)dx = \frac{-1}{x} + C$. Hence find $\int_1^y f(x)dx$. What happens as $y \to \infty$?
28. Evaluate $\int_0^1 \left(x^3 + 2x^2 - 1\right) dx$.
29. Using: $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ evaluate

$$\int_0^\pi \cos^2 x dx.$$

30. Show that the improper integral

$$\int_1^\infty \frac{dx}{x^2+1}$$

exists and and find its value.

31. Find the indefinite integral

$$\int x^3 e^{-x^4} dx.$$

Hence find the value of the improper integral

$$\int_0^\infty x^3 e^{-x^4} dx$$

32. Find

$$\int_0^\pi \frac{\cos x}{\sin^2 x} dx$$

using the substitution $u = \sin x$.

33. Find

$$\int_0^1 \sqrt{(1-x^2)} dx$$

34. For each integer $n \geq 0$ let $I_n = \int_0^\pi x^n \cos x dx$. Determine I_n in terms of I_{n-2} for $n \geq 0$.

The Gamma function

35. (a) $\Gamma\left(-\frac{1}{2}\right)$,
    (b) $\Gamma\left(-\frac{5}{2}\right)$ and
    (c) $\Gamma\left(-\frac{7}{2}\right)$.
36. Use the Gamma function to find
    (a) $\int_0^\infty e^{-x^3} dx$,
    (b) $\int_0^\infty x^{\frac{1}{4}} e^{-x^{\frac{1}{2}}} dx$ and
    (c) $\int_0^\infty y^3 e^{-2y^5} dy$.

Maximisation and minimisation in one dimension

37. Find and classify the stationary point(s) of:

$$x^2 - 6x + 8$$

38. Find and classify the stationary point(s) of:

$$x^3 - 12x + 5.$$

Vectors

39. Let $\mathbf{v} = (2, 1)$ and $\mathbf{w} = (4, -1)$. What is the scalar product of $\mathbf{v}$ and $\mathbf{w}$? What is the angle between the two vectors?
40. Let $\mathbf{v} = (2, 1, 4, -1)$ and $\mathbf{w} = (4, -1, 0, 2)$. What is the scalar product of $\mathbf{v}$ and $\mathbf{w}$? What is the angle between the two vectors?

Partial differentiation

41. Let $f(x, y) = x^3y^2 - 5e^{xy}$. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
42. Let $\phi(x, y) = x^3y + e^{xy^2}$. Find

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial x \partial y}, \frac{\partial^2 \phi}{\partial y^2}.$$

43. Let $z = x^4 + 2x^2y^2 + y^4$, where $x = r\cos\theta$ and $y = r\sin\theta$. Use the Chain Rule to show that

$$\partial z/\partial r = 4r^3 \text{ and } \partial z/\partial \theta = 0.$$

Then, as an alternative method, express z in terms of r and θ and obtain these answers more directly.

Maximisation and minimisation in higher dimensions

44. Find the critical points of the function $f(x, y) = y^3 - 3xy + x^2 - 2x$. For each one, determine whether it is a local maximum, a local minimum or a saddle point.
45. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy + 1$. Do you notice any feature of these points that you might have anticipated? For each point, determine whether it is a local maximum, a local minimum or a saddle point.

Multiple integrals

46. Let S be the region $\{(x, y) : 1 \le x \le 2, 0 \le y \le 1\}$.
    (a) Evaluate

$$\iint_S \left(x^2 - 3y^2\right) dxdy$$

    (b) Evaluate

$$\iint_S \frac{1}{\left(x + y^2\right)^2} dxdy$$

47. Let D be the region $\{(x, y) : x \ge 0, y \ge 0, x + y \le 1\}$. Evaluate the following integral either first with respect to x and then with respect to y, or else the other way round:

$$\iint_D \left(x^2 - 3y^2\right) dxdy$$

48. Let D be the region in the (x, y)-plane for which $x^2 + y^2 < 1, y \geq 0$. By changing to polar coordinates show that

$$\iint_D \frac{(x+y)^2}{1+(x^2+y^2)^2} dxdy = \frac{\pi}{4} \ln 2$$

49. Let D be the region $\left\{(x, y) : x \geq 0, 0 \leq y \leq e^x \text{ and } 1 \leq x^2 + y^2 \leq 2\right\}$. Let $u = x^2 + y^2, v = ye^{-x}$. Evaluate the Jacobian $\partial(u, v)/\partial(x, y)$ and use it to find the double integral

