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A. Citanna and M. Tvede

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Do taxspots matter?
A study of optimal tax uncertainty*

A. Citanna[†] M. Tvede [‡]
NYU AD U Sheffield

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Abstract

Should the government run an uncertain fiscal policy to finance its liabilities? We call the resulting uncertainty taxspots, and study conditions that make taxspots optimal, and recurrent, in standard Ramsey problems. We show that prudence and market incompleteness play a role in sustaining taxspots, and that equal-treatment randomizations can be decentralized via taxspots even in the absence of financial markets.

Keywords: Ramsey taxation, sunspots, lottery equilibrium.

JEL: D51, D52, D84, E62, H21

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[†]New York University Abu Dhabi, UAE, ac6609@nyu.edu
[‡]University of Sheffield, UK, m.tvede@sheffield.ac.uk

Introduction

A standard principle in public finance (see, e.g., Barro, 1979) is that, for risk averse households, taxation should be designed so as to smooth out lifetime consumption, aiming at fiscal certainty over time. Yet, tax uncertainty is recognized, usually under a negative view, as a feature of many tax systems in and outside the group of OECD countries (see, e.g., 'Tax Certainty', IMF/OECD Report for the G20 Finance Ministers, 2017), even when stemming from policy design and legislation. We study when optimal fiscal policies must involve fiscal uncertainty, or 'taxspots'. Taxspot equilibria are competitive equilibria where uncertainty comes from random tax rates. Consequently, taxspots provide new insights about the properties of optimal taxes.

We focus on Ramsey problems, where a benevolent planner chooses the income tax rates on capital and labor that maximize household welfare, taking into account the households' optimizing behavior, the government budget constraint, and feasibility. We show that for many economies optimal taxes involve taxspots. From a mathematical point of view, the reason is simple: the 'incentive constraints' reflecting optimal behavior of households need not be concave, implying that the planner's value function need not be concave. For Ramsey problems, concavity of the incentive constraints depends on the curvature of the marginal utility, itself a function of the third-order derivatives of utility. If the lack of concavity of the incentive constraints is larger than the concavity of the utility function, then uncertain tax rates can Pareto improve. From an economic point of view, if households are prudent, then adding uncertainty to disposable income increases investment or the labor supply. With sufficient prudence, risk averse households are more than compensated for the additional consumption uncertainty by an increased expected consumption.

We consider the neoclassical growth model with perfectly competitive markets and representative households. First, we study economies with complete financial markets, where the planner can issue taxspot contingent bonds. We show that taxspots can be Pareto improving. Since financial markets are complete, households can fully insure themselves against taxspots, but they will not at equilibrium, because the cost of full insurance is too high. Moreover, we show that market completeness implies that taxspots need not involve more than two episodes of random tax rates. Consequently, there is a finite date after which there is no more tax uncertainty.

We then study economies with no financial markets, where the planner cannot issue bonds and the primary deficit has to be zero. Optimal taxation in these economies has been studied by Benhabib and Rustichini (1997), Phelan and Stacchetti (2001), and Straub and Werning (2020), among others. We show that taxspots can be Pareto improving and recurrent, i.e., there is no finite date after which there is no more tax uncertainty.

Recurrent taxspots break the serial correlation between government expenditures and taxes. In contrast, a common belief is that optimal taxes smooth consumption, labor income taxes are essentially constant over the business cycle, and capital income taxes adjust only to innovations in exogenous shocks (see, e.g., Chari et al., 1994).

Finally, in the appendices we show how to change utility functions to ensure that equilibria are unchanged and the sufficient conditions for existence of taxspots are satisfied. Moreover, we provide a new turnpike theorem for the neoclassical growth model which we use to produce examples of economies with taxspots.

Related literature: The first study to highlight that tax uncertainty can be Pareto improving is Stiglitz (1982), where there are two dates and neither capital accumulation nor market completeness versus incompleteness is considered. Households face the same tax rates ex-ante but not ex-post, leading to horizontal inequality. Our taxspots make all households face the same tax rates ex-post, and do not violate horizontal equity.

Bizer and Judd (1989) study the dynamic consequences of random taxes in a neoclassical growth model with incomplete markets and exogenous randomness in taxes. They find that the welfare loss associated with random taxes can be limited.

Hagedorn (2010) considers complete markets economies without capital and with money in the utility function. Two-period cycles are found to be welfare improving in economies with no uncertainty and constant government expenditures. Our sufficient condition for taxspots to be Pareto improving resembles and extends their condition for cycles.

In their recursive formulation of Ramsey problems, Marcet and Marimon (2019) acknowledge the nonconcavity of incentive constraints. In related work on recursive formulations by Pavoni et al. (2018) as well as Cole and Kubler (2012) the technical issues arising from the nonconcavity are addressed by convexifying incentive constraints using a public randomization device. Yet no condition is given for Pareto improving lotteries.

Our general observation that uncertain policy can be Pareto improving naturally applies to models with more frictions than we consider. The frictions can come from the planner's inability to commit, as in Benhabib and Rustichini (1997) and Phelan and Stacchetti (2001), from political considerations as in Acemoglu et al. (2011), or from technology, as is the case for trade being organized through bilateral matching resulting in a sequence of incentive constraints. In fact, Phelan and Stacchetti (2001) allow for uncertainty to convexify an equilibrium correspondence. We conjecture that taxspots arising because of prudence are a robust feature of optimal fiscal policy even in the presence of additional constraints imposed on the planner.

We connect the planner's problem with taxspots to competitive equilibria with 'extrinsic' uncertainty, as in Shell and Wright (1993) and related literature on sunspots and lotteries in static economies – see Rogerson (1988) and Garratt et al. (2002), where nonconvexities come from indivisibilities, and Kehoe et al. (2002), where they come from informational problems. In Goenka and Prechac (2006) it is found that, for economies with two dates and incomplete markets, in the presence of prudence sunspots can make some consumers better off, but others worse off. We dub equilibria with uncertain fiscal policy 'taxspots' because they depend on extrinsic uncertainty. For taxspots, how equilibria depend on the extrinsic uncertainty is chosen by the planner. However, in one interpretation of fiscal policies (Ljungqvist and Sargent 2018, p. 1041) they are "a description of a system of public expectations to which the government conforms," connecting taxspots and sunspots. Our taxspots are implementable by making fiscal policy depend on 'sentiments' or higher-order beliefs circulating in the market, following the interpretation given in Angeletos and La'O (2020) and related literature.

The paper is organized as follows. Section 1 introduces the basic model. Section 2 presents the mathematical structure common to Ramsey problems and giving rise to random improvements and optimally uncertain taxes. Sections 3 and 4 focus on the cases of complete and totally incomplete financial markets, respectively. Appendix A contains some of the proofs. Appendix B presents an ancillary perturbation argument, and Appendix C details the construction of an example of economy where the Ramsey solution converges to a steady state.

1 Optimal linear taxation

Ramsey taxation problems are usually characterized by a fixed sequence of government expenditures that must be financed by linear taxes on the income. We examine the simplest case of a production economy with a representative household with preferences over consumption and leisure and a representative firm transforming capital and labor into a consumption good. Moreover, in line with a large part of the literature, we assume that the government can commit to a tax plan.

More precisely, a discrete time infinite horizon economy faces an uncertain and exogenous sequence of government expenditures and productivity shocks $(g_t, a_t)_{t\geq 0} = (\theta_t)_{t\geq 0}$ with $\theta_t \in \Theta$, where Θ is finite. A history (or date-event) is $\theta^t = (\theta_0, \dots, \theta_t)$ and $\pi_{\theta^t} \in [0,1]$ its probability. Unless stated otherwise all processes are adapted to the tree generated by $(\theta_t)_{t\geq 0}$.

There is a continuum of identical households. The representative household has an initial stock of capital k_0, one unit of time at every date and preferences represented by expected discounted utility over consumption and leisure $(c_t, x_t)_{t\geq 0}$,

$$\mathbb{E}_0 \sum_{t\geq 0} \beta^t u(c_t, x_t),$$

where $\mathbb{E}_t$ is the expectation at date t, u is the instant utility function and $\beta \in (0,1)$ is the discount factor. The representative firm has a constant returns to scale production function $f_{a_t}(\cdot)$ transforming capital and labor $(k_t, 1-x_t)$ into output subject to a productivity shock. Without loss of generality we assume that capital depreciates completely.

Utility and production functions satisfy the following assumptions:

A.1 $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R} \cup -\infty$ is continuous, thrice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$ with $u_c(0,x) = u_x(c,0) = \infty$, $Du(c,x) \in \mathbb{R}^2_{++}$ and $v^T D^2 u(c,x) v < 0$ for all vectors $v \neq 0$.

A.2 $f_a : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous, twice continuously differentiable on $\mathbb{R}^2_{++}$ with $Df_a(k, 1-x) \in \mathbb{R}^2_{++}$, and $v^T D^2 f_a(k, 1-x) v < 0$ for $v \cdot Df_a(k, 1-x) = 0$ with $v \neq 0$. Moreover, $f_a(0, 1-x) = f_a(k, 0) = 0$ and $\lim_{k\to 0} f_{ak}(k, 1-x) > \beta > \lim_{k\to\infty} f_{ak}(k, 1-x)$.

Markets are perfectly competitive. We consider two polar cases regarding financial markets: complete financial markets, and totally incomplete markets with no assets except capital.

Given initial government debt b_0, the government chooses tax rates on capital and labor $\tau_t = (\tau_t^k, \tau_t^\ell)_{t\geq 0}$ and –in case of complete financial markets– state-contingent bonds $(b_{t+1})_{t\geq 0}$. As usual in the literature, we assume that the initial capital tax rate is fixed, at zero, to avoid the trivial front loading of government expenditures via nondistortionary capital taxes on k_0.

The price at date $t = 0$ for the good at date t is p_t, and in the good at date t the return on capital is r_t and the wage is w_t. In case of complete markets, the price of bonds is $q_{t+1} = \pi_{\theta_t \theta_{t+1}} p_{t+1}/p_t$ in the absence of arbitrage.

Feasibility states that consumption, investment and government expenditures must be less than output

$$c_t + k_{t+1} + g_t \ \leq \ f_{a_t}(k_t, 1-x_t).$$

The sequential household budget constraint at date t is

$$c_t + k_{t+1} + \mathbb{E}_t \frac{p_{t+1}}{p_t} b_{t+1} \ \leq \ (1-\tau_t^k) r_t k_t + (1-\tau_t^\ell) w_t (1-x_t) + b_t.$$

In addition, the household has to satisfy the transversality conditions on capital, and on bonds –in case of complete financial markets. The sequential government budget constraint at date t is

$$g_t + b_t \ \leq \ \tau_t^k r_t k_t + \tau_t^\ell w_t (1-x_t) + \mathbb{E}_t \frac{p_{t+1}}{p_t} b_{t+1}.$$

An *equilibrium* at (Θ,π) and b_0,k_0 is prices $(p_t,r_t,w_t)_{t\geq 0}$, consumption, leisure, capital and bonds $(c_t,x_t,k_{t+1},b_{t+1})_{t\geq 0}$, and taxes $(\tau_t)_{t\geq 0}$ such that: markets clear; households maximize expected utility subject to its budget constraints and transversality conditions; firms maximize profits; and, the government satisfies its budget constraints. Initial capital k_0 and initial debt b_0, and expenditures $(g_t)_{t\geq 0}$ are assumed compatible with equilibrium existence. An equilibrium is *interior* provided $(c_t,x_t,k_{t+1})\in\mathbb{R}_{++}\times(0,1)\times\mathbb{R}_{++}$ for every $t\geq 0$.

The first-order conditions and the transversality conditions for the household are

$$\begin{cases} u_x(c_t,x_t) &= u_c(c_t,x_t)(1-\tau_t^{\ell})w_t \\ \mathbb{E}_t u_c(c_{t+1},x_{t+1})(1-\tau_{t+1}^k)r_{t+1} &= u_c(c_t,x_t) \\ \lim\limits_{t\to\infty}\mathbb{E}_0\,\beta^t u_c(c_t,x_t)k_{t+1} &= 0 \\ \lim\limits_{t\to\infty}\mathbb{E}_0\,\beta^t u_c(c_{t+1},x_{t+1})b_{t+1} &= 0. \end{cases}$$

The first-order conditions for the firm are

$$\begin{cases} r_t &= f_{a_t k}(k_t,1-x_t) \\ w_t &= f_{a_t \ell}(k_t,1-x_t). \end{cases}$$

A *Ramsey Problem* consists in finding taxes and bonds that maximize the households' expected utility over all possible taxes and bonds for which equilibria exist.

2 Pareto improving lotteries: A general framework

To study the emergence of taxspots, we offer an abstract formulation of the Ramsey Problem which will highlight its common mathematical structure across financial markets variations and beyond, and its main properties.

Random improvements

A Ramsey Problem can be represented as an optimization problem of the form:

$$\begin{aligned} &\max_{z\in Z}\quad U(z) \\ &\text{s.t.}\quad \begin{cases} \Phi_i(z) &\geq 0 \text{ for every } i\in I \\ \Psi_j(z) &\geq 0 \text{ for every } j\in J. \end{cases} \end{aligned} \tag{P}$$

where Z is an open subset of a Banach space and $U,(\Phi_i)_{i\in I},(\Psi_j)_{j\in J}\in\mathscr{C}^2(Z,\mathbb{R})$, where I and J are countable index sets and U is concave. In our applications, U is equal to the expected discounted utility of consumption and leisure, the Φ_i's include the feasibility functions and the Ψ_j's represent (some of) the incentive constraints (e.g., budgets or first order conditions).

A *random improvement* over z is $(\triangle z_1, \triangle z_2, \mu)$ such that $z+\triangle z_h \in Z$ for $h \in \{1,2\}$, $\mu \in (0,1)$, and

$$\begin{cases} \mu U(z+\triangle z_1)+(1-\mu)U(z+\triangle z_2) & > \ U(z) \\ \Phi_i(z+\triangle z_h) & \geq \ 0 \text{ for every } i \text{ and both } h \\ \mu \Psi_j(z+\triangle z_1)+(1-\mu)\Psi_j(z+\triangle z_2) & \geq \ 0 \text{ for every } j. \end{cases}$$

We assume that a solution to problem (P) exists, and focus on sufficient conditions for existence of random improvements over a solution to (P). The main idea to obtain a random improvement is twofold: construct a 'policy change' $\triangle z \in Z$ which is 'U-improving', 'Φ-feasible' and 'Ψ-unfeasible'; and, ensure that a curvature condition ensuring the policy change is more likely than opposite policy change. Hereafter, we write $\triangle z^{(2)} = (\triangle z, \triangle z)$ for any $\triangle z \in Z$. [1] Let

$$\mathfrak{Z} \; = \; \left\{ \triangle z \in Z \mid DU(z^*)\triangle z > 0, \; D\Phi_i(z^*)\triangle z \; = \; 0 \text{ for every } i \text{ and } \sup_{j \in J} D\Psi_j(z^*)\triangle z \; < \; 0 \right\}$$

be the set of 'U-improving', 'Φ-feasible' and 'Ψ-unfeasible' policy changes.

Lemma 1 *Suppose z^* is a solution to problem* (P) *and assume that there is $\triangle z \in \mathfrak{Z}$ such that:*

- $(D\Phi_i(z^*))_{i \in I \setminus H}$ *is onto for* $H = \{\, i \in I \mid \forall \gamma \in [-1,1] : \Phi_i(z^*+\gamma\triangle z) = \Phi_i(z^*) \,\}$.
- $$\inf_{j \in J} -\frac{D^2\Psi_j(z^*)\triangle z^{(2)}}{D\Psi_j(z^*)\triangle z} \; > \; -\frac{D^2U(z^*)\triangle z^{(2)}}{DU(z^*)\triangle z}. \qquad (C)$$

Then a random improvement $(\triangle z_1, \triangle z_2, \mu)$ over z^* exists with $(\triangle z_1, \triangle z_2)$ and $(\triangle z, -\triangle z)$ approximately collinear and $\mu > 1/2$.

[1] For a twice continuously differentiable function $\Gamma : Z \to \mathbb{R}$ and for $\triangle z \in Z$ and $\triangle z^{(2)} = (\triangle z, \triangle z)$, we write $D\Gamma(z)$ for the (Fréchet) derivative of Γ at z (the continuous linear functional), $D^2\Gamma(z)$ for the derivative of $D\Gamma$ at z (the bilinear continuous map), and $D\Gamma(z)\triangle z$ is the value of the continuous linear functional when applied to $\triangle z$, and similarly $D^2\Gamma(z)\triangle z^{(2)}$ is the value of $D^2\Gamma(z)$ applied to $\triangle z^{(2)}$. When the gradient or Hessian exists, we write $D\Gamma(z) \cdot \triangle z$ and $\triangle z \cdot D^2\Gamma(z) \cdot \triangle z$, respectively, where $\cdot$ is the inner product in the respective space. Further,

$$\Gamma(z+\triangle z) \; = \; \Gamma(z) + D\Gamma(z)\triangle z + \frac{1}{2} D^2\Gamma(z)\triangle z^{(2)} + o(\|\triangle z^{(2)}\|),$$

via Taylor approximation, so

$$\mu\Gamma(z+\triangle z) + (1-\mu)\Gamma(z-\triangle z) - \Gamma(z) \; = \; (2\pi-1)D\Gamma(z)\triangle z + \frac{1}{2} D^2\Gamma(z)\triangle z^{(2)} + o(\|\triangle z^{(2)}\|).$$

Since $DU(z^*)\triangle z > 0 > D\Psi_j(z^*)\triangle z$ and $D^2U(z^*)\triangle z^{(2)} < 0$, Condition (C) implies $D^2\Psi_j(z^*)\triangle z^{(2)} > 0$. Therefore, Condition (C) implies that in direction $\triangle z$ locally the Ψ_j's are more convex than U is concave. Condition (C) is not at odds with the Ψ_j's being quasi-concave or with the second-order conditions, which restrict curvatures only on the space tangent to the constraint functions.

