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Subhasish M. Chowdhury and Iryna Topolyan

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# The Attack-and-Defense Conflict with the Gun-and-Butter Dilemma *

Subhasish M. Chowdhury

[a] Department of Economics, University of Sheffield, Sheffield S1 4DT, UK

Iryna Topolyan

Department of Economics, University of Cincinnati, Cincinnati, OH 45221, USA

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**Abstract**

We analyze a general equilibrium model of attack and defense with production and conflict. One attacker and one defender allocate their fixed endowments either to produce gun or to produce butter, and the volume of guns produced determines the winner in the conflict. If the attacker wins, then it appropriates all the butter produced in the economy; otherwise, each consume only their own butter. We characterize the unique interior and unique corner equilibrium for this game. We find that (i) the defender may spend more resources on conflict than the attacker even without loss aversion or other behavioral biases, (ii) the attacker may expend all their resources only in conflict, and (iii) the interior and the corner equilibria cannot coexist.

*JEL Classification*: C72; D74; D23; Q34

*Keywords*: Conflict; Production; Gun and Butter; Attack and Defense

* Emails: Subhasish M. Chowdhury (Corresponding author) – Subhasish.chowdhury@sheffield.ac.uk; Iryna Topolyan – topolyia@ucmail.uc.edu. We thank Oliver Gürtler, Kai Konrad, Stergios Skaperdas, the participants at the European Game Theory Conference, SAET Annual Conference, Lancaster Game Theory Conference, and the seminar participants at the University of Bath for comments and suggestions. Any remaining errors are our own.

## 1. Introduction

Jack Hirshleifer (2000) once famously mentioned that "*There are two main ways of making a living: by production or by conflict. Consequently, two distinct technologies must be distinguished: the familiar technology of production and exchange on one hand, and the technology of conflict and struggle on the other*." In many such conflicts, people with limited resources allocate it between consumption (butter) and conflict (gun) (Samuelson, 1948). This has real consequences for consumption and other productive activities. For example, in 2022, the Middle East and North African countries spent 4.6% of the GDP on defense (while the world average is 2.3%) compared to 3.8% in education (world average is 4.5%). The net opportunity cost of conflict with Pakistan for India is estimated to be 2.5% of its GNP (Atlantic Council South Asia Center report, 2014). Whereas the cost of the first intifada is estimated to be about $2000 per capita per year for Israel (Horiuchi and Mayerson, 2015).

In the absence of strict property rights, which is often the case for disputed territories or in many developing countries, the means of conflict becomes important. The expenditure on conflict is usually sunk and unproductive. However, even when one is not interested or capable of appropriating others' property, they have to engage in conflict in order to defend their own. This specific form of conflict is called 'attack and defense' in which one type of agent attacks, and the other type defends. As John Stuart Mill (1848, p. 979) pointed out "[…] *the energies now spent by mankind in injuring one another, or in protecting themselves against injury* […]", both attacking others and defending oneself entail opportunity cost of not producing consumption goods. Such situations include guerrilla war, siege, terrorism, malware, bank fraud etc. Hence, the issues of production versus conflict, and attack versus defense are very important as well as related. Such an example is observed in sub-Saharan Africa, where the incidence of civil war has a negative effect on the size of the manufacturing sector (Caruso, 2010).

Both conflict vs. production (gun-and-butter) and attack-and-defense are popular research topics in economics, political science, and conflict studies. Many recent or ongoing conflicts (e.g., Maoist insurgency in India (Mahadevan, 2012), FARC conflict in Colombia (Rubiano A., 2021) etc.) follow this combination. However, there is a real scarcity of theoretical analysis to combine the two. In this study we aim to fill in this gap and contribute to both these areas of literature.

To do so, we consider a two-player attack and defense contest within a general equilibrium structure. One attacker and one defender start with their own fixed endowments and can allocate it either to produce gun or to produce butter. The guns determine the winner of the conflict through a ratio form (Tullock, 1980) contest success function. If the Attacker wins, then she appropriates the butter of the defender. Otherwise, each consumes their own butter. We fully characterize the equilibria of this game, and show that mutually exclusive unique interior solution, and unique corner solution exists. In both the equilibria, defenders may spend more resources on conflict than the attacker even without assuming loss aversion (unlike in the literature). In the corner solution, capacity constrained attackers decide only to attack and not to produce any butter. Comparative statics analyses show further non-intuitive results.