$$\iint_D \frac{\left(x+y^2\right) e^{-x}}{x^2+y^2+1} dxdy$$

Systems of linear equations; Gaussian elimination

50. Find all solutions of the following systems of equations:

(a)

$$\begin{aligned} x + 2y &= 10 \\ x + 5y &= 16 \end{aligned}$$

(b)

$$\begin{aligned} 3x - 2y &= 12 \\ y - x &= 4 \end{aligned}$$

(c)

$$\begin{aligned} x + y &= 11 \\ x - y &= 3 \\ 2x - 3y &= 5 \end{aligned}$$

51. For what values of a does the linear system

$$\begin{aligned} x + y &= 3 \\ 5x + 5y &= a \end{aligned}$$

have (a) no solution; (b) exactly one solution; (c) infinitely many solutions?

52. For the three sets of equations i), ii) and iii):

(a) form the augmented matrix ($A \mid \mathbf{b}$).

(b) reduce ($A \mid \mathbf{b}$) to row echelon form ($H \mid \mathbf{c}$) where H is upper triangular.

(c) find all the solutions of the equations.

i.

$$\begin{aligned} x + 2y + z &= 2 \\ 2x + 3y + z &= 4 \\ x + y - z &= 3 \end{aligned}$$

ii.

$$\begin{aligned} x + 2y + t &= 3 \\ 2x + z + 2t &= 5 \\ x + 3y + z - t &= -2 \\ -5y + 2t &= 4 \end{aligned}$$

iii.

$$\begin{aligned} x+y+z&=2\\ x-y+2z&=4\\ 2y-z&=2 \end{aligned}$$

Matrices

53. Given the matrices

$$A=\begin{pmatrix}1 & -2\\ -1 & 1\\ 0 & 2\end{pmatrix},\quad B=\begin{pmatrix}3 & 2\\ -6 & -4\end{pmatrix},\quad C=\begin{pmatrix}5 & -4\\ 1 & -2\end{pmatrix}$$

(a) Evaluate each of the following, where possible. If evaluation is not possible, explain why not.
i) $C-3A$ ii) $C-3B$ iii) AB iv) BA v) BC vi) CB
(b) Confirm by evaluation that $\mathrm{tr}(BC)=\mathrm{tr}(CB)$.
(c) Determine the matrix $(AC)^T$ and verify that $(AC)^T=C^TA^T$.

54. (a) Form the product BA where

$$A=\begin{bmatrix}1 & 3 & 4\\ 5 & 6 & 2\end{bmatrix},\quad B=\begin{bmatrix}2 & 7\\ -1 & 2\end{bmatrix}$$

(b) Explain why the product AB cannot be formed and show that A^TA is symmetric.

55. Use elementary row operations to determine A^{-1} where

$$A=\begin{pmatrix}1 & 3 & -1\\ -2 & -5 & 1\\ 4 & 11 & -2\end{pmatrix}$$

56. Given that

$$A^{-1}=\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\quad B^{-1}=\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$

determine $(AB)^{-1}$.

57. (a) Evaluate the following determinant

$$\begin{vmatrix}4 & 1 & 0\\ 0 & 2 & -1\\ 2 & 3 & 1\end{vmatrix}$$

by i) expanding by the first row ii) expanding by the first column.
(b) Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

58. (a) Calculate

$$\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 6 \end{vmatrix}$$

(b) Use the results of (a) and the properties of determinants to write down the values of:

i. $\begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 6 \\ 0 & 2 & 1 \end{vmatrix}$

ii. $\begin{vmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 6 \end{vmatrix}$

iii. $\begin{vmatrix} 2 & 3 & 3 \\ 3 & 3 & 6 \\ 0 & 6 & 1 \end{vmatrix}$

Eigenvalues and eigenvectors

59. Find the eigenvalues and eigenvectors of the following matrices

i) $\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

ii) $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 6 \\ 3 & 1 & 1 \end{pmatrix}$

iii) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Confirm that the trace is the sum of the eigenvalues in each cases.

60. Write down the eigenvalues of the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and determine the corresponding eigenvectors.

Diagonalisation of symmetric matrices

61. Let A be the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 6 \\ 3 & 1 & 1 \end{pmatrix}$$

In Exercise 59(ii) you found the eigenvalues and eigenvectors of A; use this to show how A can be diagonalised.

62. Let P be the matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(This is the transition matrix of a simple Markov chain, for those of you who know what this means.) Also let Q be the matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

which has inverse

$$Q^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

Use the fact that

$$Q^{-1}PQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

to find P^{12}.