We note that the surjectivity condition on $D\Phi(z^*)$ used in Lemma 1 is not more demanding than regularity of z^*,[2] itself sufficient to obtain necessity of the Kuhn-Tucker conditions, and often assumed in applications as Slater conditions may not be available because the problem may not be convex. Interesting conditions yielding regularity of a solution, and summable Kuhn-Tucker multipliers, will be provided when the model is specialized, below.

With regularity and summable multipliers $(\lambda_i)_{i\in I}$ and $(\alpha_j)_{j\in J}$, the following *sufficient second-order condition* characterizes solutions to Problem (P) as local maximizers,

$$D^2U(z^*)\triangle z^{(2)} + \sum_i \lambda_i D^2\Phi_i(z^*)\triangle z^{(2)} + \sum_j \alpha_j D^2\Psi_j(z^*)\triangle z^{(2)} \ < \ 0$$

for all $\triangle z \neq 0$ with $D\Phi_i(z^*)\triangle z = 0$ for every i and $D\Psi_j(z^*)\triangle z = 0$ for every j.

Ramsey Problems have the common feature that the functions Ψ_j depends on marginal utilities, often derived from U. Thus, our curvature conditions in Theorem 1 relate to the second- and third-order derivatives of utilities – it is here prudence is going to come into play. Further, whether there is a 'Φ-feasible' $\triangle z$ with the property $DU(z^*)\triangle z > 0$ crucially relates to taxes, as we show below.

The case where there is a unique Ψ–function turns out to be of particular interest, because it is easier to give lotteries a policy interpretation coherent with other market restrictions.

Optimal lotteries transience and recurrence

When the conditions for Lemma 1 hold, lotteries can be optimal. Let $\triangle(Z)$ be the set of (Borel) probability measures over Z –once lotteries are considered, there is no reason to restrict attention to two-point lotteries. Associated to problem (P) we construct the optimization problem which involves lotteries over z, that is,

$$\max_{\mu\in\Delta(Z)} \quad \mathbb{E}_\mu U(z) \qquad \text{s.t.} \quad \begin{cases} \Phi_i(z) \geq 0 \text{ for every } i \in I \\ \mathbb{E}_\mu \Psi_j(z) \geq 0 \text{ for every } j \in J. \end{cases} \tag{L-P}$$

[2] A point $z \in Z$ is *regular* if $(D\Phi_{i\in I}(z), D\Psi_{j\in J}(z))$ is onto $\mathbb{R}^I \times \mathbb{R}^J$.

Again, for the time being we assume that a solution μ^* to problem (L-P) exists, and instead focus on a particular property of its solutions.

In our settings the set Z has a time dimension. Indeed, there is an open subset of a Euclidean space A such that $Z = \prod_{t\geq 0} Z_t$ where Z_t is the set of controls or functions from Θ^t to A. A probability distribution μ over Z can be written as a sequence of conditional probability distribution μ_t on Z_t. Let $\omega^t = (\theta^t, z^{t-1})$ be a history of states up to date t and controls up to date $t-1$ with $\omega^0 = \theta_0$ and let $(\mathscr{F}_t)_{t\geq 0}$ be the filtration generated by the histories ω^t. Hence, in problem (L-P) the planner chooses $(\mu_t)_{t\geq 0}$: at every date $t > 0$ for every history of states θ^t and all histories controls z^{t-1}, z_t is distributed according to the probability distribution $\mu_t(\omega^t)$.

Randomizations $(\mu_t)_{t\geq 0}$ are *transient* provided that there is T such that μ_t is a Dirac measure for every $t \geq T$, and *recurrent* provided for every T there $t > T$ such that μ_t is not a Dirac measure. We say that μ is transient (recurrent) if $(\mu_t)_{t\geq 0}$ is transient (recurrent). Let $\operatorname{supp}\mu$ denote the support of μ. The next lemma states sufficient conditions that rule out recurrent randomizations.

Lemma 2 *Suppose J is finite. Let μ^* be a solution to Problem* (L-P)*. Assume $\operatorname{supp}\mu^*$ is product compact and $(U, (\Phi_i)_i, (\Psi_j)_j)$ are product continuous on $\operatorname{supp}\mu^*$. Then there is a transient solution μ^{**} to Problem* (L-P) *with no more than $|J|+1$ conditional randomizations on any given path.*

A usual concern for the use of lotteries in allocation problems is their possible decentralization (a theme explored in other, static, contexts by, e.g., Shell and Wright (1993), Kehoe et al. (2002), and related literature): the question is whether the randomizations from lotteries μ_t^* can occur in markets via a coordination device (a 'taxspot'). Further, averaging out the Ψ constraints via μ may require additional policy tools to transfer purchasing power across the lottery realizations, and may be unattainable. Whether or not additional tools are needed depends on the details of the policy change $\triangle z$ and of the underlying economy, thus we will examine this second issue in the sections below. Instead, here we settle the first issue, of the optimal lottery representation via taxspot states.

Let $s = (s_t)_{t\geq 0}$ be a process with serially uncorrelated values s_t uniformly distributed in $S = [0,1]$, all $t \geq 0$, and $\hat{z} = (\hat{z}_t)_{t\geq 0}$ be a process adapted to the filtration $(\hat{\mathscr{F}}_t)_{t\geq 0}$ generated by all histories $\hat{\omega}^t = (\theta^t, s^t)$, and with values $\hat{z}_t(\theta^t, s^t) \in A$.

Lemma 3 *Let μ^* be a solution to Problem* (L-P) *with $\operatorname{supp}\mu^*$ product compact. Then,*

$$\mathbb{E}_{\mu^*} U(z) \;=\; \mathbb{E}_{\nu^*} U(\hat{z}),\ \textit{and}\ \mathbb{E}_{\mu^*} \Psi_j(z) \;=\; \mathbb{E}_{\nu^*}\ \Psi_j(\hat{z})\ \textit{for every}\ j,$$

for some (Borel) probability measure ν^* on processes $\hat{z} = (\hat{z}_t)_{t\geq 0}$.

The proof of Lemma 3 uses standard measure-theoretic tools (in particular, Kuratowski's Isomorphism Theorem for Borel sets, and Skorokhod's Representation Theorem; see, e.g., Parthasarathy, 1967, Ch. 3) to map lottery distributions μ_t over z_t into random variables over $(S, \mathscr{B}(S), Leb)$, where $\mathscr{B}(S)$ is the Borel sigma-algebra and *Leb* the Lebesgue measure, and is therefore omitted. We now go back to the two variants we have introduced above, and see how the theorems apply.

3 Taxspots with complete markets

Here, using market completeness and the transversality conditions, the equilibrium sequential budget constraints can be equivalently compressed into a single intertemporal constraint, eliminating portfolios. As standard, the Ramsey Problem can then be expressed in primal, relaxed form, after substitution of the equilibrium first order conditions, as

$$
\begin{aligned}
&\max_{(c_t,x_t,k_{t+1})_{t\geq 0}} \quad \mathbb{E}_0 \sum_{t\geq 0} \beta^t u(c_t,x_t) \\
&\text{s.t.} \quad \begin{cases} f_t(k_t, 1-x_t) - c_t - k_{t+1} - g_t \;\geq\; 0 \;\text{ for } t \geq 0, \\ \mathbb{E}_0 \sum_{t\geq 0} \beta^t [u_c(c_t,x_t)c_t - u_x(c_t,x_t)(1-x_t)] \\ \qquad\qquad \geq\; u_c(c_0,x_0)(f_{a_0 k}(k_0,1-x_0)k_0 + b_0) \\ \lim_{t\to\infty} \mathbb{E}_0 \beta^t u_c(c_t,x_t)k_{t+1} \;=\; 0. \end{cases}
\end{aligned}
\tag{RRP}
$$

The last inequality in Problem (RRP) is the intertemporal *incentive constraint*, coming from the intertemporal budget constraint and first-order conditions of the household. The weak inequality implies that, over the lifetime, the household can spend more than its after-tax income.

The controls are $z_t = (c_t, x_t, k_{t+1})$. Interest rate r_t and wage rate w_t can then be derived from the first-order conditions of the firm. The tax rates can be found from the first-order conditions of the household. The *ex-ante capital tax* at date t for date $t+1$ and the labor tax at date t are

$$
\begin{cases} \bar{\tau}^k_{t+1} \;=\; 1 - \dfrac{u_c(c_t,x_t)}{\beta \mathbb{E}_t\, u_c(c_{t+1},x_{t+1}) f_{a_t k}(k_{t+1}, 1-x_{t+1})} \\[2ex] \tau^\ell_t \;=\; \dfrac{u_c(c_t,x_t) f_{a_t \ell}(k_t,1-x_t) - u_x(c_t,x_t)}{u_c(c_t,x_t) f_{a_t \ell}(k_t,1-x_t)}. \end{cases}
\tag{1}
$$

As shown by Zhu (1992), the capital tax at every date is determined only up to a martingale transformation at every date t, making $\bar{\tau}^k_{t+1}$ the relevant capital tax rate.

In line with the literature, to make the taxation problem interesting we assume that the present value of government assets is not sufficient to finance future government expenditures at solutions to Problem (RRP) without the incentive constraint. Let $\hat{z}$ be the solution to Problem (RRP) without the incentive constraint, and ψ_t be defined by

$$\psi_t(z_t) \;=\; \begin{cases} u_c(c_0,x_0)c_0 - u_x(c_0,x_0)(1-x_0) - u_c(c_0,x_0) f_{a_0k}(k_0,1-x_0)k_0 & \text{for } t=0 \\ u_c(c_t,x_t)c_t - u_x(c_t,x_t)(1-x_t) & \text{for } t>0 \end{cases}$$

with the derivatives with respect to (c_t,x_t) denoted (ψ_{ct},ψ_{xt}), and $D^2\psi_t$ its Hessian.

Lemma 4 *Suppose that*

$$\mathbb{E}_0 \sum_{t\geq 0} \beta^t u_c(\hat{c}_t,\hat{x}_t) g_t + u_c(\hat{c}_0,\hat{x}_0) b_0 \;>\; 0.$$

If z^* is a solution to Problem (RRP)*, then the incentive constraint is satisfied with equality. If $\psi^*_{xt} > 0$ for some history, then the resource constraint is satisfied with equality at that history.*

Since $\psi_{xt} = u_{xt} + u_{xct}c_t - u_{xxt}(1-x_t)$ for $t>0$, separability of u implies $\psi_{xt} > 0$ so feasibility is satisfied with equality at every date.

Theorem 1 *Suppose z^* is an interior solution to Problem (RRP). Assume there is a date $\tilde{t} \geq 0$ and history $\tilde{\theta}^{\tilde{t}}$ with $\pi_{\tilde{\theta}^{\tilde{t}}} > 0$ such that at least one of the following two conditions is satisfied:*

- *The labor tax rate is positive and Condition (C) is satisfied for $\triangle z^{\ell}$ with*

$$\triangle z^{\ell}_t(\theta^t) \;=\; \begin{cases} (f^*_{\ell\tilde{t}}(\tilde{\theta}^{\tilde{t}}),-1,0) & \textit{for } (t,\theta^t) = (\tilde{t},\tilde{\theta}^{\tilde{t}}) \\ 0 & \textit{otherwise}. \end{cases}$$

- *The ex-ante capital tax rate is positive and Condition (C) is satisfied for $\triangle z^k$ with*

$$\triangle z^{k}_t(\theta^t) \;=\; \begin{cases} (-1,0,1) & \textit{for } (t,\theta^t) = (\tilde{t},\tilde{\theta}^{\tilde{t}}) \\ (f^*_{k\tilde{t}+1}(\tilde{\theta}^{\tilde{t}},\theta),0,0) & \textit{for } (t,\theta^t) = (\tilde{t}+1,(\tilde{\theta}^{\tilde{t}},\theta)) \textit{ for some } \theta\in\Theta \\ 0 & \textit{otherwise}. \end{cases}$$

Then there is a random improvement over z^*.

Proof: The theorem is established as an application of Lemma 1 to (RRP) with $I = \cup_{t\geq 0}\Theta^t$ and $J = \{1\}$, where U, $(\Phi_{\theta^t})_{\theta^t\in I}$ and Ψ are identified as:

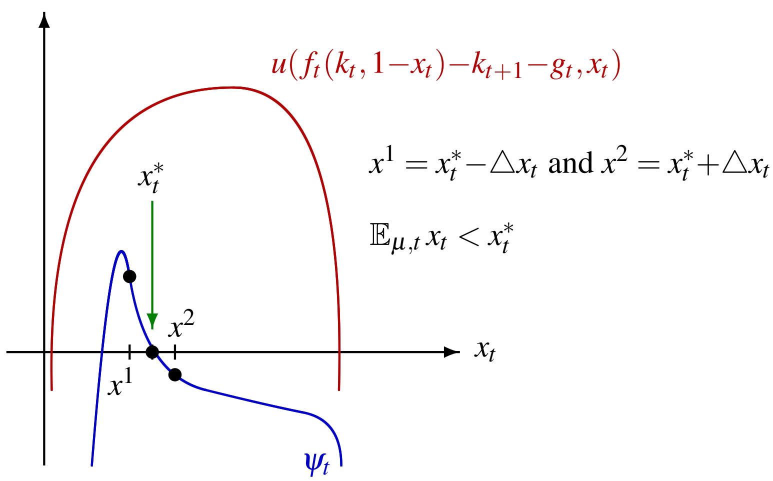
$$\begin{cases} U(z) &= \mathbb{E}_0 \sum_{t\geq 0} \beta^t u(c_t, x_t) \\ \Phi_{\theta^t}(z) &= f_{a_t}(k_t(\theta^{t-1}), 1-x_t(\theta^t)) - c_t(\theta^t) - k_{t+1}(\theta^t) - g_t \text{ for every } \theta^t \\ \Psi(z) &= \mathbb{E}_0 \sum_{t\geq 0} \beta^t \psi_t(z_t). \end{cases}$$

The differential $(D\Phi_{\theta^t}(z^*))_{\theta^t}$ is onto, because the differential $(D_c\Phi_{\theta^t}(z^*))_{\theta^t}$ is minus the identity. Clearly,

$$\begin{aligned} DU^*\triangle z^\ell &= -\beta^{\tilde{t}}\pi(\tilde{\theta}^{\tilde{t}})[u^*_{c\tilde{t}}(\tilde{\theta}^{\tilde{t}})f^*_{\ell\tilde{t}}(\tilde{\theta}^{\tilde{t}}) - u^*_{x\tilde{t}}(\tilde{\theta}^{\tilde{t}})]\triangle x^\ell_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}) > 0 \\ DU^*\triangle z^k &= \beta^{\tilde{t}}\pi(\tilde{\theta}^{\tilde{t}})\Big[\beta\mathbb{E}_{\tilde{t}}[u^*_{c\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}, \theta_{\tilde{t}+1})f^*_{k\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}, \theta_{\tilde{t}+1})] - u^*_{c\tilde{t}}(\tilde{\theta}^{\tilde{t}})\Big]\triangle k^k_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}) > 0. \end{aligned}$$

For both $\triangle z$'s, by construction, $D\Phi^*\triangle z = 0$. Since Condition (C) is assumed to be satisfied at these $\triangle z$'s, it is $D\Psi^*\triangle z \neq 0$. Since z^* is optimal for (RRP), it is $D\Psi^*\triangle z < 0$. Thus, $\triangle z \in \mathfrak{Z}$. □

Figure 1 below illustrates the theorem in case the labor tax is positive, $\tau^{*\ell}_t > 0$.



Theorem 1 shows that there are random improvements with the following effects on average: consumption and labor supply increase at date $\tilde{t}$ in case of a positive labor tax; investment increases at date $\tilde{t}$, and consumption increases at date $\tilde{t}+1$ in case of a positive ex-ante capital tax. The increases in averages more than compensate for the increase in consumption-leisure volatility. However, the result depends on Condition (C) being satisfied.

The assumption in Lemma 4 ensuring that at any solution to Problem (RRP) the incentive constraint holds as equality also implies that taxes are positive at some date-event. When taxes are positive the incentive constraint prevents the planner from increasing the labor supply or the capital investment further. However, lotteries can help because their uncertainty increases labor supply or capital investment without violating the incentive constraint.