Both the areas of research have separately attracted adequate attention. Seminal studies by Skaperdas (1992), Hirshleifer (1995) and the ones following them (e.g., Grossman and Kim (1995), Neary (1997), Durham et al. (1998), Noh (2002) among others) focus on various aspects of tension between consumable production (butter) versus resources in conflict (butter) under budget constraints.  Other notable contributions in the game of production and conflict include Hausken (2005), Hafer (2006), and Kolmar (2008). Hausken (2005) compares the production and conflict model with rent seeking model and find the effects of group size in both. Hafer (2006) theoretically shows the emergence of two specific types of agents (haves and have-nots) in a steady state equilibrium. Kolmar (2008) uses a model of sequential attack and defense in Tullock contest to find conditions for endogenously arising property rights. He finds that even when perfectly secure property rights emerge, the incentive to produce remains inefficient. Kornienko (2020) analyzes an $n$-player all-pay auction in which the reward is the residual of resources of all players after their bid. Under certain conditions the equilibrium payoff becomes identical to the one in single-object independent private value auctions. In these studies, however, the players are either not defined as attacker and defender, or they are defined so only in terms of their budget.

There is also a long literature on attack and defense both within and outside economics. Bester and Konrad (2004) find the asymmetry between an attacker and a defender to be responsible for delay in contests. Clark and Konrad (2007) consider a multi-battle contest in which the attacker needs to win at most one battlefield, whereas the defender will have to defend (win) all the battlefields. Similar structures are used in Arce et al. (2012) and Kovenock and Roberson (2018). Bose and Konrad (2020) study the effects of observability in a situation in which multiple attackers attack multiple defenders. When defense effort is observable, then defenders compete between themselves to not to become a weak target. In case of unobservability, this competition disappears. Dziubiński and Goyal (2013) use a model of network where a defender forms costly links among a number of nodes and invests on defending a subset of them, whereas an attacker tries to eliminate the nodes. They find that the defendable network can be either sparse or dense depending on whether the cost is small or large. Similarly, Goyal and Vigier (2014) investigate contagion in a network with attack and defense. They find conditions under which certain specific types of defense network emerges.

There is also a stream of experimental studies on this topic, but those also do not consider production. Deck and Sheremeta (2012) use an all-pay auction structure similar to Clark and Riis (2007). They show theoretically and experimentally that when the valuation of the winning is not high enough, the defender stops fighting, whereas it tries to fight all the battlefields in case the valuation is high. Chowdhury et al. (2018), instead, consider theoretically and experimentally a standard contest in which the defender owns the prize to begin with (attacker starts with nothing), and loses it to the attacker if he loses the contest. They find that defenders expend more effort than the attackers and explain such behavior in terms of loss aversion. Neuroscience experiments (e.g., De Dreu et al., 2021) investigate the neurocognitive and hormonal foundations of attack and defense. De Dreu et al. (2019), in a series of experiments, show two psychological pathways through which a lack of attack can be explained. They find that people with higher level of social preferences (such as empathy) and people who make cautious decisions attack less. These pathways, however, do not explain the defender behavior. See Kovenock et al. (2018) for further experimental evidence.

Attack and defense is also analyzed in group settings. Chowdhury and Topolyan (2016a, b) analyze group contests in which the attacker group has a best-shot impact function whereas the defender group has a weakest link impact function. Aloni and Sela (2012) consider pairwise individual contests between two groups. The attackers need to win one such contest, whereas the defenders will need to win all. Biologists and psychologists consider attack and defense mostly within a prisoner's dilemma set up. See, for example, De Drew and Gross (2019) for a survey on various behavioral aspects of attack-and-defense. None of these studies, again, consider production while analyzing attack and defense.

The only study that combines these two areas of literature and the closest to ours is by Yektaş et al. (2019). They use a similar structure in which the whole butter of the defender and a fraction of butter of the attacker constitute the prize. They focus on the effect of that fraction (when it is 1, i.e., the standard case, and when it is less than 1), and the difference in production function on the behavior of the players. They make several assumptions to achieve unique Nash equilibrium in this set up and find only interior equilibrium (attacker never expends the whole resource in producing guns). They also focus on the choice of being an attacker or a defender that we do not do, since many of the attackers and defenders are exogenously and historically determined.

## 2. Model

Consider a setting with 2 risk-neutral players: identified as the Attacker ($A$) and the Defender ($D$). Each Player $i(= A, D)$ has their own fixed endowment $E_i > 0$ that they can allocate either to create gun (i.e., conflict effort: $x_i$) or to produce butter (i.e., consumption good: $y_i$). The conversion from endowment to consumption good is assumed to be a direct one-to-one, whereas the constant marginal cost of conflict is $c_i > 0$ for Player $i$ (and hence, $y_i \leq E_i - c_i x_i$). Any endowment that is not allocated to produce either guns or butter is wasted. The utility or payoff of each player depends directly on the amount of butter they consume. The conflict effort in itself does not provide any utility to the players, hence the final payoff depends only on own consumption.