As already mentioned, $D^2\Psi^*\triangle z^{(2)}$ must be positive for Condition (C) to be satisfied. Some tedious but straightforward calculations show that

$$\begin{cases} D^2\Psi^*(\triangle z^\ell)^{(2)} &= \beta^{\tilde{t}}\pi_{\tilde{t}}\Big[[2u^*_{cc\tilde{t}}+u^*_{ccc\tilde{t}}c^*_{\tilde{t}}-u^*_{ccx\tilde{t}}(1-x^*_{\tilde{t}})](f^*_{\ell\tilde{t}})^2 \\ & \quad -2[u^*_{cx\tilde{t}}+u^*_{ccx\tilde{t}}c^*_{\tilde{t}}-u^*_{cxx\tilde{t}}(1-x^*_{\tilde{t}})]f^*_{\ell\tilde{t}} \\ & \quad +2u^*_{xx\tilde{t}}+u^*_{xxc\tilde{t}}c^*_{\tilde{t}}-u^*_{xxx\tilde{t}}(1-x^*_{\tilde{t}})\Big] \\ D^2\Psi^*(\triangle z^k)^{(2)} &= \beta^{\tilde{t}}\pi_{\tilde{t}}\Big[2u^*_{cc\tilde{t}}+u^*_{ccc\tilde{t}}c^*_{\tilde{t}}-u^*_{ccx\tilde{t}}(1-x^*_{\tilde{t}}) \\ & \quad +\mathbb{E}_{\tilde{t}}\beta[2u^*_{cc\tilde{t}+1}+u^*_{ccc\tilde{t}+1}c^*_{\tilde{t}+1}-u^*_{ccx\tilde{t}+1}(1-x^*_{\tilde{t}+1})](f^*_{k\tilde{t}+1})^2\Big]. \end{cases}$$

Arguments in functions are dropped: consumption and leisure in utility; capital and labor in production; and histories in probability, consumption, leisure, and capital. Obviously, Condition (C) cannot be satisfied for quadratic separable utility because $u_{ccc}=0$ and $D^2\Psi^*(\triangle z)^{(2)}<0$. However, if $u^*_{ccc\tilde{t}}$ is positive and large, then Condition (C) can be satisfied. Indeed, $u^*_{ccc\tilde{t}}>0$ is also necessary when the instantaneous utility is separable.

As Rothschild and Stiglitz (1971) and Kimball (1990) pointed out, a positive third-order derivative u_{ccct} represents prudence, and leads to precautionary savings and labor supply. Our analysis shows that, in the presence of sufficient prudence, the positive effects of additional uncertainty can more than compensate for its negative effects.

Condition (C) holds if and only if

$$\frac{D^2\Psi(z^*)\triangle z^{(2)}}{D\Psi(z^*)\triangle z} < \frac{D^2U(z^*)\triangle z^{(2)}}{DU(z^*)\triangle z}.$$

Therefore, other things equal, Condition (C) is always more likely to hold the higher the initial tax, since for the specified policies $DU(z^*)\triangle z$ is equal to the tax. Further, $|\, D\Psi(z^*)\triangle z\,|$ is equal to $\beta^{\tilde{t}}\pi^{\tilde{t}}(\psi^*_{x\tilde{t}}-f_{\ell t}\psi^*_{c\tilde{t}})$ for $\triangle z^\ell$, and to $\beta^{\tilde{t}}\pi^{\tilde{t}}[\psi^*_{c\tilde{t}}-\beta\mathbb{E}_t\psi^*_{c\tilde{t}+1}f^*_{k\tilde{t}+1}]$ for $\triangle z^k$. Now ψ_{xt} is proportional to $u_{xt}(1+1/\eta^F_t)$, where η^F_t is the Frisch labor elasticity, while ψ_{ct} is proportional to $u_{ct}(\sigma_{ct}-1)$, where σ_{ct} is the consumption relative risk aversion (hereafter, RRA) coefficient. Then, Condition (C) is also more likely to hold when σ_{ct} is greater than one; and when the labor tax is positive, the larger η^F_t.

Whether Condition (C) holds is an empirical matter once it has been shown that it can be satisfied theoretically. It turns out that under regularity, if Condition (C) does not hold, then u_{ccct} can be increased without changing anything else at the solution z^*. Consequently, as we now show, Condition (C) is not vacuous.

First, to consider changes in u_{ccct} we need a notion of convergence for utility functions. We say that a sequence of utility functions $(u_n)_{n\in\mathbb{N}}$ *converges to u* if it does so in the Whitney $\mathscr{C}^2$-topology: there is a compact subset such that u_n and u are identical outside that compact set for every n, and (u_n, Du_n, D^2u_n) converges uniformly to (u, Du, D^2u). Obviously, if $(u_n)_{n\in\mathbb{N}}$ converges to u, then there is a $N \in \mathbb{N}$ such that $n > N$ implies u_n is differentially strongly monotonic and strictly concave. Second, changes at a point $z^*_{\tilde{t}}(\theta^{\tilde{t}})$ must not affect other points of the solution z^*. Point $z^*_{\tilde{t}}(\theta^{\tilde{t}})$ is *locally isolated* provided there is $\varepsilon > 0$ such that either $\|z^*_t(\theta^t) - z^*_{\tilde{t}}(\theta^{\tilde{t}})\| > \varepsilon$ or $\|z^*_t(\theta^t) - z^*_{\tilde{t}}(\theta^{\tilde{t}})\| = 0$ for every (t, θ^t). Thus, at a locally isolated point, u_{ccc} can be increased without changing z^*. It turns out that, if a second order condition is satisfied at u, z^* remains a solution even after the perturbation.

Theorem 2 *Let z^* be an interior and regular solution to Problem (RRP). Suppose the sufficient second-order condition holds at z^* and $z^*_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})$ is locally isolated with $\tau^{*\ell}_{\tilde{t}}(\tilde{\theta}^{\tilde{t}}) > 0$ or $\bar{\tau}^{*k}_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}) > 0$ for some $\tilde{\theta}^{\tilde{t}}$ with $\pi_{\tilde{\theta}^{\tilde{t}}} > 0$. Then there is $(u_n)_{n\in\mathbb{N}}$ converging to u such that z^* is a solution to Problem (RRP) at u_n and Condition (C) is satisfied by some $\triangle z \in \mathfrak{Z}$.*

For homothetic utilities, the perturbed utility in Theorem 2 can always be made $\mathscr{C}^3$-arbitrarily close. In finite economies, regular locally isolated solutions to (RRP) are easily shown to be generic in utility via standard repeated applications of the Parametric Transversality Theorem. Infinite horizon economies with solutions to (RRP) with locally isolated points, and positive taxes, exist. In Appendix C we prove the following result, which constructs a robust example of an economy satisfying the assumptions.

Suppose productivity is constant and $u(c,x) = u(c) + v(x)$ is separable with RRA σ_{ct} greater than one and $u(0) = -\infty$. Then, for small but positive g, solutions to (RRP) are regular, and converge to a unique interior globally stable steady state with $\tau^{*\ell} > 0$. Therefore, all of the assumptions of Theorem 2 hold, and up to a utility perturbation Condition (C) is satisfied by some $\triangle z \in \mathfrak{Z}$. By Lemma 1 there is a random improvement over z^*. Hence, at the very least the utility perturbation method allows us to claim that the set of economies where u_{ccc} is large enough, and Condition (C) holds, is non-vacuous.

Consider the lottery version of the relaxed Ramsey problem, where the functions Φ and Ψ are defined as in Problem (RRP):

$$\max_{\mu\in\Delta(Z)} \quad \mathbb{E}_\mu U(z)$$
$$\text{s.t.} \quad \begin{cases} \Phi_t(z) \geq 0 & \text{for every } t \geq 0 \\ \mathbb{E}_\mu \Psi(z) \geq 0. \end{cases} \tag{L-RRP}$$

To ensure existence of solutions is not an issue, we need two additional assumptions. According to Assumption (A.2) there is $\bar{c},\bar{k} > 0$ such that consumption and capital are bounded from above by $(\bar{c},\bar{k})$.

A.3 There exists $\tilde{z} \in \times_{t,\theta^t}(0,\bar{c})\times(0,1)\times(0,\bar{k})$ with $\infty > \mathbb{E}_0 \sum_t \beta^t |\psi_t(\tilde{z})| \geq \Psi(\tilde{z}) > 0$ and $U(\tilde{z}) > -\infty$.

A.4 There exists $\varepsilon > 0$ such that for every (t,θ^t) such that $c_t < \varepsilon$ or $x_t < \varepsilon$ implies there is $\triangle z_t$ such that $D\psi_t(z_t)\cdot\triangle z_t > 0$ for some $\triangle z_t$ with $z_t + \triangle z_t \in (0,\bar{c})\times(0,1)\times(0,\bar{k})$, $D\phi_t \cdot (0,\triangle z_t) = 0$, and $Du_t(z_t)\cdot\triangle z_t \geq \varepsilon_t$.

These two assumptions in addition to the two assumption already made ensure problem (L-RRP) has a solution μ^* with compact support and $(c_t,x_t,k_{t+1}) \in \mathbb{R}_{++}\times(0,1)\times\mathbb{R}_{++}$ with μ^*-probability one.[3] As we require market clearing for every realization of uncertainty and not 'on average', there is only one function Ψ over which randomizations take place – the intertemporal incentive constraint function. Using Lemma 2 we conclude that for all solutions to Problem (L-RRP) there are equivalent solutions for which randomizations disappear after some date $\bar{t} > 0$.

Proposition 1 *Let μ^* be a solution to Problem* (L-RRP). *Then there is a solution μ^{**} for which randomizations are transient with μ^*-probability one.*

For deterministic economies Proposition 1 combined with Condition (C) has strong implications for the long-run tax on capital. Indeed, suppose Condition (C) is satisfied at a steady state $(\bar{c},\bar{x},\bar{k})$. Then Proposition 1 implies that there exists a solution to Problem L-RRP which randomizes over three paths $(c_t,x_t,k_{t+1})_{t\in\mathbb{N}_0}$ with the property that $\|(c_t,x_t,k_{t+1})-(\bar{c},\bar{x},\bar{k})\| > \varepsilon$ for every $t > \bar{t}$, for some $\bar{t} \in \mathbb{N}$ and some $\varepsilon > 0$. Consequently, convergence to steady states does not imply that the long-run tax on capital is zero.

We now come to the issue of decentralization of the optimal lottery solving (L-RRP). Let $(s_t)_{t\geq 0}$ be a process of serially uncorrelated shocks with values uniformly distributed on $S = [0,1]$. Processes $\left(\hat{c}_t,\hat{x}_t,\hat{k}_{t+1},\hat{b}_{t+1},\hat{p}_t,\hat{w}_t,\hat{r}_t,\hat{\tau}_t\right)_{t\geq 0}$ are now adapted to the filtration $(\hat{\mathscr{F}}_t)_{t\geq 0}$ generated by histories $\hat{\omega}^t = (\theta^t,s^t)$. The household and government budgets, and feasibility, do not change, but now they must be satisfied at each $\hat{\omega}^t = (\theta^t,s^t)$, and there are bonds for every contingency θ_{t+1},s_{t+1}. Let $(\nu_t)_{t\geq 0}$ be the distributions of $\left(\hat{c}_t,\hat{x}_t,\hat{k}_{t+1},\hat{b}_{t+1},\hat{p}_t,\hat{w}_t,\hat{r}_t,\hat{\tau}_t\right)$ conditional on the realization of $\hat{\omega}^{t-1},\theta_t$, i.e., the *taxspot* distribution. We say that the taxspot distribution is *trivial* if ν_t is a Dirac measure for all $t \geq 0$.

[3] A proof is available from the authors upon request.

An *equilibrium with taxspots* is a process $\left(\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}, \hat{b}_{t+1}, \hat{p}_t, \hat{w}_t, \hat{r}_t, \hat{\tau}_t\right)_{t\geq 0}$ of consumption, leisure, capital, bond holdings, prices, wages, interest rates and taxes, adapted to $(\hat{\mathscr{F}}_t)_{t\geq 0}$ and such that: markets clear; $(\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}, \hat{b}_{t+1})_{t\geq 0}$ maximizes utility subject to sequential budgets and transversality conditions; firms maximize profits; and, the government satisfies its budget constraints.

An equilibrium with taxspots process *decentralizes* a μ over Z if the corresponding taxspot distribution process $(\nu_t)_{t\geq 0}$ satisfies

$$\nu_t(\{(\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}) \in B\}) \;=\; \mu_t(\{(c_t, x_t, k_{t+1}) \in B\})$$

for every t and Borel set $B \subset \mathbb{R}^3$.

For fixed μ^* solving (L-RRP), we derive the decentralizing equilibrium with taxspots using Lemma 3 and the maps (1) defining taxes as functions of consumption, leisure and capital. The rest of the argument consists in backing up prices and bond holdings in a straightforward manner due to market completeness.

Proposition 2 *If μ^* is a lottery solving problem (L-RRP), then there exists an equilibrium with taxspots process $\left(\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}, \hat{b}_{t+1}, \hat{p}_t, \hat{w}_t, \hat{r}_t, \hat{\tau}_t\right)_{t\geq 0}$ decentralizing μ^*.*

Hence, when Theorem 2 (thus, Lemma 1) holds for problem (RRP) (a nonvacuous situation, as we showed), no solution to (L-RRP) can be a Dirac measure so there are nontrivial taxspots at equilibrium. While markets are complete, and households can buy insurance to hedge against taxspots, the planner implicitly chooses the bond supply so that full taxspot insurance is not priced at fair odds, and households optimally choose to hold some taxspot risk.

If $D^2\psi_t$ is negative definite for all (c_t, x_t), as in the separable quadratic case, there are no taxspots. Moreover, if the process $(\theta_t)_{t\geq 0}$ has first-order Markovian transition, optimal fiscal policy is a function of solely of the current state θ_t. This is the well-known 'Markovian' aspect of optimal taxation – see, e.g., Ljundqvist and Sargent (2018). However, if instead Condition (C) is satisfied (e.g., by $\triangle z^\ell$ or $\triangle z^k$), $D^2\psi_t$ is positive definite and optimal taxes are random and not deterministic functions of the current state. Hence, the optimal fiscal policy creates randomness which conflicts with the presumed smoothing role of taxes. In other words, the serial correlation properties of taxes and of government expenditures (or of productivity shocks) are different.

Combining Propositions 1 and 2, every optimal lottery is payoff equivalent to a taxspot equilibrium where tax uncertainty disappears in finite time. Taxspot uncertainty, represented by a nontrivial $(\nu_t^\tau)_{t\geq 0}$ with finite support, need not have more than two taxspots episodes. Therefore, at the optimal taxspot equilibrium at most finitely many taxspot-contingent bonds are needed to decentralize the optimal lottery.

4 Taxspots with no bonds

The absence of bonds makes the reduction of household and government sequential budget constraints to a single intertemporal constraint impossible. Therefore, the primal relaxed Ramsey Problem takes the form

$$\max_{(c_t,x_t,k_{t+1})_{t\geq 0}} \quad \mathbb{E}_0 \sum_{t\geq 0} \beta^t u(c_t,x_t)$$

$$\text{s.t.} \quad \begin{cases} f_t(k_t,1-x_t)-c_t-k_{t+1}-g_t \ \geq\ 0 \ \text{ for every } t\geq 0 \\ c_0+k_1-\dfrac{u_x(c_0,x_0)}{u_c(c_0,x_0)}(1-x_0)-f_{k0}(k_0,1-x_0)k_0-b_0\geq 0 \\ \beta\mathbb{E}_{t-1}\left[u_c(c_t,x_t)(c_t+k_{t+1})-u_x(c_t,x_t)(1-x_t)\right] \\ \qquad -u_c(c_{t-1},x_{t-1})k_t\geq 0 \ \text{ for every } (t,\theta^t) \text{ with } t\geq 1 \\ \lim_{t\to\infty} \mathbb{E}_0\,\beta^t u_c(c_t,x_t)k_{t+1} \ =\ 0. \end{cases} \tag{RRP-I}$$

There is an incentive constraint for every history, resulting in additional restrictions on the decentralization of Pareto improving random taxes, as we will explain below.

We first present sufficient conditions to ensure an ancillary technical property, i.e., that any solution to Problem (RRP-I) solves the Ramsey primal problem with all constraints as equalities. It turns out that merely ruling out the unconstrained optimum $\hat{z}$ as a solution to (RRP-I) is not sufficient to ensure equality for the constraints. We obtain equality under more stringent, but common parametric restrictions.

Lemma 5 *Suppose u is separable with $\sigma_{ct}>1$ and $g_t\geq 0$ with $\mathbb{E}_{t-1}\,g_t>0$ for every t. Then every constraint is satisfied with equality at all solutions to Problem* (RRP-I).

Separable utility case with $\sigma_{ct}>1$ and $g_t>0$ will feature prominently in the analysis below as a special and relevant case.

Throughout, we require that solutions to (RRP-I) are regular. Below we will construct economies where regularity holds.

To show existence of a random improvement upon solutions to problem (RRP-I), we apply Lemma 1. Key to the existence of decentralizable random improvements is the construction of policy changes that are Pareto improving but are 'Ψ-unfeasible' at a single date. If such policy changes also satisfy Condition (C), then we can apply Lemma 1 to claim the optimality of randomizations. To this end, for every date-event θ^t consider the $3(1+|\Theta|)$-dimensional vector $(z_t(\theta^t),(z_{t+1}(\theta^t,\theta))_{\theta\in\Theta})$, and when we omit reference to θ^t

we write (z_t, z_{t+1}). For every $t \geq 0$, let ψ_t^f be defined by

$$\psi_t^f(z_t, z_{t+1}) \; = \; \beta \mathbb{E}_t \left[u_c(c_{t+1}, x_{t+1})(c_{t+1} + k_{t+2}) - u_x(c_{t+1}, x_{t+1})(1 - x_{t+1})\right] - u_c(c_t, x_t) k_{t+1},$$

with derivative $D\psi_t^f = (D\psi_{t,t}^f, D\psi_{t,t+1}^f)$ and Hessian $D^2 \psi_t^f$. We further define $\psi_{-1}^f(z_0; b_0, k_0)$ as the incentive (or budget) constraint at date $t = 0$. Next, let

$$\phi_t(z_{t-1}, z_t) \; = \; f_t(k_t, 1 - x_t) - c_t - k_{t+1} - g_t.$$

and consider $\hat{\phi}_t = (\phi_t, \phi_{t+1})$ and $\hat{\psi}_t = (\psi_{t-1}, \psi_t, \psi_{t+1})$ as functions of $(c_t, x_t, k_{t+1}, c_{t+1}, x_{t+1})$, keeping the other controls fixed. Then at a date-event θ^t, vectors $(D\hat{\phi}_t, D\hat{\psi}_t)$ form a 3+2$|\Theta|$-dimensional square matrix.

Hereafter, we let $\mathfrak{Z}_{\tilde{t}}$ to be the set $\mathfrak{Z}$ for $\Psi = \psi_{\tilde{t}}^f$ and $\Phi = ((\phi_t)_{t \geq 0}, (\psi_t^f)_{t \neq \tilde{t}})$ so every incentive constraint except the one at date $\tilde{t}$ is treated as a feasibility constraint.

Theorem 3 *Let z^* be an interior and regular solution to Problem (RRP-I) of an economy for which leisure is a differentiable strictly normal good, and $g_t \geq 0$ with $\mathbb{E}_{t-1} g_t > 0$ for every t. Suppose there is $(\tilde{t}, \tilde{\theta}^{\tilde{t}})$ with $\tilde{t} > 0$ and $\pi_{\tilde{\theta}^{\tilde{t}}} > 0$ such that $(D\hat{\phi}_{\tilde{t}}(z^*), D\hat{\psi}_{\tilde{t}}(z^*))$ has full rank and at one of the following two conditions is satisfied:*

(ℓ) The labor tax rate is positive.

(k) The labor tax rate is zero and the ex-ante capital tax rate is positive.

Then there exists $\triangle z \in \mathfrak{Z}_{\tilde{t}-1}$. If $\triangle z$ satisfies Condition (C), then a random improvement over z^* exists.

Theorem 3 finds 'Ψ-unfeasible' Pareto improving policy changes $\triangle z$ which violate a household budget in the corresponding competitive equilibrium, yielding a government deficit. This is not surprising, since z^* is optimal in Problem (RRP-I). Crucially for decentralization, there is a budget deficit at a single date-event. Indeed, the lottery can be chosen such that

$$\beta \mathbb{E}_{\mu,t} \left[u_c(c_{t+1}, x_{t+1})(c_{t+1} + k_{t+2}) - u_x(c_{t+1}, x_{t+1})(1 - x_{t+1})\right] - u_c(c_t, x_t) k_{t+1} = 0$$

at all $t \geq 0$. It is easily checked that if $\tau_{\tilde{t}}^{*\ell}, \bar{\tau}_{\tilde{t}+1}^{*k}, \tau_{\tilde{t}+1}^{*\ell} > 0$, then either $\mathbb{E}_{\mu,\tilde{t}} x_{\tilde{t}} < x_{\tilde{t}}^*$ or $\mathbb{E}_{\mu,\tilde{t}+1} x_{\tilde{t}+1} < x_{\tilde{t}+1}^*$ or $\mathbb{E}_{\mu,\tilde{t}} k_{\tilde{t}+1} < k_{\tilde{t}+1}^*$. Otherwise the general effects of the considered policy changes are ambiguous.

Theorem 3 does not cover economies where labor is subsidized at every date-event, i.e., $\tau_t^{*\ell} < 0$ for every (t, θ^t) with $t > 0$. However, much of the literature focuses on economies with solutions converging to steady states with positive labor taxes, a case covered by the theorem.

Theorem 3 requires that Condition (C) be satisfied by the Pareto improving Ψ-unfeasible policy changes $\triangle z$. This holds provided the curvature of $\psi^{*f}_{\tilde{t}-1}$ in the direction of the Pareto improving change is sufficiently large. If $\triangle c_{\tilde{t}} \neq 0$, then it is the case provided $u^*_{ccc\tilde{t}} > 0$ is sufficiently large. Hence, as for complete markets, random taxes improve on non-random taxes provided prudence is sufficiently large.

As in the previous section, we can make $u^*_{ccc\tilde{t}} > 0$ sufficiently large to ensure Condition (C) is satisfied some $\triangle z \in \mathfrak{Z}_{\tilde{t}-1}$ without changing the equilibrium, provided the solution to Problem (RRP-I) contains locally isolated points.

Theorem 4 *Let z^* be an interior and regular solution to Problem* (RRP-I) *satisfying the sufficient second-order condition of an economy for which leisure is a differentiable strictly normal good and $g_t \geq 0$ with $\mathbb{E}_{t-1}\, g_t > 0$ for every t. Suppose there is $(\tilde{t}, \theta^{\tilde{t}})$ with $\tilde{t} > 0$ and $\pi_{\tilde{\theta}^{\tilde{t}}} > 0$ such that:*

- *$z^*_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})$ is a locally isolated point.*
- *$\tau^{*\ell}_{\tilde{t}}(\tilde{\theta}^{\tilde{t}}) > 0$ or $\tau^{*\ell}_{\tilde{t}}(\tilde{\theta}^{\tilde{t}}) = 0$ and $\bar{\tau}^{*k}_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}) > 0$.*
- *$(D\hat{\phi}_{\tilde{t}}(z^*), D\hat{\psi}_{\tilde{t}}(z^*))$ has full rank.*

Then there is $(u_n)_{n\in\mathbb{N}}$ converging to u such that z^* is a solution to Problem (RRP-I) *at u_n and Condition (C) is satisfied by some $\triangle z \in \mathfrak{Z}_{\tilde{t}-1}$.*

The conditions stated in Theorem 4 are not vacuous for economies where solutions to Problem (RRP-I) converge to a steady state. As an example, consider the deterministic economy we constructed in Section 3 with positive government expenditures $g > 0$, and separable utility with RRA greater than one for every t. A straightforward extension of the argument in Appendix C shows that the economy has a regular solution to Problem (RRP-I), and a unique interior, globally stable steady state for g sufficiently small. Combining feasibility and the incentive constraint, steady states for Problem (RRP-I) are solutions to the following equation:

$$f(k, 1-x) - g \;=\; \frac{u_x(x)}{u_c(f(k,1-x)-g-k)}(1-x) + \frac{1}{\beta}k,$$

where $f_k = 1/\beta$, and $f_\ell > u_x/u_c$. Hence, at the steady state for Problem (RRP-I) capital taxes are zero and labor taxes are positive.

Under separability, a second-order derivatives utility perturbation of u_{xx} at the steady state ensures both that $(D\hat{\phi}_t(z^*), D\hat{\psi}_t(z^*))$ has full rank and $\triangle c_t \neq 0$, for every t sufficiently large. Therefore, since $\tau_t^{*\ell} > 0$ for all large t, and convergence to a steady state also implies optimal paths contain locally isolated points for t large, the conditions of Theorem 4 are satisfied.[4] We conclude that there is a random improvement over z^*.

For deterministic economies with positive government expenditures $g > 0$ recurrence is possible in the absence of bonds. Indeed, Lemma 2 does not apply, because in Problem (L-RRP-I) the number of constraints over which there can be lotteries is infinite. At steady states capital taxes are zero, so labor taxes must be positive. If all solutions to Problem (RRP-I) converge to steady states, Theorem 4 can be used to show that, modulo a utility perturbation, the curvature and ancillary regularity conditions needed for Condition (C) to be satisfied can be met by some $\triangle z \in \mathfrak{Z}_t$, all large enough t. Then there are taxspots according to Lemma 1. These taxspots must be recurrent because if they were transient, then there would be convergence to steady states and the argument can be repeated.

Proposition 3 *Let μ^* be a solution to Problem* (L-RRP-I) *for a deterministic economy with $g > 0$. Suppose there are finitely many steady states and all solutions to Problem* (RRP-I) *converge to a steady state. Then there is $(u_n)_{n\in\mathbb{N}}$ converging to u such that the corresponding solutions $(\mu_n^*)_{n\in\mathbb{N}}$ to Problem* (L-RRP-I) *are recurrent.*

Proposition 3 implies that either all solutions to Problem RRP-I converge to steady states, and then taxspots are recurrent, or the dynamics are complex. This is in contrast to Chamley (1986), where optimal tax equilibria lead to convergence to a steady state, and to consumption smoothing.

In light of Theorem 4, we introduce the following relaxed optimal random taxation problem,

$$\max_{\mu\in\triangle(Z)} \quad \mathbb{E}_{\mu,0}\sum_{t\geq 0}\beta^t u(c_t,x_t)$$

$$\text{s.t.}\quad \begin{cases} f_t(k_t,1-x_t)-c_t-k_{t+1}-g_t \;\geq\; 0 \text{ for every } t\geq 0 \\ c_0+k_1-\dfrac{u_x(c_0,x_0)}{u_c(c_0,x_0)}(1-x_0)-f_{k0}(k_0,1-x_0)k_0-b_0 \;\geq\; 0 \\ \mathbb{E}_{\mu,t-1}\,\beta[u_c(c_t,x_t)(c_t+k_{t+1})-u_x(c_t,x_t)(1-x_t)] \\ \qquad -u_c(c_{t-1},x_{t-1})k_t \;\geq\; 0 \text{ for every } (t-1,\theta^{t-1}) \text{ with } t>0 \\ \lim\limits_{t\to\infty}\mathbb{E}_0\,\beta^t u_c(c_t,x_t)k_{t+1} \;=\; 0. \end{cases} \tag{L-RRP-I}$$

Relative to (L-P), we have put restrictions on the randomizations: we are focusing on lotteries over processes $z = (z_t)_{t\geq 0}$ that are not averaging the household or government)

[4] The rank condition here provided: $f^*_{\ell t+1}u^*_{ct}[1+\sigma^*_{ct}\kappa^*_{t+1}(1-f^*_{\ell t}\frac{\psi^{*f}_{kt-1,t}-\psi^{*f}_{ct-1,t}}{\psi^{*f}_{xt-1,t}-f^*_{\ell t}\psi^{*f}_{ct-1,t}})] \neq f^*_{kt+1}\psi^{*f}_{xt,t+1}$.

budget at any date. Existence of a solution is guaranteed with the assumptions already made.

An equilibrium with taxspots can be defined as for the case of complete markets. Importantly, feasibility and the sequential budgets are satisfied at every history and no bonds are introduced. An equilibrium with taxspots decentralizing the lottery μ^* solving Problem (L-RRP-I) can be constructed using Lemma 3.

Proposition 4 *If μ^* is a lottery solving problem (L-RRP-I), then there exists an equilibrium with taxspots process $\left(\hat{c}_t, \hat{x}_t, \hat{k}_{t+1}, \hat{w}_t, \hat{r}_t, \hat{\tau}_t\right)_{t\geq 0}$ decentralizing μ^*.*

Proposition 4 finds a taxspot equilibrium decentralizing the optimal lottery satisfying feasibility and the budget constraint at every history without bonds, so budget constraints are satisfied history by history. Under the conditions of Theorem 4, the taxspot equilibrium must be nontrivial: if μ^* solves Problem (L-RRP-I), then μ^* cannot be a Dirac measure so there are nontrivial taxspots at some date-event.

The reason why budget constraint is satisfied at every date-event is that even if

$$\begin{aligned} &\beta \mathbb{E}_{t-1,s_t}\left[u_c(c_t,x_t)(c_t+k_{t+1})-u_x(c_t,x_t)(1-x_t)\right] \\ &\qquad -u_c(c_{t-1}(\hat{\omega}^{t-1}),x_{t-1}(\hat{\omega}^{t-1}))k_t(\hat{\omega}^{t-1}) \quad < \quad 0 \end{aligned}$$

at some history $\omega^{t-1} = (\theta^{t-1}, s^{t-1})$ and for a set of current taxspot states s_t with positive probability, this is consistent with the budget constraint at (θ_t, s_t):

$$\begin{aligned} c_t(\hat{\omega}^{t-1},\hat{\omega}_t)+k_{t+1}(\hat{\omega}^{t-1},\hat{\omega}_t) \quad = \quad & (1-\tau_t^{\ell}(\hat{\omega}^{t-1},\hat{\omega}_t))w_t(\hat{\omega}^{t-1},\hat{\omega}_t)(1-x_t(\hat{\omega}^{t-1},\hat{\omega}_t)) \\ & +(1-\tau_t^{k}(\hat{\omega}^{t-1},\hat{\omega}_t))r_t(\hat{\omega}^{t-1},\hat{\omega}_t)k_t(\hat{\omega}^{t-1}) \end{aligned}$$

for every s_t and (θ^t, ω^{t-1}), because the Euler equation for capital k_t at ω^{t-1} must be satisfied ν_t^*-average. Since feasibility is satisfied history by history, the government budget is balanced at every history and taxspot realization: the randomization does not produce any budget deficit or violation of the household's budget.

Instead, if in Problem (L-RRP-I) we considered lotteries over Z with constraints of the form

$$\mathbb{E}_{\mu}\, \psi_t^f(z_t, z_{t+1}) \;\geq\; 0$$

at every t, we would need 'taxspot insurance' bonds to decentralize the optimal lottery. Allowing the government to organize taxspot insurance can be difficult to justify in the present setting where we have assumed that financial markets are totally incomplete. Hence, the absence of taxspot insurance at the optimal equilibrium with taxspots can be seen as a policy advantage in the case the government is supposed not to have had the ability to issue bonds.

Workers and capitalists

Above we have built economies where taxspots occur on labor income taxes. This is not a necessary feature. For a prominent example where taxspots affect capital tax rates, we consider the Judd (1985)-inspired variant of a no bond economy with workers and capitalists studied by Straub and Werning (2020).

There are households owning capital k_0 and the production technology ('capitalists'), and those with no capital but who supply labor inelastically ('hand-to-mouth workers'). Assume that productivity is constant, and let $g_t = g > 0$, all t, and the capitalists' utility u be isoelastic with RRA coefficient $\sigma_u > 1$. Labor can still be taxed or subsidized, but in a lump-sum fashion. Letting T_t be this transfer, the government budget becomes $g + T_t = \tau_t^k r_t k_t$. The workers' consumption is $\hat{c}_t$, and their per-period utility is $v(\hat{c}_t)$, which is a twice continuously differentiable, differentially increasing and strictly concave function. Hereafter, we write $f(k_t) \equiv f(k_t, 1)$. The workers' budget is $\hat{c}_t = f(k_t) - r_t k_t + T_t$, the capitalists' budget is $c_t + k_{t+1} = (1 - \tau_t^k) r_t k_t$, and market clearing is $f(k_t) = \hat{c}_t + c_t + k_{t+1} + g_t$. Assuming here for simplicity that the Ramsey planner cares only about workers' welfare, the relaxed primal Ramsey problem becomes

$$\max_{(c_t, \hat{c}_t, k_{t+1})_{t\geq 0}} \sum_{t\geq 0} \beta^t v(\hat{c}_t)$$
$$\text{s. to} \quad \begin{cases} f(k_t) - \hat{c}_t - c_t - k_{t+1} - g_t \ \geq \ 0, \ t \geq 0, \\ \beta u_c(c_t)(c_t + k_{t+1}) - u_c(c_{t-1}) k_t \geq 0, \ t > 0, \\ \lim\limits_{t\to\infty} \beta^t u_c(c_t) k_{t+1} = 0. \end{cases} \tag{RRP-WC}$$

where we let T_0, τ_0^k be derived from $(c_0, \hat{c}_0, , k_0)$ to make the workers' and the government budgets at $t = 0$ hold, and $r_t = f_{kt}$. Under the stated assumptions, all solutions to Problem (RRP-WC) are interior and regular, and constraints are satisfied with equalities. A steady state is $(c, \hat{c}, k) = \lim_{t\to\infty}(c_t, \hat{c}_t, k_{t+1})$. Straub and Werning (2020) show that any solution to Problem (RRP-WC) converges to a unique steady state with $c > 0$, $\hat{c} = 0$, $k > 0$ and positive capital tax in the limit.