The players can be asymmetric in terms of their level of endowment ($E_i$) as well as in terms of their conflict cost ($c_i$). Nonetheless, the main difference between the players arises from their nature in the conflict: Player $A$ (she) attacks and Player $D$ (he) defends. If the attacker wins, then she appropriates the defender's butter (along with her own). However, if the defender wins or there is no conflict, then each player consumes only the butter they produce on their own. One can view the attacker and the defender as a terrorist organization and a defending government, or an attacking country and a country defending its land from the attacker, or a group of bandits and a village hiring mercenaries (as in the movie *Seven Samurais*) etc.

The winner of the conflict is determined by the amount of guns produced by each player with a ratio form (Tullock, 1980) contest success function.[1] In particular, when at least one of the players

[1] Another popular CSF that is implemented in the attack-and-defence is the All-pay auction (see, for example, Aloni and Sela, 2012; and Chowdhury and Topolyan, 2016a). We choose the ratio form (Tullock, 1980) since it captures the stochastic nature of a conflict outcome and allows us to obtain closed form solution in interpretable pure strategies.

decides to produce guns, then the probability that the defender wins is given by $p_D = x_i/(x_A + x_D)\ if\ (x_A, x_D) \neq (0,0)$. However, if no player decides to produce guns, then by default the players consume their own butter; or in other words, $p_D = 1,\ if\ (x_A, x_D) = (0,0)$. The probability that the attacker wins is given always by $p_A = 1 - p_D$, i.e., no stalemate is possible.

Hence, the payoff functions for the Attacker and the Defender, respectively, are:

$$u_A = \begin{cases} (E_D - c_D x_D)\frac{x_A}{x_A + x_D} + (E_A - c_A x_A) & if\ (x_A, x_D) \neq (0,0) \\ E_A & if\ (x_A, x_D) = (0,0) \end{cases} \tag{1}$$

$$u_D = \begin{cases} (E_D - c_D x_D)\frac{x_D}{x_A + x_D} & if\ (x_A, x_D) \neq (0,0) \\ E_D & if\ (x_A, x_D) = (0,0) \end{cases} \tag{2}$$

Eq. (1) states that the attacker always consumes her own butter $(E_A - c_A x_A)$, and also consumes the butter of the defender $(E_D - c_D x_D)$ when she wins with the probability $p_A$. Eq. (2), on the other hand, states that the defender can consume only his own butter $(E_D - c_D x_D)$, only when he wins with the probability $p_D$.

This attack-and-defense game is defined as $\Gamma(A, D; \Omega)$, where $\Omega = \{E_A, E_D, c_a, c_D\}$ is the set of parameters. The objective function for Player $i(= A, D)$ in this attack-and-defense game $\Gamma$ is:

$$\max_{x_i} u_i \text{ subject to the budget constraint } x_i \in [0, E_i/c_i],\ i = A, D. \tag{3}$$

We show later in Table 1 that $(x_A, x_D) = (0,0)$ can never be an equilibrium. For the time being, provided that $(x_A, x_D) \neq (0,0)$, the players' marginal utilities are:

$$\frac{\partial u_A}{\partial x_A} = \frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} - c_A, \tag{4}$$

$$\frac{\partial u_D}{\partial x_D} = \frac{x_A E_D - x_D c_D (2x_A + x_D)}{(x_A + x_D)^2}. \tag{5}$$

It is easy to note that $\frac{\partial u_i}{\partial x_i}$ is decreasing in $x_i$ $(i = A, D)$, so each Player $i$'s payoff function $u_i$ is strictly concave in their own decision variable $x_i$. Consequently, Player $i$'s best response is unique for any $x_j \neq 0$, $j \neq i$. Note that Player $A$'s best response is undefined when $x_D = 0$, because the attacker always wants to deviate to an infinitesimally small yet positive effort. Player $A$'s best response is zero when $x_D \geq \frac{E_D}{c_A+c_D}$ (this follows from the fact that $\frac{\partial u_A}{\partial x_A}$ is decreasing in $x_A$ and $\frac{\partial u_A}{\partial x_A} \leq 0$ when $x_A = 0$ and $x_D \geq \frac{E_D}{c_A+c_D}$). Using a similar reasoning, we conclude that Player *D*'s best response is zero when $x_A = 0$ and strictly positive when $x_A > 0$.

In what follows in our characterization of equilibria, we distinguish between interior equilibria, i.e., those with $x_i \in (0, E_i/c_i)$ for each $i = A, D$; and corner equilibria, such that for some $i$, $x_i = 0$ or $x_i = E_i/c_i$. From Eq. (3) – (5), 9 such possible cases can arise in which both the players either exert no effort, or the highest effort possible, or something in between. However, it is easy to show that many of such cases cannot be an equilibrium. Table 1 below summarizes all the cases.