Hereafter, to ease the notation we write $z_t = (c_t, \hat{c}_t, k_{t+1})$ and $z = (z_t)_{t\geq 0}$, and keep denoting as ψ_{t-1}^f the left-hand side of the incentive inequality, and use ϕ_t^e for the feasibility constraint function. Correspondingly, we let $\mathfrak{Z}_t^{WC}$ be the previously introduced set of policy changes $\mathfrak{Z}_t$ but where functions ϕ_t^e, instead of ϕ_t, are used.

Theorem 5 *Let z^* be a solution to Problem* (RRP-WC). *Then there is date $t > 0$ and a change $\triangle z \in \mathfrak{Z}_t$ such that $\triangle z_{t'} = 0$ for $t' \notin \{t+1, t+2\}$, and $\triangle k_{t+2} \neq 0$. There is $(u_n)_{n\in\mathbb{N}}$ converging to u such that z^* is a solution to Problem* (RRP-WC) *at u_n, Condition (C) is satisfied for $\triangle z$, and optimal nontrivial taxspots are recurrent.*

Theorem 5 claims the suboptimality of a deterministic fiscal policy in economies $\mathscr{C}^2$-arbitrarily close to those considered by Straub and Werning (2020). As in Section 3, the intuition for Theorem 5 is that prudence can be exploited to increase the capitalists' savings with taxspot uncertainty. As in the previous representative agent economy, when we run the Pareto improving lottery, and decentralize it via taxspots, we obtain budget-balance at all lottery realizations.

Conclusions

We have shown that taxspots arise in response to the need to spur the agents' incentives to work or to save. The possibility that random taxes are Pareto improving depends on the comparison between the benefit and the cost of the increased uncertainty, or on the relative size of the precautionary effect of prudence and of risk aversion.

All taxspots occur without violating horizontal equity. Taxspots can occur also with complete markets, but essentially vanish in finite time. When there are no financial assets, taxspots can be recurrent unless the dynamics of optimal taxation are complex, that is, violate regularity and display some elements of chaotic behavior.

Appendix A: Proofs

Proof of Lemma 1: The requirement

$$2\mu - 1 \;<\; -\frac{1}{2}\,\frac{D^2\Psi_j(z^*)\triangle z^{(2)}}{D\Psi_j(z^*)\triangle z}$$

implies $\mu\Psi_j(z^*{+}\triangle z){+}(1{-}\mu)\Psi_j(z^*{-}\triangle z) > \Psi_j(z^*)$ because $D\Psi_j(z^*)\triangle z < 0$ and

$$\mu\Psi_j(z^*{+}\triangle z) + (1{-}\mu)\Psi_j(z^*{-}\triangle z) - \Psi_j(z^*) \;\approx\; (2\mu{-}1)D\Psi_j(z^*)\triangle z + \frac{1}{2}D^2\Psi_j(z^*)\triangle z^{(2)}.$$

Since $DU(z^*)\triangle z > 0$,

$$2\mu - 1 \;>\; -\frac{1}{2}\,\frac{D^2U(z^*)\triangle z^{(2)}}{DU(z^*)\triangle z}$$

similarly implies $\mu U(z^*{+}\triangle z){+}(1{-}\mu)U(z^*{-}\triangle z) > U(z^*)$.

The requirements on $2\mu{-}1$ can be satisfied if Condition (C) holds, as assumed. Because $D\Phi_i(z^*)_{i\in I}$, is onto $\mathbb{R}^I$, and $D\Phi_i(z^*)\triangle z = 0$, it can immediately be seen that $(\triangle z_1, \triangle z_2) \approx (\triangle z, -\triangle z)$ can be chosen such that also $\Phi_i(z^*{+}\triangle z_h) > 0$ for $h \in \{1,2\}$ and for all i such that $\Phi_i(z^* + \triangle z) \neq \Phi_i(z^*)$. Then, $(\triangle z_1, \triangle z_2, \mu)$ is a random improvement over z^*, and it is verified that $\mu > 1/2$ as wanted. □

Proof of Lemma 2: First, observe that $\operatorname{supp}\mu^* \subset \{z \in Z : \Phi_i(z) \geq 0,\ i \in I\}$. Since $Z = \prod_{t,\theta^t} Z_{\theta^t}$, and Z_{θ^t} is open in $\mathbb{R}^m$, Z is separable in the product topology, and supp μ^* is nonempty (see, e.g., Aliprantis and Border (2005), Thm 12.14). Let

$$E \;=\; \{\,(u,r) \in \mathbb{R}^{1+|J|} \mid \exists z \in \text{ supp } \mu^* : u = U(z) \text{ and } r_j = \Psi_j(z) \text{ for every } j\,\}$$

Then E itself is compact. Second, let $(u^*,0)$ be the point $u^* = \int U(z)\,d\mu^*$ and $0 = \int \Psi_j(z)\,d\mu^*$. Then, since Z is metrizable, by Theorem 15.10 in Aliprantis and Border (2005) $\lim_{n\to\infty}(u_n,r_n) = (u^*,0)$ for $u_n = \int U(z)\,d\mu_n$ and $r_n = \int_Y \Psi_j(z)\,d\mu_n$ where $\lim_{n\to\infty}\mu_n = \mu^*$ in the weak* topology, and μ_n has finite support for every n. Thus (u_n,r_n) is in the convex hull of E, $\operatorname{co}E$. Further, since $\operatorname{co}E$ is closed because E is closed, it is $(u^*,0) \in \operatorname{co} E$. Since E is a subset of $\mathbb{R}^{1+|J|}$, by Carathéodory's Convexity Theorem, all points in $\operatorname{co}E$ can be expressed as the convex combination of at most $2+|J|$ points in E. Thus, there is μ^{**} with support on at most $2+|J|$ points in E corresponding to $2+|J|$ elements of supp μ^*.

Then, μ^{**} has finite support z_j, $j = 1,...,|J|+2$. Let $\mathscr{J}(\theta^{t-1})$ be the set of j that are consistent with history (θ^{t-1}, z^{t-1}), that is,

$$\mathscr{J}(\theta^{t-1}) = \{j = 1,...,|J|+2 : z^{t-1} = z_j^{t-1}(\theta^{t-1})\}.$$

As

$$\text{supp } \mu_t^{**}(\theta^t, z^{t-1}) \subset \{z \in Z_{\theta^t} : (z^{t-1}, z) = z_j^t(\theta^t), \text{ some } j \in \mathscr{J}(\theta^{t-1})\},$$

and $\mathscr{J}(\theta^{t-1})$ is nonincreasing in the history length, there cannot be more than $|J|+1$ randomizations μ_t^* over processes z_j, and there exists $\bar{t}$ such that $\mu_t^* \in Z_t$, all $t \geq \bar{t}$, as wanted. □

Proof of Lemma 4: Suppose not, and then

$$\mathbb{E}_0 \sum_{t\geq 0} \beta^t [u_{ct}^* c_t^* - u_{xt}^*(1-x_t^*)] > u_{c0}^*[f_{k0}k_0 + b_0].$$

This makes the solution z^* to (RRP) equal to $\hat{z}$. As market clearing then holds as equality, the intertemporal incentive inequality holds only if

$$\mathbb{E}_0 \sum_{t\geq 0} \beta^t u_c(\hat{c}_t, \hat{x}_t) g_t + u_c(\hat{c}_0, \hat{x}_0) b_0) \leq 0,$$

a contradiction to the assumption.

Now suppose $\psi_{xt}^* > 0$ at some date-event $\hat{\theta}^t$ and, by contradiction, that $f_{a_t}(k_t^*(\hat{\theta}^{t-1}), 1-x_t^*(\hat{\theta}^t)) - c_t^*(\hat{\theta}^t) - k_{t+1}^*(\hat{\theta}^t) - \hat{\theta}_t > 0$. Consider $\triangle z$ with $\triangle x_t(\hat{\theta}^t) > 0$, and $\triangle c_t(\hat{\theta}^t) = \triangle k_{t+1}(\hat{\theta}^t) = 0$, while $\triangle z_t(\theta^t) = 0$ at all other date-events θ^t, and further $\triangle z_{t'} = 0$ at all $t' \neq t$. Since $\psi_{xt} > 0$, it is $D\Psi^* \triangle z > 0$, thus the intertemporal incentive equation is satisfied, while all other constraints are unchanged, and $DU(z^*)\triangle z = \beta^t\pi(\hat{\theta}^t)u_{xt}\triangle x_t(\hat{\theta}^t) > 0$, a contradiction to the optimality of z^*.

Proof of Theorem 2: Let an interior solution z^* to problem (RRP) and $\tilde{\theta}^{\tilde{t}}, \tilde{t} \geq 0$ be given where $z^*_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})$ is locally isolated and either tax is positive.

By regularity, and Theorem 1 in Luenberger (1969, p. 249), there is a separating positive continuous linear functional. Moreover, the derivative of the constraint functions is a bounded linear operator which is matrix representable and upper-diagonal. Thus, by Rustichini (1998, Theorem 5.5 and its Corollary) the separating functional is representable by a summable nonnegative multipliers process $(\hat{\lambda}^*_t)_{t\geq 0}$ (for feasibility) and by $\alpha^* \geq 0$ (for the incentive constraint). Letting (u^*_{ct}, u^*_{xt}) be the derivative of u with respect to c_t, x_t, and $\hat{\lambda}^*_t = \beta^t\pi^t\lambda^*_t$, the following first order conditions hold:

$$\begin{cases} u^*_{ct} - \lambda^*_t + \alpha^*\psi^*_{ct} &= 0 \\ u^*_{xt} - \lambda^*_t f^*_{\ell t} + \alpha^*\psi^*_{xt} &= 0 \\ -\lambda^*_t + \beta\mathbb{E}_t\lambda^*_{t+1}f^*_{kt+1} &= 0. \end{cases}$$

If $\tau^{*\ell}_{\tilde{t}}(\tilde{\theta}^{\tilde{t}}) > 0$, then immediately from the first two equations $\alpha^* > 0$. If $\bar{\tau}^{*k}_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}}) > 0$, consider the first equation in the FOC at $(\tilde{\theta}^{\tilde{t}})$ and at $(\tilde{\theta}^{\tilde{t}}, \theta)$, any $\theta \in \Theta$, respectively. Then, rearranging terms and dropping the history arguments we obtain

$$\alpha^*[\psi^*_{c\tilde{t}} - \beta\mathbb{E}_{\tilde{t}}\psi^*_{c\tilde{t}+1}f^*_{k\tilde{t}+1}] = \beta\mathbb{E}_{\tilde{t}}u^*_{c\tilde{t}+1}f^*_{k\tilde{t}+1} - u^*_{c\tilde{t}} > 0$$

implying again $\alpha^* > 0$.

Hereafter, we drop reference to $\tilde{\theta}^{\tilde{t}}$ whenever possible and illustrate the proof for the case when $\tau^{*\ell}_{\tilde{t}} > 0$, and leave the analogue case of a positive ex-ante capital tax to the reader.

By regularity, Condition (C) is satisfied by some $\triangle z \in \mathfrak{Z}$ provided

$$[\alpha^* D^2\Psi^* + D^2U^*](\triangle z)^{(2)} > 0,$$

where $\alpha^* = -DU^*\triangle z/D\Psi^*\triangle z$.

The sufficient second order condition for z^* to be a local maximizer is

$$\beta^{\tilde{t}}\pi^{\tilde{t}}[\alpha^* D^2\psi^*_{\tilde{t}} + D^2U^*_{\tilde{t}}](\triangle z_{\tilde{t}})^{(2)} + \mathbb{E}_0\sum_{t\neq\tilde{t}}\beta^t[\alpha^* D^2\psi^*_t + D^2U^*_t](\triangle z_t)^{(2)}$$
$$+\sum_t \hat{\lambda}^*_t D^2\phi^*_t(\triangle z_{t-1}, \triangle z_t)^{(2)} < 0 \quad (2)$$
$$\mathbb{E}_0\sum_t \beta^t D\psi^*_t \cdot \triangle z_t = 0 \quad (3)$$
$$D\Phi^* \cdot \triangle z = 0 \quad (4)$$

all $\triangle z \neq 0$.

Suppose that Condition (C) fails at all $\triangle z \in \mathfrak{Z}$. Pick $\triangle \hat{z}$ satisfying (3) and (4) and such that $\triangle \hat{z}_{\tilde{t}-1} = \triangle \hat{z}_{\tilde{t}+1} = 0$, while $\triangle \hat{c}_{\tilde{t}} \neq 0$. This can be done by regularity. Then, $D\phi^*_{\tilde{t}} \cdot (0, \triangle \hat{z}_{\tilde{t}}) = 0$, $D\phi^*_{\tilde{t}+1} \cdot (\triangle \hat{z}_{\tilde{t}}, 0) = 0$ and $\tau^{*\ell}_{\tilde{t}} > 0$ imply, by $\alpha^* > 0$, that $D\psi^*_{\tilde{t}} \cdot \triangle \hat{z}_{\tilde{t}} \neq 0$. If $[\alpha^* D^2\psi^*_{\tilde{t}} + D^2 U^*_{\tilde{t}}](\triangle \hat{z}_{\tilde{t}})^{(2)} > 0$, then the vector $\triangle \hat{z}'$ where $\triangle \hat{z}'_t = 0$ at all $t \neq \tilde{t}$, and with $\pm \triangle \hat{z}_{\tilde{t}}$ as $\tilde{t}$ coordinate would have the two properties: $\triangle \hat{z}' \in \mathfrak{Z}$ and Condition (C) satisfied, a contradiction. Thus,

$$[\alpha^* D^2\psi^*_{\tilde{t}} + D^2 U^*_{\tilde{t}}](\triangle \hat{z}_{\tilde{t}})^{(2)} \leq 0$$

for all such vectors $\triangle \hat{z}_{\tilde{t}}$.

Next, using the perturbation of Appendix B, we change $D^2\psi^*_{\tilde{t}}$ by increasing $u^*_{ccc\tilde{t}}$ by a term $\zeta_c > 0$, and thereby increasing $[\alpha^* D^2\psi^*_{\tilde{t}} + D^2 U^*_{\tilde{t}}](\triangle \hat{z}_{\tilde{t}})^{(2)}$, in a neighborhood $N_{z^*_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})} \subset \mathbb{R}^3$ of $z^*_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})$ without upsetting the derivatives at any other $z^*_t(\theta^t)$, $t \neq \tilde{t}$, by local isolation, or $D^2 U^*_t$ and $D^2\phi^*_t$ at any t.

As the involved expressions are continuous in ζ_c and monotonically increasing, there exists $\bar{\zeta}_c > 0$ large enough so that $[\alpha^* D^2\psi^*_{n\tilde{t}}(\bar{\zeta}_c) + D^2 U^*_{\tilde{t}}] \cdot (\triangle \hat{z}_{\tilde{t}})^{(2)} = 0$. Then, Condition (C) is satisfied by $\triangle \hat{z}$ with $\zeta_c = \bar{\zeta}_c + \varepsilon$ any $\varepsilon > 0$.

Consider the problem

$$\max_{\triangle z : ||\triangle z|| = 1} \mathbb{E}_0 \sum_t \beta^t [\alpha^* D^2\psi^*_{nt}(\zeta_c) + D^2 U^*_t](\triangle z_t)^{(2)} + \hat{\lambda}^* D^2 \Phi^*(\triangle z)^{(2)} \quad \text{s.to (3) and (4).}$$

This problem is continuous in $D^2\psi^*_{nt}(\zeta_c)$, that is, in ζ_c. Let $\triangle z(\zeta_c)$ denote its maximizer, and $V(\zeta_c)$ its value, which is increasing in ζ_c as long as it is below zero, and $V(0) < 0$.

To guarantee that z^* is still a solution at a ζ_c, we need to make sure that $V(\zeta_c)$ is negative. Let $\zeta^*_c > 0$ be the infimum level of ζ_c for which $V(\zeta_c) = 0$.

Without loss of generality $(\triangle k_t(\zeta^*_c), \triangle x_t(\zeta^*_c)) \neq 0$ for some t. Then, if $\hat{\lambda}^*$ is strictly positive, $\hat{\lambda}^* D^2 \Phi^*(\triangle z(\zeta^*_c))^{(2)} < 0$ by strict concavity of f, and as a result

$$\mathbb{E}_0 \sum_{t \geq 0} \beta^t [\alpha^* D^2\psi^*_{nt}(\zeta^*_c) + D^2 U^*_t](\triangle z_t(\zeta^*_c))^{(2)} \; > \; 0. \tag{5}$$

By continuity, $V(\zeta^*_c - \varepsilon) < 0$ for $\varepsilon > 0$ small enough, while (5) still holds. Then, by regularity we can find $\triangle z'$ close to $\triangle z(\zeta^*_c - \varepsilon)$ such that $\triangle z' \in \mathfrak{Z}$ while Condition (C) holds and so does the sufficient second order condition.

Suppose instead that $\hat{\lambda}^* D^2 \Phi^*(\triangle z(\zeta^*_c))^{(2)} = 0$. Then, (5) holds as a weak inequality, implying that $\zeta^*_c > \bar{\zeta}_c$. Then, $V(\bar{\zeta}_c) < 0$, and by continuity $V(\bar{\zeta}_c + \varepsilon) < 0$ for some $\varepsilon > 0$ small enough. Then, $\triangle \hat{z} \in \mathfrak{Z}$ and satisfies Condition (C) at $\bar{\zeta}_c + \varepsilon$, while the second order condition is satisfied, concluding the proof. □

Proof of Lemma 5: Suppose that $\hat{z}$ is a solution to (RRP-I). Then, $\hat{f}_{\ell t}\hat{u}_{ct} = \hat{u}_{xt}$ and $\beta \mathbb{E}_{t-1} \hat{u}_{ct} \hat{f}_{kt} = \hat{u}_{ct-1}$ at every $t > 0$. Substituting, and using the resource constraint at every t, the incentive constraint at $t-1$ is $-\beta \mathbb{E}_{t-1} \hat{u}_{ct} g_t \geq 0$, a contradiction.

Next, let u be separable and $\sigma_{ct} > 1$. Suppose that at a solution z^* to (RRP-I) it is $\psi^f_{t-1}(z^*_{t-1}, z^*_t) > 0$ for some θ^{t-1}, and $t > 1$.

If $f^*_{\ell t} u^*_{ct} > u^*_{xt}$ for some $\theta^t = (\theta^{t-1}, \theta_t)$, then consider a change $\triangle z$ everywhere zero except for at θ^t, where $\triangle c_t > 0$ and $\triangle x_t = -\frac{\triangle c_t}{f^*_{\ell t}}$, $\triangle k_{t+1} = 0$. Then, it is $DU(z^*) \cdot \triangle z > 0$ while no constraint is violated, as $D\psi^{*f}_t(\triangle z_t, 0) > 0$, a contradiction.

If $f^*_{\ell t} u^*_{ct} \leq u^*_{xt}$ for all θ_t, then

$$\left[\beta \mathbb{E}_{t-1} u^*_{ct} f^*_{kt} - u^*_{ct-1}\right] k^*_t \geq \beta \mathbb{E}_{t-1} u^*_{ct} g_t > 0.$$

Thus, consider a change $\triangle z$ everywhere zero except for at θ^{t-1} and all (θ^{t-1}, θ_t), where $\triangle k_t = -\triangle c_{t-1} > 0$, $\triangle c_t = f^*_{kt} \triangle k_t$, $\triangle x_{t-1} = \triangle x_t = \triangle k_{t+1} = 0$. Letting $D\psi^f_{t-2,t-1}$ be the derivative of ψ^f_{t-2} with respect to the vector z_{t-1} it is

$$\psi^{*f}_{ct-2,t-1} \triangle c_{t-1} + \psi^{*f}_{xt-2,t-1} \triangle k_t = [\psi^{*f}_{xt-2,t-1} - \psi^{*f}_{ct-2,t-1}] \triangle k_t > 0$$

because $\sigma_{ct-1} > 1$ and u is separable. Thus, all constraints are satisfied, while

$$DU(z^*) \cdot \triangle z = u^*_{ct-1} \triangle c_{t-1} + \beta \mathbb{E}_{t-1} u^*_{ct} \triangle c_t = \left[\beta \mathbb{E}_{t-1} u^*_{ct} f_{kt} - u^*_{ct-1}\right] \triangle k_t > 0,$$

a contradiction to optimality.

Finally, suppose that at a solution z^* it is $\phi_t(z^*_{t-1}, z^*_t) > 0$ for some θ^t, and $t \geq 0$. Consider a change $\triangle z$ which has all coordinates zero except for at θ^t, where we denote it simply as $\triangle z_t$. From $D\psi^{*f}_{t,t} \cdot \triangle z_t = 0$, using separability we get

$$u^*_{ct} \triangle k_{t+1} = -u^*_{cct} k^*_{t+1} \triangle c_t$$

and from $D\psi^{*f}_{t-1,t} \cdot \triangle z_t = 0$, we get

$$\triangle x_t = \frac{u^*_{cct} k^*_{t+1} - \psi^{*f}_{ct-1,t}}{\psi^{*f}_{xt-1,t}} \triangle c_t$$

which applied to the utility gradient yields

$$u^*_{ct} \triangle c_t + u^*_{xt} \triangle x_t = \left[1 - \frac{u^*_{xt}}{\psi^{*f}_{xt-1,t}} (1 - \sigma^*_t))\right] u^*_{ct} \triangle c_t$$

and given the assumptions the term in brackets is positive, so setting $\triangle c_t > 0$ gives $\triangle k_{t+1} > 0$, and $D\phi^*_{t+1} \cdot (\triangle z_t, 0) > 0$, a contradiction to optimality. □

Proof of Theorem 3: Drop the superscript * throughout. Let

$$\begin{cases} U(z) &= \mathbb{E}_0 \sum_t \beta^t u(c_t, x_t) \\ \Phi^e_{\theta^t}(z) &= \phi_t(z_{t-1}(\theta^{t-1}), z_t(\theta^t), g_t) \\ \Phi^f_{\theta^t}(z) &= \psi^f_t(z_t(\theta^t), z_{t+1}(\theta^t, \theta)_{\theta\in\Theta}) \text{ for all } t \neq \tilde{t}-1, \text{ all } \theta^{\tilde{t}-1} \neq \tilde{\theta}^{\tilde{t}-1}; \\ \Psi(z) &= \psi^f_{\tilde{t}-1}(z_{\tilde{t}-1}(\tilde{\theta}^{\tilde{t}-1}), z_{\tilde{t}}(\tilde{\theta}^{\tilde{t}-1}, \theta)_{\theta\in\Theta}), \end{cases}$$

It is

$$D\psi^f_t \;= D_{(z_t, z_{t+1})}\psi^f_t = \;(-k_{t+1}(u_{cct}, u_{cxt}), -u_{ct}, ..., \psi^f_{ct,t+1}, \psi^f_{xt,t+1}, \psi^f_{kt,t+1}, ...),$$

and

$$\begin{aligned} \psi^f_{ct,t+1} &= \beta\pi_{t+1,t}[u_{ct+1}\left(1-\sigma_{t+1}\left(\tfrac{c_{t+1}+k_{t+2}}{c_{t+1}}\right)\right) - u_{cxt+1}(1-x_{t+1})], \\ \psi^f_{xt,t+1} &= \beta\pi_{t+1,t}[u_{xt+1} - u_{xxt+1}(1-x_{t+1}) + u_{cxt+1}(c_{t+1}+k_{t+2})], \\ \psi^f_{kt,t+1} &= \beta\pi_{t+1,t}u_{ct+1}, \end{aligned}$$

where $\pi_{t+1,t} = \pi(\theta_{t+1}|\theta^t)$, and $\sigma_t = -u_{cct}c_t/u_{ct}$ is the RRA coefficient.

Consider $\triangle z = (\triangle z_t)_{t\geq 0}$, with $\triangle z_t = 0$ except for at $\tilde{\theta}^{\tilde{t}}$ and its (immediate) successors. Then, the nonzero changes in the ϕ_t functions occur only at $t = \tilde{t}, \tilde{t}+1, \tilde{t}+2$ (at histories $\tilde{\theta}^{\tilde{t}}, (\tilde{\theta}^{\tilde{t}}, \theta), (\tilde{\theta}^{\tilde{t}}, \theta, \theta')$ for $\theta, \theta' \in \Theta$). From $D\phi_{\tilde{t}+2}\cdot(\triangle z_{\tilde{t}+1}, 0) = 0$ we obtain $\triangle k_{\tilde{t}+2} = 0$. The nonzero changes in functions ψ^f_t occur only for $t = \tilde{t}-1, \tilde{t}, \tilde{t}+1$ (i.e., at histories $\tilde{\theta}^{\tilde{t}-1}, \tilde{\theta}^{\tilde{t}}, (\tilde{\theta}^{\tilde{t}}, \theta)$ for $\theta \in \Theta$).

The vectors $D\hat{\phi} = (D\phi_t,\ t = \tilde{t}, \tilde{t}+1)$ and $D\psi^f_t,\ t = \tilde{t}-1, \tilde{t}, \tilde{t}+1$ of derivatives at the given histories with respect to the variables $(\triangle z_{\tilde{t}}, \triangle z_{\tilde{t}+1})$, excluding $\triangle k_{\tilde{t}+2}$, form a $3 + 2 \mid \Theta \mid$ square matrix, which by assumption has full rank. Thus, there exists a nonzero $(\triangle z_{\tilde{t}}, \triangle z_{\tilde{t}+1}) \in \mathbb{R}^3 \times \mathbb{R}^{2|\Theta|}$ (with $\triangle k_{\tilde{t}+2} = 0$) such that

$$\begin{gathered} D\hat{\phi}\cdot(0, \triangle z_{\tilde{t}}, \triangle z_{\tilde{t}+1}, 0) = 0, \\ D\psi^f_{\tilde{t}-1}\cdot(0, \triangle z_{\tilde{t}}) < 0, \\ D\psi^f_{\tilde{t}}\cdot(\triangle z_{\tilde{t}}, \triangle z_{\tilde{t}+1}) = 0, \\ D\psi^f_{\tilde{t}+1}\cdot(\triangle z_{\tilde{t}+1}, 0) = 0, \end{gathered}$$

and by optimality $DU(z)\triangle z \geq 0$. We are left to rule out $DU(z)\triangle z = 0$.

Since z^* is a regular interior solution to (RRP-I), then, and using Rustichini (1998), given the matrix-representable and upper-diagonal form of the derivative of the constraint functions, the Kuhn-Tucker first order conditions hold for some summable multiplier process $(\lambda^*_t)_{t\geq 0}$ (for ϕ_t) and $(\alpha^*_t)_{t\geq 0}$ (for ψ^f_t), with $\lambda^*_t \geq 0$ and $\alpha^*_t \geq 0$.

The tangency condition $D\hat{\phi}\cdot(0,\triangle z_{\tilde{t}},\triangle z_{\tilde{t}+1},0)=0$ is

$$\begin{aligned}\triangle c_{\tilde{t}} &= -f_{\ell\tilde{t}}\triangle x_{\tilde{t}}-\triangle k_{\tilde{t}+1},\\ \triangle c_{\tilde{t}+1}(\theta) &= f_{k\tilde{t}+1}\triangle k_{\tilde{t}+1}-f_{\ell\tilde{t}+1}\triangle x_{\tilde{t}+1}(\theta).\end{aligned}$$

Thus, when restricted to $\triangle z$ as above, $DU(z)\triangle z$ is proportional to $D\hat{u}_{\tilde{t},\tilde{t}+1}\cdot(\triangle x_{\tilde{t}},\triangle k_{\tilde{t}+1},\triangle x_{\tilde{t}+1}(\theta)_{\theta\in\Theta})$, where

$$D\hat{u}_{\tilde{t},\tilde{t}+1} = (u_{x\tilde{t}}-f_{\ell\tilde{t}}u_{c\tilde{t}},\beta\mathbb{E}_{\tilde{t}}u_{c\tilde{t}+1}f_{k\tilde{t}+1}-u_{c\tilde{t}},\ldots,\beta\pi_{\tilde{t}+1,\tilde{t}}(u_{x\tilde{t}+1}-f_{\ell\tilde{t}+1}u_{c\tilde{t}+1}),\ldots).$$

Similarly, let D be the matrix of derivatives $(D\psi^f_{\tilde{t}},\ldots,D\psi^f_{\tilde{t}+1},\ldots)$ applied to $\triangle z$ restricted to the same tangency conditions, i.e., D is

$$\begin{pmatrix} k_{\tilde{t}+1}(u_{cc\tilde{t}}f_{\ell\tilde{t}}-u_{cx\tilde{t}}) & \beta\mathbb{E}_{\tilde{t}}\psi^f_{c\tilde{t},\tilde{t}+1}f_{k\tilde{t}+1}-(u_{c\tilde{t}}-k_{\tilde{t}+1}u_{cc\tilde{t}}) & \ldots & \psi^f_{x\tilde{t},\tilde{t}+1}-f_{\ell\tilde{t}+1}\psi^f_{c\tilde{t},\tilde{t}+1} & \ldots \\ & \vdots & & \vdots & \\ 0 & -k_{\tilde{t}+2}u_{cc\tilde{t}+1}f_{k\tilde{t}+1} & \ldots & k_{\tilde{t}+2}(u_{cc\tilde{t}+1}f_{\ell\tilde{t}+1}-u_{cx\tilde{t}+1}) & \ldots \\ & \vdots & & \vdots & \end{pmatrix}$$

where the columns are in the order of the variables $\triangle x_{\tilde{t}},\triangle k_{\tilde{t}+1},\triangle x_{\tilde{t}+1}(\theta)_{\theta\in\Theta}$, respectively.

Hence, to ensure that $DU(z)\triangle z>0$, by the Kuhn-Tucker conditions it suffices to exclude that

$$D\hat{u}_{\tilde{t},\tilde{t}+1}+\alpha(\tilde{\theta}^{\tilde{t}})D_{\tilde{\theta}^{\tilde{t}}}+\sum_{\theta\in\Theta}\alpha(\tilde{\theta}^{\tilde{t}},\theta)D_\theta = 0$$

for some $\alpha\geq 0$, where $D_{\tilde{\theta}^{\tilde{t}}},(D_\theta)_{\theta\in\Theta}$ are the rows of D.

If (ℓ), i.e., $\tau^\ell_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})>0$, the first coordinate of the $D\hat{u}_{\tilde{t},\tilde{t}+1}$ vector is negative, and under strict normality $u_{cc\tilde{t}}f_{\ell\tilde{t}}-u_{cx\tilde{t}}<0$, implying $\alpha(\tilde{\theta}^{\tilde{t}})<0$, a contradiction.

If (k), i.e., $\tau^\ell_{\tilde{t}}(\tilde{\theta}^{\tilde{t}})=0$, as the first row and column entry of D is negative by normality, $\alpha(\tilde{\theta}^{\tilde{t}})=0$. Since in the second column all entries but the first are positive, and since by $\bar{\tau}^k_{\tilde{t}+1}(\tilde{\theta}^{\tilde{t}})>0$ the second coordinate of $D\hat{u}_{\tilde{t},\tilde{t}+1}$ is positive, it is $\alpha(\tilde{\theta}^{\tilde{t}},\theta)<0$ for some contingency θ at $\tilde{t}+1$, a contradiction. Then, $\triangle z\in\mathfrak{Z}_{\tilde{t}-1}$.

Finally, we observe that the rank condition on $(D\hat{\phi}_{\tilde{t}}(z^*),D\hat{\psi}_{\tilde{t}}(z^*))$ gives us the rank condition for $D\Phi^*_i$ for $i\in I\setminus H$, and from Lemma 1 we conclude that a random improvement over z^* exists. □

Proof of Proposition 3: We want to show that for every $t>0$ there exists $t'\geq t$ such that $\mu^*_{t'}$ is not a Dirac measure over $A\subset\mathbb{R}^3$. If not, suppose μ^* is such that $\mu^*_{t'}$ is Dirac, all $t'\geq t$, with positive μ^* probability. Then, μ^* solves problem (RRP-I) starting at t with capital $k_t>0$ given. Let $z^*_{t'},t'\geq t$ be the corresponding continuation solution.