**Table 1.** Possible types of equilibria in $\Gamma(A, D; \Omega)$

| $x_A^*$ | $x_D^*$ | Note | Explanation |
|---|---|---|---|
| 0 | 0 | Not Possible | $A$ can increase payoff by exerting small $x_A^* = \varepsilon > 0$. |
| 0 | $E_D/c_D$ | Not Possible | $D$ can increase payoff by exerting $x_D^* = 0$. |
| 0 | $(0, E_D/c_D)$ | Not Possible | If $x_A^* = 0$, then $D$ wants to decrease $x_D^*$ to 0. |
| $E_A/c_A$ | 0 | Not Possible | $A$ can increase payoff by exerting small $x_A^* = \varepsilon > 0$. |
| $E_A/c_A$ | $E_D/c_D$ | Not Possible | $A$ can increase payoff by exerting $x_A^* \in (0, E_A)$. |
| $(0, E_A/c_A)$ | 0 | Not Possible | $A$ can increase payoff by exerting small $x_A^* = \varepsilon > 0$. |
| $(0, E_A/c_A)$ | $E_D/c_D$ | Not Possible | $D$ can increase payoff by exerting $x_D^* \in (0, E_D)$. |
| $(0, E_A/c_A)$ | $(0, E_D/c_D)$ | Possible | Standard interior solution. |
| $E_A/c_A$ | $(0, E_D/c_D)$ | Possible | $A$'s low budget may make it feasible. |

Hence, in the continuation, we study only the equilibria possible to attain. First, we investigate possible interior equilibrium and find a unique equilibrium that is characterized in Proposition 1.

**Proposition 1.** There exists a unique interior equilibrium characterized by $x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right)$ and $x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)$, if and only if $\frac{E_D}{4c_D}\left(\sqrt{1+\frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}$.

**Proof:** See the Appendix. ■

From Eq. (8) in the Appendix, it is easy to show that as long as interior equilibrium exists, the defender's best response $x_D$ is increasing in $x_A$, indicating strategic complementarity (Amir, 2005). However, the defender's best response $x_D$ is, in general, non-monotone in $x_A$. The following diagram depicts best responses and (interior) equilibrium when $E_A = 1.5,\ E_D = 1,\ c_A = c_D = 1$. Note that $(0,0)$ is not an equilibrium since best responses are discontinuous at zero.

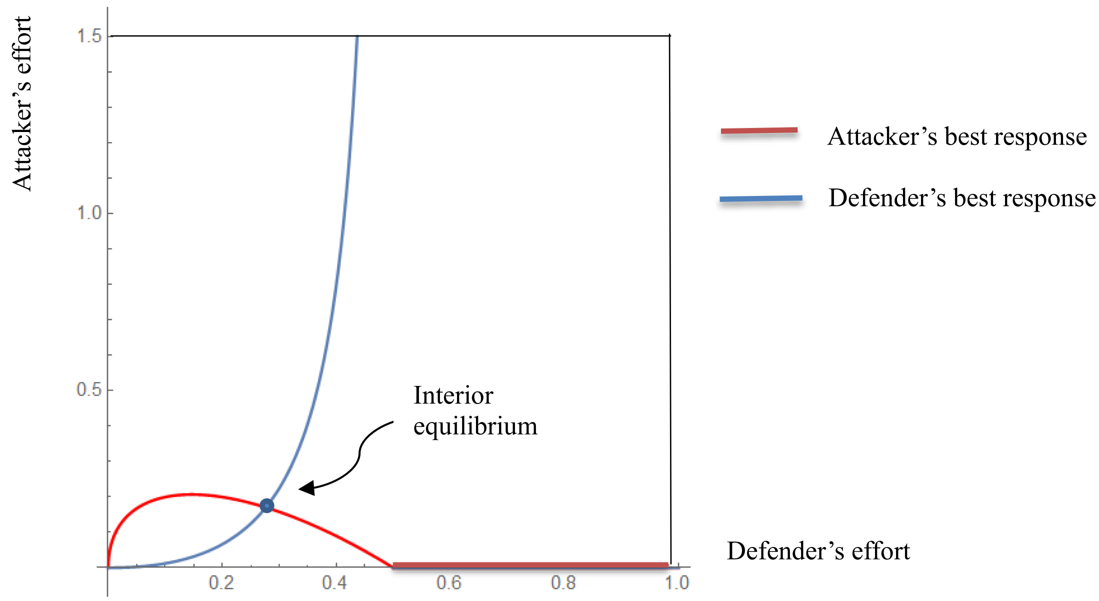


**Figure 1.** Best responses and the interior equilibrium

Note that the interior equilibrium exists only if the attacker has either a sufficiently large endowment relative to the defender, or a sufficiently small cost of conflict – when the endowment is not large enough. This matches with the field observations in which a fringe terrorist group, a small group of bandits (or a lone wolf), having a very small endowment compared to the defending government, often expend all their resources in conflict. However, a relatively larger terrorist group, such as the Maoists in India (Mahadevan, 2012) spend their resources both to arrange their own economic system, as well as to engage into conflict with the government.