As for all admissible $k_0 > 0$ there is $(c^*, x^*, k^*) \in \mathbb{R}_{++} \times (0,1) \times \mathbb{R}_{++}$ such that $\lim_{t\to\infty}(c_t^*, x_t^*, k_{t+1}^*) = (c^*, x^*, k^*)$ for $(c_t^*, x_t^*, k_{t+1}^*)_{t\geq 0}$ solving (RRP-I), the continuation converges to a steady state. Then, points $z_{t''}^*, t'' \geq t$, are locally isolated. Considering the finitely many incentive and feasibility equations up to t, and time and separability of the utility function, by Carathéodory's Convexity Theorem there is a finite collection of distinct points in supp $\mu_{t'}^*$ for $t' < t$, and we can assume that the $z_{t''}^*$ are also locally isolated from any of the points in supp $\mu_{t'}^*$ for $t' < t$. Through a round of second-order derivatives perturbations at the steady state, $(D\hat{\phi}_{t''}(z^*), D\hat{\psi}_{t''}(z^*))$ has full rank for at least one t''. Since labor taxes are positive at the steady state, a change $\Delta z \in \mathfrak{Z}_{t''-1}$ as in Theorem 4 then exists, with $\triangle c_{t''} \neq 0$, satisfying Condition (C). By local isolation, no other point $z_{t'}^*$ is affected by the perturbation, and the original sequence $z_{t'}^*, t' \geq t$, is also a solution to (RRP-I) starting at t, also for the perturbed economy and, by the same token, μ^* is still a solution. Yet, we get a two-point lottery that Pareto improves over $z_{t'}^*, t' \geq t$, and then over μ^*, a contradiction. □

Proof of Theorem 5: Hereafter, we write σ for σ_u, and drop the superscript *. Let κ_t be the capital-to-consumption ratio. Pick a $t > 0$ and let $\triangle z_{t'} = 0$ for all $t' < t+1$, and $\triangle z_{t'} = 0$ for $t' > t+2$. Then, $\triangle z \in \mathfrak{Z}$, and only three incentive equations, at dates t, $t+1$ and $t+2$, are affected by $\triangle z$. If we neglect the incentive equation at t, then $(\triangle c_{t+1}, \triangle k_{t+2})$ and $(\triangle c_{t+2}, \triangle k_{t+3})$ satisfy the incentive equations at dates $t+1$ and $t+2$ with $(\triangle c_{t+3}, \triangle k_{t+4}) = 0$:

$$\begin{bmatrix} \beta(1-\sigma(1+\kappa_{t+3})) & \beta \\ \sigma\kappa_{t+3} & -1 \end{bmatrix} \begin{bmatrix} \triangle c_{t+2} \\ \triangle k_{t+3} \end{bmatrix} = \begin{bmatrix} m_{t+2}^{-1}(-\sigma\kappa_{t+2}\triangle c_{t+1} + \triangle k_{t+2}) \\ 0 \end{bmatrix} \qquad (*)$$

where $m_t \equiv u_{ct}/u_{ct-1}$. Thus, $D\psi_{t'}^f \cdot \triangle z = 0$ all $t' \neq t$. The determinant of the matrix on the l.h.s. of (*) is $-\beta(1-\sigma)$, and for $\sigma > 1$ the matrix is invertible. Then, let $\triangle_{t+1} \subset \mathbb{R}^4$ be the set of $(\triangle c_{t+1}, \triangle k_{t+2}) \in \mathbb{R}^2$ and $(\triangle c_{t+2}, \triangle k_{t+3}) \in \mathbb{R}^2$, where $(\triangle c_{t+1}, \triangle k_{t+2})$ is arbitrary and $(\triangle c_{t+2}, \triangle k_{t+3})$ is uniquely determined as a function of $(\triangle c_{t+1}, \triangle k_{t+2})$, via (*).

Straightforward calculations show that $\hat{c}$ changes only at dates $t+1$, $t+2$ and $t+3$, and $\triangle \hat{c}_{t'}$ are determined via the feasibility equations at dates $t+1$, $t+2$ and $t+3$ so that $D\phi_{t'}^e \cdot \triangle z = 0$ for all $t' \geq 0$. As $\lim_{t\to\infty} z_t^*$ exists, and at this limit $\kappa = \beta/(1-\beta)$, for t large enough these changes in $\hat{c}$ are approximately equal to

$$\begin{aligned}
\triangle \hat{c}_{t+1} &= -\triangle c_{t+1} - \triangle k_{t+2} \\
\triangle \hat{c}_{t+2} &= \frac{1}{\beta(1-\sigma)}[f_k\beta(1-\sigma)\triangle k_{t+2} - (1+\sigma\kappa)(-\sigma\kappa\triangle c_{t+1} + \triangle k_{t+2})] \\
\triangle \hat{c}_{t+3} &= \frac{1}{\beta(1-\sigma)} f_k\sigma\kappa(-\sigma\kappa\triangle c_{t+1} + \triangle k_{t+2}).
\end{aligned}$$

We now prove the following intermediate step. Let

$$Dv = (0,...,0,v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3},0...).$$

Auxiliary Claim: *Suppose that $\sigma > 1$, $\kappa = \beta/(1-\beta)$ and $f_k > 1/\beta$. Then, there exist a date $t > 0$ and a change in $\triangle_{t+1}$ such that $Dv \cdot \triangle\hat{c} \neq 0$.*

Proof: For $t > 0$ large enough and a change in $\triangle_{t+1}$, the derivatives of $\hat{c}$ with respect to c_{t+1} and k_{t+2} are

$$\begin{bmatrix} -1 & -1 \\ a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \equiv \begin{bmatrix} -1 & -1 \\ \frac{1}{\beta(1-\sigma)}(1+\sigma\kappa)\sigma\kappa & \frac{1}{\beta(1-\sigma)}[f_k\beta(1-\sigma)-(1+\sigma\kappa)] \\ -\frac{1}{\beta(1-\sigma)}f_k(\sigma\kappa)^2 & \frac{1}{\beta(1-\sigma)}f_k\sigma\kappa \end{bmatrix}.$$

The vector $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is orthogonal to the two columns in the matrix of derivatives if and only if

$$\begin{bmatrix} -1 & a_1 \\ -1 & b_1 \end{bmatrix}\begin{bmatrix} v_{\hat{c}t+1} \\ \beta v_{\hat{c}t+2} \end{bmatrix} = -\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\beta^2 v_{\hat{c}t+3}.$$

Since $\sigma > 1$ implies $a_1 < 0$ and $b_1 > 0$, the determinant of the l.h.s. matrix is not zero, so the matrix is invertible. Since $\sigma > 1$ implies $a_2 > 0$ and $b_2 < 0$, the ratios $v_{\hat{c}t+1}/\beta v_{\hat{c}t+2}$ and $\beta v_{\hat{c}t+2}/\beta^2 v_{\hat{c}t+3}$ are well defined, and equal to

$$\begin{cases} \dfrac{v_{\hat{c}t+1}}{\beta v_{\hat{c}t+2}} = -\dfrac{a_1 b_2 - a_2 b_1}{a_2 - b_2} = \dfrac{\sigma\kappa}{1+\sigma\kappa} \\ \dfrac{v_{\hat{c}t+2}}{\beta v_{\hat{c}t+3}} = -\dfrac{a_2-b_2}{a_1-b_1} = \dfrac{f_k(1+\sigma\kappa)\sigma\kappa}{(1+\sigma\kappa)^2 - f_k\beta(1-\sigma)}. \end{cases}$$

If the vector $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is orthogonal to the columns in the matrix *at every date t* (large enough), then the two ratios have to be identical, and

$$f_k = \frac{(1+\sigma\kappa)^2}{(1+\sigma\kappa)^2+\beta(1-\sigma)}.$$

Observe that $(1+\sigma\kappa)^2+\beta(1-\sigma) > 0$. Now $f_k > 1/\beta$ is equivalent to $(1-\beta)(1+\sigma\kappa)^2 < \beta(\sigma-1)$. Clearly, for $\kappa = \beta/(1-\beta)$ the inequality is violated. Consequently, there is $t > 0$ such that $(v_{\hat{c}t+1},\beta v_{\hat{c}t+2},\beta^2 v_{\hat{c}t+3})$ is not orthogonal to the columns in the matrix. □

It is now an immediate consequence of the Auxiliary Claim that if $\tau^{k*} > 0$, i.e., $f_k^*\beta > 1$, then there is a date $t > 0$ and a change in $\triangle_{t+1}$ such that $Dv^* \cdot \triangle\hat{c} \neq 0$. Thus, there exists

$\triangle z \in \mathfrak{Z}$ such that $DU^* \cdot \triangle z = Dv^* \cdot \triangle \hat{c} > 0$, and $D\psi^{*f}_{t'} \cdot \triangle z = 0$ all $t' \neq t$, $D\phi^e_{t'} \cdot \triangle z = 0$ all $t' \geq 0$. By optimality of z^*, it cannot be that $D\psi^{*f}_t \cdot \triangle z \geq 0$. It is immediately checked that $\triangle c_{t+1} \neq 0$, and $D^2\psi^{*f}_{t,t+1}(\triangle z)^{(2)}$ is proportional to $\beta[2u_{cct+1} + u_{ccct+1}(c_{t+1} + k_{t+2})]$. This proves the statement. □

Appendix B: A perturbation argument

Lemma 6 *For all $u \in \mathscr{C}^3(\mathbb{R}^2_{++}, \mathbb{R})$, $(\bar{c}, \bar{x}) \in \mathbb{R}_{++} \times (0,1)$, $\zeta_c, \zeta_x \in \mathbb{R}$ and $\varepsilon > 0$ there exists $(u_n)_{n\in\mathbb{N}} \in \mathscr{C}^\infty(\mathbb{R}^2_{++}, \mathbb{R})$ converging to u in the Whitney $\mathscr{C}^2$-topology such that*

- *$|c - \bar{c}|, |x - \bar{x}| \geq \varepsilon$ implies $u_n(c,x) = u(c,x)$.*
- *$D^m u_n(\bar{c}, \bar{x}) = D^m u(\bar{c}, \bar{x})$ for every $m \geq 0$, except $u_{nccc}(\bar{c}, \bar{x}) = u_{ccc}(\bar{c}, \bar{x}) + \zeta_c$ and $u_{nxx}(\bar{c}, \bar{x}) = u_{xx}(\bar{c}, \bar{x}) + \zeta_x / n$.*

Proof: Let the bump function $\chi \in \mathscr{C}^\infty(\mathbb{R}, [0,1])$ satisfy

$$\chi(\xi) \;=\; \begin{cases} 1 & \text{for } \xi \in [-1/2, 1/2] \\ 0 & \text{for } \xi \notin [-1,1]. \end{cases}$$

For $(\zeta_c, \zeta_x, \varepsilon) \in \mathbb{R}^3$ and every $n \geq 1$ let the function $u_n \in \mathscr{C}^\infty(\mathbb{R}^2_{++}, \mathbb{R})$ be defined by

$$u_n(c,x) \;=\; u(c,x) + \chi(n\triangle_c)\,\frac{\zeta_c}{6}\,\triangle_c^3 + \chi(\varepsilon\triangle_x)\,\frac{\zeta_x}{2n}\triangle_x^2.$$

where $\triangle_c = c - \bar{c}$ and $\triangle_x = x - \bar{x}$. Then,

$$\begin{cases}
u_{nc}(c,x) - u_c(c,x) &=& n\chi'(n\triangle_c)\,\frac{\zeta_c}{6}\,\triangle_c^3 + \chi(n\triangle_c)\,\frac{\zeta_c}{2}\triangle_c^2 \\
u_{ncc}(c,x) - u_{cc}(c,x) &=& n^2\chi''(n\triangle_c)\,\frac{\zeta_c}{6}\,\triangle_c^3 + n\chi'(n\triangle_c)\zeta_c\triangle_c^2 \\
&& +\chi(n\triangle_c)\zeta_c\triangle_c \\
u_{nccc}(c,x) - u_{ccc}(c,x) &=& n^3\chi'''(n\triangle_c)\,\frac{\zeta_c}{6}\,\triangle_c^3 + n^2\chi''(n\triangle_c)\,\frac{3\zeta_c}{2}\,\triangle_c^2 \\
&& +3n\chi'(n\triangle_c)\,\triangle_c + \chi(n\triangle_c)\zeta_c \\
u_{nx}(c,x) - u_x(c,x) &=& \varepsilon\chi'(\varepsilon\triangle_x)\,\frac{\zeta_x}{2n}\,\triangle_x^2 + \chi(\varepsilon\triangle_x)\,\frac{\zeta_x}{n}\,\triangle_x \\
u_{nxx}(c,x) - u_{xx}(c,x) &=& \varepsilon^2\chi''(\varepsilon\triangle_x)\,\frac{\zeta_x}{2n}\,\triangle_x^2 + 2\varepsilon\chi'(\varepsilon\triangle_x)\,\frac{\zeta_x}{n}\,\triangle_x \\
&& +\chi(\varepsilon\triangle_x)\,\frac{\zeta_x}{n}
\end{cases}$$

and the sequence $(u_n)_{n\in\mathbb{N}}$ has the properties described in the statement. □

Appendix C

Let $z_{-1} \equiv (c_{-1}, x_{-1}, k_0)$, at arbitrary c_{-1}, x_{-1}. Let $z(0; z_{-1}) \in Z \subset L(\mathbb{R}^3)$ be the solution to the optimal growth problem for $g = 0 = b_0$, of

$$\begin{aligned} V(k_0) \; = & \max_{(c_t, x_t, k_{t+1})_{t \geq 0}} \quad \sum_{t \geq 0} \beta^t u(c_t, x_t) \\ & \text{s.t.} \qquad f(k_t, 1-x_t) - c_t - k_{t+1} \; \geq \; 0, \text{ all } t \geq 0, \end{aligned} \tag{G}$$

given initial capital k_0. Under our maintained assumptions there exists a unique interior solution to (G), and the value function V uniquely satisfies the Bellman equation,

$$\begin{aligned} V(k_t) \; = & \max_{(c_t, x_t, k_{t+1})} \quad u(c_t, x_t) + \beta V(k_{t+1}) \\ & \text{s.t.} \qquad f(k_t, 1-x_t) - c_t - k_{t+1} \; \leq \; 0. \end{aligned}$$

and V is concave.

We are first going to show the existence of a unique, globally stable steady state. Then, by our separability and σ_{ct} assumptions, we show that $z(0; z_{-1})$ as well as all points uniformly interior satisfying the constraints with equality are regular for problem (RRP), leading to the solutions to be (sup norm) continuous in g in a neighbourhood of $g = 0$. Combining these two observations, for positive but small g the dynamics in (RRP) display the same behavior as when $g = 0$. As the stated assumptions imply that the first order conditions apply, and there must be labor taxes at some date-event, the multiplier α is positive, and labor taxes are then nonzero at every t.

Existence, uniqueness and global stability of the steady state at $g = 0$

In the present subsection we weaken the assumptions on utility to:

- $\lim_{x \to 0} \dfrac{u_c(w(1-x), x)}{u_x(w(1-x), x)} \; = \; 0$ and $\lim_{x \to 1} \dfrac{u_c(w(1-x), x)}{u_x(w(1-x), x)} \; = \; \infty$ for all $w > 0$.
- $u_c(c, x) f_\ell(k, 1-x) - u_x(c, x) = 0$ implies $u_{cc}(c, x) f_\ell(k, 1-x) - u_{cx}(c, x) \leq 0$.

The first assumption is about the marginal rate of substitution between consumption and leisure. The second assumption implies leisure is a differentiable normal good.

Lemma 7 (Existence and uniqueness of the interior steady state) *There is a unique steady state* $\bar{z} = (\bar{c}, \bar{x}, \bar{k}) \in \mathbb{R}_{++} \times (0, 1) \times \mathbb{R}_{++}$.

Proof: Clearly, $(\bar{c},\bar{x},\bar{k})$ is a steady state if and only if

$$\begin{cases} f(\bar{k},1-\bar{x})-\bar{c}-\bar{k} &= 0 \\ u_c(\bar{c},\bar{x})f_\ell(\bar{k},1-\bar{x})-u_x(\bar{c},\bar{x}) &= 0 \\ \beta f_k(\bar{k},1-\bar{x}) &= 1. \end{cases}$$

Use the first equation to eliminate c in the second equation. Since there are constant returns to scale, there is a unique capital intensity $\bar{v}=k/(1-x)$ such that the third equation is satisfied. By substitution, the second equation becomes

$$\frac{u_c(((f(\bar{v},1)-\bar{v})(1-x),x)}{u_x(((f(\bar{v},1)-\bar{v})(1-x),x)} f_\ell(\bar{v},1)-1 \;=\; 0.$$

By assumption the expression inside the parentheses converges to minus one when $x \to 0$, and tends to infinity when $x \to 1$. Hence, by the Intermediate Value Theorem there is $\bar{x} \in (0,1)$ such that the equation is satisfied. If $\bar{x}$ is unique, then we set $\bar{k}=\bar{v}\bar{x}$ and $\bar{c}=f(\bar{k},1-\bar{x})-\bar{k}$ for the unique steady state. We now show uniqueness of $\bar{x}$.

By the Implicit Function Theorem, $\beta f_k(k,1-x)=1$ implies $dk/dx=f_{\ell k}/f_{kk}<0$ because $f_{\ell k}>0>f_{kk}$ by constant returns to scale. Constant returns to scale also imply $f_{kk}f_{\ell\ell}-f_{\ell k}f_{k\ell}=0$. Moreover, $u_{cc}f_\ell f_\ell-(u_{xc}+u_{cx})f_\ell+u_{xx}<0$ because D^2u is negative definite. Thus, the derivative of the second equation with respect to x at (x,k) with $\beta f_k(k,1-x)=1$ is positive, because $u_{cc}f_\ell-u_{cx}\le 0$ by assumption and $f_k=1/\beta>1$. Consequently, there is a unique $\bar{x}$ such that the second equation is satisfied, as wanted. □

Since $\lim_{k\to 0} f(k,1-x)=0$ and $\lim_{c\to 0} u_c(c,x)=\lim_{x\to 0} u_x(c,x)=\infty$, $\lim_{k\to 0} V'(k)=\infty$. Let

$$\rho(k,k')=u(f(k,1-x(k,k'))-k',x(k,k')),$$

be the return function, where $x(k,k')$ is the twice-differentiable optimal static leisure policy for (k,k') with $f(k,1)-k'>0$. Strict concavity of utility u and production function f imply that ρ is concave. Further, some tedious but straightforward calculations show that

$$\det D^2\rho(k,k') \;=\; \frac{u_c f_{kk} \det D^2 u}{u_c f_{\ell\ell}+u_{cc}f_\ell f_\ell-u_{xc}f_\ell-u_{cx}f_\ell+u_{xx}} \;>\; 0.$$

Since ρ is strictly concave and $\det D^2\rho(k,k')>0$, $D^2\rho(k,k')$ is negative definite.