Note also that the endowment of the attacker, $E_A$ does not enter in the equilibrium conflict allocation of either the attacker or the defender. This is because it is a fixed amount that is not contested, whereas due to the conflict over the residual of $E_D$ and $x_D$, $E_D$ enters the calculation. As expected, the larger the defender's endowment that is redistributed in conflict, the higher both the attacker's and the defender's efforts in conflict. We provide below some further results derived from this proposition in Corollary 1.

**Corollary 1.** Consider the unique interior equilibrium described by Proposition 1.

(i) The defender exerts more effort than the attacker ($x_D > x_A$) if and only if $c_D < (3/4)c_A$.

(ii) Each player's effort ($x_i, i = A, D$) is decreasing in their own marginal cost of effort ($c_i$), as well as their rival's marginal cost of effort ($c_j, j \neq i$).

(iii) The probability of each player's success ($p_i, i = A, D$) is decreasing in their own marginal cost of effort ($c_i$) and increasing in the rival's marginal cost of effort ($c_j, j \neq i$).

(iv) The payoff of the attacker is increasing in their own endowment but increasing in defender's endowment only if $2c_D > c_A$. The payoff of the defender is increasing in their own endowment, but independent of the attacker's endowment.

**Proof.** See the Appendix.

The first result in the corollary is of great interest. This shows that it is possible for the defender to exert even more effort than the attacker if he has enough cost advantage. In this case the defender can produce an adequate amount of butter for consumption, but at the same time can also produce enough guns to protect his butter. This provides a more intuitive and arguably a more general support of the observations from the field where defenders indeed exert more conflict effort. Earlier literature in Attack and Defense (without production) has derived such result either due to network externalities (Clark and Konrad, 2007) or due to the loss aversion of the defender (Chowdhury et al., 2018). We show here that while optimizing between consumption and conflict, defender may be more conflictual than the attacker even without loss aversion or externalities. When the attacker has more endowment than the defender, then the 'weaker' defender wins with higher probability – reflecting the 'paradox of power' (Hirshleifer, 1991).

The second result shows that an increase in either own conflict cost or opponent's conflict cost reduces own conflict effort. While the effect of own conflict cost is intuitive, the effect of opponent's cost is counterintuitive, especially in the context of standard contest models (e.g., Baik, 2004). This is because unlike standard contests with no production, an increase in the opponent's cost of conflict provides higher incentives for the opponent to produce more butter. This prompts the player concerned also to exert less conflict effort and produce more butter themselves.

The third result is straightforward. Own (opponent) marginal cost has a negative (positive) effect on own winning. Combining the second and the third results it can be observed that since own costly effort and own probability of winning get opposite effect from marginal costs, the overall effect on payoff is ambiguous.

The fourth result is one of the more interesting ones. It shows that an increase in own endowment always increases own payoff – a result similar to standard contests. However, due to the production – consumption tradeoff in the game, an increase in defender's endowment increases attacker's payoff only when they have a relative cost advantage. Moreover, due to the attack and defense aspect of the game, the endowment of the attacker does not affect the payoff of the defender.

We now aim to investigate the possibility of corner equilibria. We find that a unique corner solution exists in which the attacker spends all her endowment in conflict, whereas the defender allocates only a part of his endowment in conflict. We also find that the interior and the corner equilibria do not coexist. This is summarized in the proposition below.

**Proposition 2.** There exists a unique corner equilibrium characterized by $x_A^* = \frac{E_A}{c_A}$ and $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$, if and only if $\frac{E_D}{4c_D}\left(\sqrt{1+\frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$ and $E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A+2c_D}{\sqrt{c_A+4c_D}} - \sqrt{c_A}\right)$.

**Proof.** See the Appendix.

This result is straightforward and intuitive. When the attacker is constrained by a very low endowment, she decides not to produce butter at all and allocates the entire endowment to conflict. As discussed earlier, this reflects situations such as a group of bandits or mercenaries who live off conflict, or a hacker who earns a living by hacking – p in contrast to an organization that engages in its own work as well as hacking. The defender, however, expends only a portion of his endowment on conflict. Interestingly, when the defender's endowment increases, he increases his production of both guns and butter, but this does not affect the decision of the already constrained attacker. However, unlike in an interior equilibrium, if the attacker's endowment increases within a certain range, it triggers the defender to allocate more resources to producing guns and defending his butter.

The following diagram depicts best responses and the corner equilibrium when $E_A = 1,\ E_D = 8$, $c_A = c_D = 1$. As before, note that (0,0) is not an equilibrium since best responses are discontinuous at zero.

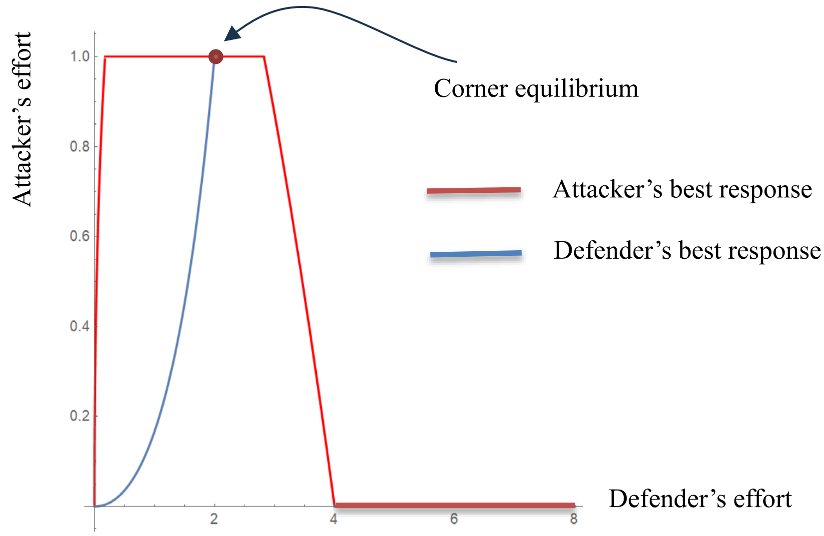


**Figure 2.** Best responses and the corner equilibrium

It is interesting to further investigate which player exerts higher conflict effort, and how the cost structure affects their decisions. These are included in the following corollary.

**Corollary 2.** Consider the unique corner equilibrium described by Proposition 2.

(i) The defender exerts more effort than the attacker if and only if $(E_D/c_D) > 3(E_A/c_A)$.

(ii) Each player's effort is decreasing in their marginal cost of effort. Moreover, the defender's effort is decreasing in the attacker's marginal cost.

(iii) The probability of each player's success is decreasing in own marginal cost of effort and increasing in the opponent's marginal cost of effort.

(iv) Each player's payoff is decreasing in own marginal cost of effort and increasing in the opponent's marginal cost of effort.

**Proof.** See the Appendix.

Note that when the attacker allocates her entire budget to conflict, the defender exerts more conflict effort than the attacker only if his budget (normalized by cost) is very large (at least three times the normalized budget of the attacker). Once again, this result shows that it is possible for the defender to be more conflictive even without loss aversion or network effects. It is interesting that even when the corner equilibrium arises and the attacker's effort is restricted by her low endowment, there is a possibility that the attacker exerts more effort than the defender, as the following example demonstrates. This also shows that part (i) of Corollary 2 is non-trivial.

**Example 1.** Suppose $E_A = 1$, $E_D = 9.5$, $c_A = 1$, and $c_D = 4$, It is easy to check that the condition for the corner equilibrium given in Proposition 2 is satisfied. At the same time, calculations show that $x_A^* = 1$ while $x_D^* = 0.83$.

We now summarize the results in the following theorem.

**Theorem.** In the attacker-defender game $\Gamma(A, D; \Omega)$, there exists a unique interior equilibrium characterized by $x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right)$ and $x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)$, if and only if $\frac{E_D}{4c_D}\left(\sqrt{1+\frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}$; and a unique corner equilibrium characterized by $x_A^* = \frac{E_A}{c_A}$ and $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$, if and only if $\frac{E_D}{4c_D}\left(\sqrt{1+\frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$ and $E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A+2c_D}{\sqrt{c_A+4c_D}} - \sqrt{c_A}\right)$.

**Proof.** Comes directly from Table 1, Proposition 1, and Proposition 2. ■

It can be shown easily that the conditions required for the existence of the interior equilibrium, and the corner equilibrium cannot simultaneously hold. As a result, interior and corner equilibria do not coexist. At the same time, it is possible to find a range of parameters for which neither interior nor corner equilibria exist, as the following example shows.

**Example 2.** Suppose $E_A = 1$ and $E_D = 8$. Fig. 3 depicts the regions in $(c_A, c_D)$ plane where a unique interior and a unique corner equilibrium exists. Note that the two regions do not overlap and for some parameter values, no equilibrium exists (green regions in the diagram).