For the next two lemmas we assume the value function V is twice continuously differentiable. Afterwards we show that the assumption is not needed.

Let $h:\mathbb{R}_{++}\to\mathbb{R}^3_{++}$ denote the policy correspondence so $(\bar{c},\bar{x},\bar{k})\in h(\bar{k})$ and (c_t,x_t,k_{t+1}) is a solution to the Bellman problem at capital k_t if and only if $(c_t,x_t,k_{t+1})\in H(k_t)$.

Lemma 8 (Monotonicity of the dynamics) *The policy correspondence h is a continuously differentiable function $h = (h^c, h^x, h^k)$, and the capital dynamics are monotonic with $h^{k\prime}(k_t) > 0$ for all $k_t > 0$.*

Proof: There is a function $h \in \mathscr{C}(\mathbb{R}_{++}, \mathbb{R}^3_{++})$ such that (c_t, x_t, k_{t+1}) is a solution to the Bellman problem if and only if $(c_t, x_t, k_{t+1}) = h(k_t)$, where existence and continuity follow by Berge's Maximum Theorem, and single-valuedness by strict concavity. Inada conditions, marginal productivity conditions and $\lim_{k\to 0} V'(k) = \infty$ imply that for all $k_t > 0$ there is $\varepsilon \in (0,1)$ such that for if $c_t < \varepsilon$, $x_t \notin [\varepsilon, 1-\varepsilon]$ or $k_{t+1} < \varepsilon$, then (c_t, x_t, k_{t+1}) is not a solution to the Bellman problem. Thus, and after substitution of material balance in the utility u, the first-order conditions of the problem are

$$\begin{cases} -u_c(f(k_t, 1-x_t) - k_{t+1}, x_t) f_\ell(k_t, 1-x_t) + u_x(f(k_t, 1-x_t) - k_{t+1}, x_t) &= 0 \\ -u_c(f(k_t, 1-x_t) - k_{t+1}, x_t) + \beta V'(k_{t+1}) &= 0. \end{cases}$$

The matrix of derivatives of the first-order conditions with respect to (x_t, k_{t+1}) is:

$$A = \begin{pmatrix} u_c f_{\ell\ell} + u_{cc} f_\ell f_\ell - (u_{xc} + u_{cx}) f_\ell + u_{xx} & u_{cc} f_\ell - u_{cx} \\ u_{cc} f_\ell - u_{xc} & u_{cc} + \beta V'' \end{pmatrix}$$

where arguments and dates are dropped for convenience. The determinant of A is $|A| > 0$ because $V'' \leq 0$. Therefore, by the Implicit Function Theorem, the derivatives of (x_t, k_{t+1}) with respect to k_t are

$$\begin{pmatrix} h^{x\prime} \\ h^{k\prime} \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} u_c u_{cc} f_{k\ell} + [u_c f_{kl} + (u_{cc} f_\ell - u_{cx}) f_k] \beta V'' \\ u_c[-(u_{cc} f_\ell - u_{cx}) f_{k\ell} + u_{cc} f_k f_{\ell\ell}] + |D^2 u| f_k \end{pmatrix}$$

so $h^{k\prime}(k_t) > 0$ is positive because $u_{cc} f_\ell - u_{cx} \leq 0$ by assumption. □

Next, we combine the results on uniqueness of the steady state and monotonicity to show that the steady state is globally stable.

Lemma 9 (Global stability of the steady state) *Dynamics are monotone, $h^k(k) \in (k, \bar{k})$ for all $k < \bar{k}$ and $h^k(k) \in (\bar{k}, k)$ for all $k > \bar{k}$.*

Proof: By Lemma 8 there is a function $h^k \in \mathscr{C}^1(\mathbb{R}_{++}, \mathbb{R}_{++})$ such that $k_{t+1} = h^k(k_t)$ and $h^{k\prime}(k_t) > 0$ for all $k_t > 0$. By Lemma 7 the function h^k has a unique fixed point $h^k(\bar{k}) = \bar{k}$. Therefore: either $h^k(k) < k$ for all $k < \bar{k}$ or $h^k(k) > k$ for all $k < \bar{k}$; and, either $h^k(k) < k$ for all $k > \bar{k}$ or $h^k(k) > k$ for all $k > \bar{k}$.

If $k_t > \bar{k}$, then: $\bar{k} = h^k(\bar{k}) < h^k(k_t)$ because $h^{k\prime}(k) > 0$ for all $k > 0$; and, $h^k(k_t) < k_t$ because $k_{t+1} < f(k_t,1)$ and $\lim_{k\to\infty} f_k(k,1) < \beta$ so there is $K > 0$ such that $k > K$ implies $f(k,1) < k$.

If $k_t < \bar{k}$, then: $\bar{k} = h^k(\bar{k}) > h^k(k_t)$ because $h^{k\prime}(k) > 0$ for all $k > 0$. Suppose $h^k(k_t) < k_t$. Then $V'(k_t) \le V'(k_{t+1})$. By the Envelope Theorem, $V'(k_t) = u_{ct} f_{kt}$, and by the first order conditions $V'(k_{t+1}) = u_{ct}/\beta$. Thus, $\beta f_k(k_t, 1-h^x(k_t)) < 1$. As a result, $f_k(k_t, 1-h^x(k_t)) < f_k(\bar{v},1)$ and $f_\ell(k_t, 1-h^x(k_t)) > f_\ell(\bar{v},1)$, and $\lim_{k_t\to 0} 1-h^x(k_t) = 0$. On the other hand, the first-order condition with respect to x_t for the maximization problem is:

$$-u_c[f(k_t,1-h^x(k_t))-h^k(k_t),h^x(k_t)]f_\ell(k_t,1-h^x(k_t)) \\ +u_x[f(k_t,1-h^x(k_t))-h^k(k_t),h^x(k_t)] \;=\; 0.$$

Since $\lim_{k_t\to 0} u_c(f(k_t,1-h^x(k_t))-h^k(k_t),h^x(k_t)) = \infty$, $f_\ell(k_t,1-h^x(k_t)) > f_\ell(\bar{v},1)$ implies $\lim_{k_t\to 0} u_x(f(k_t,1-h^x(k_t))-h^k(k_t),h^x(k_t)) = \infty$, so $\lim_{k_t\to 0} h^x(k_t) = 0$. A contradiction is obtained as $\lim_{k_t\to 0} 1-h^x(k_t) = 0$ and $\lim_{k_t\to 0} h^x(k_t) = 0$ so $h^k(k_t) > k_t$ for all $k_t < \bar{k}$.

Now let $h^{k,0}(k) = k$ and $h^{k,t} = h^k \circ h^{k,t-1}(k)$ for every $t \ge 1$. Then the sequence $(h^{k,t}(k))_{n\in\mathbb{N}}$ converges to the steady state $\bar{k}$ for all $k > 0$. □

Lemmas 8 and 9 were obtained using the assumption that the value function is twice continuously differentiable, but that assumption is not needed. Indeed, consider iteration of the functions $(V_n)_{n\in\mathbb{N}}$ so

$$V_{n+1}(k) \;=\; \max_{k'} \rho(k,k') + \beta V_n(k').$$

Then $(V_n)_{n\in\mathbb{N}}$ converges to V and $(h^k_n)_{n\in\mathbb{N}}$ converges to h^k. Suppose V_n is concave and twice continuously differentiable so the policy function h^k_n is continuously differentiable with the property that $h^k_n(k) \in (k,\bar{k})$ for all $k \in (0,\bar{k})$ according to Lemma 8. Hence, V_{n+1} concave and twice continuously differentiable so the policy function h^k_{n+1} is continuously differentiable with the property that $h^k_{n+1}(k) \in (k,\bar{k})$ for all $k \in (0,\bar{k})$. Consequently, $h(k) \in [k,\bar{k}]$ for all $k \in (0,\bar{k})$. However, according to Lemma 7 there is a unique steady state so $h^k(k) \in (k,\bar{k}]$ for all $k \in (0,\bar{k})$ implying $(h^{k,n}(k))_{n\in\mathbb{N}}$ converges to $\bar{k}$ for all $k > 0$.

Continuity, and local isolation

The optimum correspondence $Z(g;z_{-1})$ is nonempty for all small enough g: this comes from product upper semicontinuity of the utility and product compactness of the domain. For $O \subset \mathbb{R}$ a neighborhood of zero, let $\Xi : O \rightrightarrows L(\mathbb{R}^3)$ be the correspondence

$$\Xi(g) = \{z \in L(\mathbb{R}^3) : (\Phi,\Psi)(z) \ge y_g\},$$

where $(\Phi, \Psi)(z) \geq y_g$ corresponds to

$$\phi_t(z_{t-1}, z_t; g) \geq 0, \; \sum_{t \geq 0} \beta^t \psi(z_t, z_{t+1}) - u_{c0} r_0 k_0 \geq 0$$

and y_g is the vector with 0 at its first coordinate, and g otherwise. Let $\partial \Xi$ denote the boundary of the correspondence, where inequalities are exact equalities.

We observe that $z(0; z_{-1})$ is also a solution to problems (RRP) at $g = 0$, and that it is in $\Xi(0)$. In fact, under our assumptions $Z(g; z_{-1}) \subset \partial \Xi(g)$, and we just need to show that $\partial \Xi$ is continuous at $g = 0$. To this end, we say that a point z is *uniformly interior* if $\inf c_t > 0$, $\inf k_{t+1} > 0$ and $0 < \inf x_t$ and $\sup x_t < 1$. Since the solution $z(0; z_{-1})$ converges to an interior steady state, it is uniformly interior, and we can restict attention to uniformly interior points of the domain.

Lemma 10 (Regularity) *Under the stated assumptions, the continuous linear map $D(\Phi, \Psi)(z)$ is onto at every uniformly interior $z \in \partial \Xi(0)$.*

Proof: Hereafter, all functions are evaluated on the Banach space $L(\mathbb{R}^3)$ of sup-norm bounded processes with values in $\mathbb{R}^3$, and the set of bounded sequences is ℓ_∞. Given the assumptions that u is separable, $a_t = 1$ and $\sigma_{ct} > 1$, all t, it is $D_x \psi_t^* - f_{\ell t} D_c \psi_t^* \neq 0$ at all θ^t. We can generate the vector $(0, ..., 0, 1)$ via changes $\triangle z$ where only $\triangle z_t$ is nonzero at that date event, $\triangle k_{t+1} = 0$, and $\triangle c_t = -f_{\ell t}^* \triangle x_t$. The derivative of the constraint functions applied to this $\triangle z$ has nonzero coordinates only corresponding to the feasibility constraint at θ^t and the function ψ at the same date-event, where it is

$$\begin{bmatrix} -f_{\ell t}^* & -1 \\ D_x \psi_t^* & D_c \psi_t^* \end{bmatrix}$$

with nonzero determinant. Vectors $(0, ..., 0, 1, 0, ..., 0)$, with 1 corresponding to the feasibility constraint at some θ^t can be generated with changes $\triangle z$ where only $\triangle z_t$ is nonzero at that date event, $\triangle k_{t+1} = \triangle x_t = 0$ and $\triangle c_t \neq 0$.

Further, since we are computing derivatives for points z uniformly interior, $\mathbb{E}_0 \sum_t \beta^t u_{ct}^*$ exists and is finite, and any indicator function of countable subsets of the nonnegative integers can be generated in the tangent space of the domain of the feasibility constraint functions, using $\triangle c_t = -1$ for every t in the subset, and adjusting the effect on the intertemporal incentive constraint with the perturbation identified above. Thus, as any other vector in $\ell_\infty \times \mathbb{R}$ is the image of limits of simple functions over indicators of subsets of the nonnegative integers, they can all be generated in $L(\mathbb{R}^3)$. We conclude that the derivative of the constraint functions is onto. $\square$

Lemma 11 (Continuity) *The solution correspondence is sup-norm continuous at $g = 0$:*

$$\text{for every } \varepsilon > 0 \text{ there exists } N_\varepsilon \text{ such that } \|z(g_n;z_{-1}) - z(0;z_{-1})\| < \varepsilon, \text{ all } n > N_\varepsilon,$$

for every $z(g_n;z_{-1}) \in Z(g_n;z_{-1})$ and $g_n \to 0$ as $n \to \infty$.

Proof: To obtain the continuity property, we show that $\partial\Xi$ is continuous in g at $g = 0$, in the sup-norm topology. We then apply Berge's Maximum Theorem to obtain the conclusion.

For l.h.c., as by regularity the continuous linear map $D(\Phi,\Psi)(z)$ is onto, by Liusternik's Theorem (see Luenberger 1969, Thm 1, p. 240) there exists $B > 0$ such that for every neighborhood N_0 of $0 \in L(\mathbb{R})$ and every point $y \in N_0$ it is $(\Phi,\Psi)(z') = y$ for all $z' \in N_z \subset L(\mathbb{R}^3)$ and $\|z' - z\|_\infty \leq B||y||_\infty$. Choose $y_n = (...,g_n,...,0)$. Then, $(\Phi,\Psi)(z_n) = y_n$ and $z_n \in \partial\Xi(g_n)$, while from $\|z_n - z\|_\infty \leq B||y_n||_\infty$ we get that $z_n \to z$ in sup-norm when $g_n \to 0$ (and, thus, $||y_n|| \to 0 \equiv y_0$).

Now, for u.h.c., suppose $g_n \to 0$, and let $z_n \in \partial\Xi(g_n)$. Then, $\phi_t(z_{n,t-1},z_{n,t};g_n) = 0$, and $\sum_{t\geq 0}\beta^t\psi(z_{n,t},z_{n,t+1}) = 0$.

If $\|z_n - z\|_\infty \to 0$, some $z \in L(\mathbb{R}^3)$, then the coordinates $t-1,t$ converge, and since ϕ_t is continuous (as a finite dimensional map), we have that $\lim\phi_t(z_{n,t-1},z_{n,t};g_n) = \phi_t(z_{t-1},z_t;0) = 0$, and since $\sum_{t\geq 0}\beta^t\psi(.)$ is product continuous on the domain, $\lim\sum_{t\geq 0}\beta^t\psi(z_{n,t},z_{n,t+1}) = \sum_{t\geq 0}\beta^t\psi(z_t,z_{t+1}) = 0$. Thus, $z \in \partial\Xi(0)$, as we wanted to show.

The only thing left to prove is that $\|z_n - z\|_\infty \to 0$.

Since $D(\Phi,\Psi)$ is onto at all $z'^{-1}(y_0)$, it is $\|z_n - z'\|_\infty \leq B\left\|y_{g_n} - y_0\right\|_\infty$ for $z_n \in (\Phi,\Psi)^{-1}(y_{g_n})$, for all $n \geq N_1$. Thus, given $\varepsilon > 0$, take $n,m > N_2 \geq N_1$ such that $\max(\left\|y_{g_n} - y_0\right\|_\infty, \left\|y_{g_m} - y_0\right\|_\infty) \leq \varepsilon/2B$. Then,

$$\|z_n - z_m\|_\infty \leq \left\|z_n - z'\right\|_\infty + \left\|z_m - z'\right\|_\infty \leq B\left\|y_{g_n} - y_0\right\|_\infty + \left\|y_{g_m} - y\right\|_\infty < \varepsilon$$

for all $n,m > N$, and $\{z_n\}_{n\geq N}$ is a Cauchy sequence in the sup norm. Since $L(\mathbb{R}^3)$ is complete, z_n converges to $z \in L(\mathbb{R}^3)$, in the sup norm. □

As $z_t(g_n;z_{-1})$ is close in the sup-norm to $z_t(0;z_{-1})$ by Lemma 11, by Lemma 10 it is regular (an open property), and it satisfies the first order conditions for some $\alpha^*(g_n) > 0$, as in Theorem 2. Consider problem

$$\begin{array}{ll} \max\limits_{(c_t,x_t,k_{t+1})_{t\geq 0}} & \sum\limits_{t\geq 0}\beta^t[u(c_t,x_t) + \alpha(g_n)\psi_t(z)] \\ \text{s.to} & f(k_t,1-x_t) - g_n - c_t - k_{t+1} \;\geq\; 0, \text{ all } t \geq 0. \end{array} \tag{GP}$$

With $\hat{u}(c_t,x_t) = u(c_t,x_t) + \alpha(g_n)\psi_t(c_t,x_t)$ and $\hat{f}(k_t,1-x_t) = f(k_t,1-x_t) - g_n$, for $\alpha(g_n)$ (i.e., g_n) small enough $\hat{u},\hat{f}$ preserve the properties of u,f at $g = 0$. Then, problem (GP) is an instance of problem (G). It is convex, the first order conditions are necessary and sufficient, and $z_t(g_n; z_{-1})$ solves them. Then, $z_t(g_n; z_{-1})$ is a solution to (GP). However, since problem (GP) is an instance of problem (G), we conclude that for a small enough $g > 0$, $z_t(g; z_{-1})$ converges to an interior, globally stable steady state, and points can be found which are locally isolated.

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