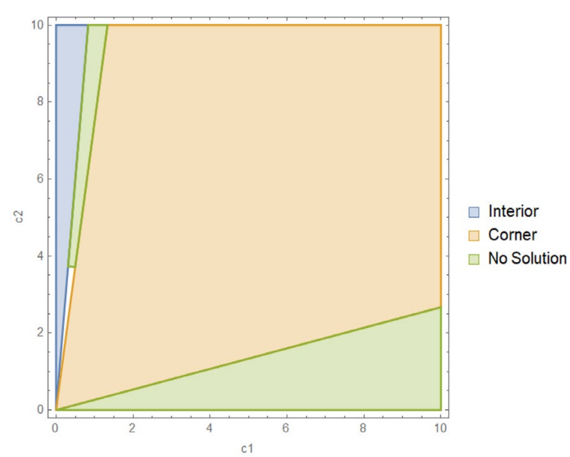


**Figure 3.** Existence of interior and corner equilibria

## 3. Discussion

We analyze a two-player attack and defense model in which both the attacker and the defender can allocate their endowments between production and conflict. The main difference of this model with the existing literature lies in the outcome of the conflict. In the status quo, or if the defender wins, both players consume their own production of butter. However, if the attacker wins, she appropriates all the butter produced in the economy. We show that there are two types of equilibrium. If the attacker is highly budget constrained, then she spends the whole endowment in conflict. In another case, both the attacker and the defender allocate only a part of the endowment in conflict. We find that, depending on the cost of conflict, the defender may be more conflict-prone than the attacker, even without loss aversion or network externalities.

These findings contribute to both the gun and butter and the attack and defense literature by addressing the often-overlooked asymmetry in the objectives and outcomes of conflicts between attackers and defenders. This paper fills a gap in existing research by examining these differences. Moreover, the comparative statics results from this model demonstrate that due to the outcome asymmetry, changes in one player's cost of conflict or endowment can have asymmetric effects on the opponent. These new results provide further understanding in these areas of research.

This paper also contributes to the literature on psychology/neuroscience and economics experiments in attack and defense. As pointed out by Chowdhury (2019), economics literature focuses on theoretical benchmarks to understand behavioral mechanisms, whereas neuroscientists use tools such as fMRI for the same purpose. Moreover, while economics experiments on attack and defense do not consider production, psychology and neuroscience experiments consider consumption but do not follow a rigorous theoretical framework. This paper can be considered as a bridge between these two literatures.

This study can be extended in various ways, both theoretically and in practical applications. A natural extension would be to include more than two players. Group dynamics in the current setting would be another important extension. Considering network externalities would make the structure more realistic. It will also be useful to include social preference aspects such as spite or retaliation. All these aspects as well as the original model results can also be investigated in a laboratory setting. We defer these ideas for exploration in future research endeavors.

## References


Aloni, E., & Sela, A. (2012). The assortative matching scheme in a survival battle. *Economics Letters*, 117(1), 272-275.

Amir, R. (2005). Supermodularity and complementarity in economics: An elementary survey. *Southern Economic Journal*, 71(3), 636-660.

Arce, D. G., Kovenock, D., & Roberson, B. (2012). Weakest-link attacker-defender games with multiple attack technologies. *Naval Research Logistics (NRL)*, 59(6), 457-469.

Baik, K. H. (2004). Two-player asymmetric contests with ratio-form contest success functions. *Economic Inquiry*, 42(4), 679-689.

Bester, H., & Konrad, K. A. (2004). Delay in contests. *European Economic Review*, 48(5), 1169-1178.

Bose, G., & Konrad, K. A. (2020). Devil take the hindmost: Deflecting attacks to other defender. *Reliability Engineering & System Safety*, *204*, 107205.

Caruso, R. (2010). Butter, Guns, and Ice cream: Theory and Evidence from Sub-Saharan Africa. *Defence and Peace Economics*, 21(3), 269-283.

Chowdhury, S. M. (2019). The Attack and Defense Mechanisms – Perspectives from Behavioral Economics and Game Theory, *Behavioral and Brain Sciences*, 42, e121.

Chowdhury, S. M., Jeon, J. Y., & Ramalingam, A. (2018). Property rights and loss aversion in contests. *Economic Inquiry*, 56(3), 1492-1511.

Chowdhury, S. M., & Topolyan, I. (2016a). Best-shot versus weakest-link in political lobbying: an application of group all-pay auction. *Social Choice and Welfare*, 47(4), 959-971.

Chowdhury, S. M., & Topolyan, I. (2016b). The Attack-And-Defense Group Contests: Best Shot Versus Weakest Link. *Economic Inquiry*, 54(1), 548-557.

Clark, D. J., & Konrad, K. A. (2007). Asymmetric conflict: weakest link against best shot. *Journal of Conflict Resolution*, 51(3), 457-469.

De Dreu, C. K., Giacomantonio, M., Giffin, M. R., & Vecchiato, G. (2019). Psychological constraints on aggressive predation in economic contests. *Journal of Experimental Psychology: General*, 148(10), 1767.

De Dreu, C. K., & Gross, J. (2019). Revisiting the Form and Function of Conflict: Neurobiological, Psychological and Cultural Mechanisms for Attack and Defense Within and Between Groups. *Behavioral and Brain Sciences*, 42.

Dechenaux, E., Kovenock, D., & Sheremeta, R. M. (2015). A survey of experimental research on contests, all-pay auctions and tournaments. *Experimental Economics*, 18(4), 609-669.

Deck, C., & Sheremeta, R. M. (2012). Fight or flight? Defending against sequential attacks in the game of siege. *Journal of Conflict Resolution*, 56(6), 1069-1088.

Durham, Y., Hirshleifer, J., & Smith, V.L. (1998). Do the Rich Get Richer and the Poor Poorer? Experimental Tests of a Model of Power. *American Economic Review*, 88 (4), 970-83.

Dziubiński, M., & Goyal, S. (2013). Network design and defence. *Games and Economic Behavior*, 79, 30-43.

Goyal, S., & Vigier, A. (2014). Attack, defence, and contagion in networks. *Review of Economic Studies*, 81(4), 1518-1542.

Grossman, H. I., & Kim, M. (1995). Swords or plowshares? A theory of the security of claims to property. *Journal of Political Economy*, 103(6), 1275-1288.

Hafer, C. (2006). On the origins of property rights: Conflict and production in the state of nature. *Review of Economic Studies*, 73(1), 119-143.

Hausken, K. (2005). Production and conflict models versus rent-seeking models. *Public Choice*, 123(1-2), 59-93.

Hirshleifer, J. (1991). The technology of conflict as an economic activity. *American Economic Review*, 81(2), 130-134.

Hirshleifer, J. (1995). Theorizing about conflict. In K. Hartley and T. Sandler (Eds.) *Handbook of Defense Economics*, 1 (pp. 165-189). North Holland.

Hirshleifer, J. (2000). The macrotechnology of conflict. *Journal of Conflict Resolution*, 44(6), 773-792.

Horiuchi, Y., & Mayerson, A. (2015). The opportunity cost of conflict: Statistically comparing Israel and synthetic Israel. *Political Science Research and Methods*, 3(3), 609-618.

Kolmar, M. (2008). Perfectly secure property rights and production inefficiencies in Tullock contests. *Southern Economic Journal*, 441-456.

Kornienko, T. (2020). Expropriation Auctions: Predation Contests and Wallet Games, Mimeo.

Kovenock, D., & Roberson, B. (2012). Conflicts with multiple battlefields. In M.R. Garfinkel, & S. Skaperdas (Eds.), *The Oxford Handbook of the Economics of Peace and Conflict* (pp. 503 – 531). Oxford University Press, New York.

Kovenock, D., & Roberson, B. (2018). The optimal defense of networks of targets. *Economic*

*Inquiry*, 56(4), 2195-2211.

Kovenock, D., Roberson, B., & Sheremeta, R. M. (2019). The attack and defense of weakest-link networks. *Public Choice*, 179(3-4), 175-194.

Mahadevan, P. (2012). The Maoist insurgency in India: between crime and revolution. *Small Wars & Insurgencies*, 23(2), 203-220.

Mill, J.S. (1848). *Principles of political economy*. Oxford University Press, Oxford, UK:.

Neary, H. M. (1997). Equilibrium structure in an economic model of conflict. *Economic Inquiry*, 35(3), 480-494.

Noh, S. J. (2002). Production, appropriation, and income transfer. *Economic Inquiry*, 40(2), 279-287.

Rubiano A., M. P. (2021). How to Feed 10,000 Rebel Fighters for 50 Years. *Gastro Obscura*. Available at: https://www.atlasobscura.com/articles/what-farc-ate-colombia

Samuelson, P. (1948). Military expenditure and capitalism, in *Economics: An Introductory Analysis*, 1, 61-76. McGraw-Hill, New York.

Skaperdas, S. (1992). Cooperation, conflict, and power in the absence of property rights. *American Economic Review*, 720-739.

Tullock, G. (1980). Efficient Rent Seeking. In J.M. Buchanan, R.D. Tollison, & G. Tullock, (Eds.), *Toward a Theory of the Rent-Seeking Society* (pp. 97-112). Texas A&M University Press, College Station, TX.

Yektaş, H., Hoffmann, M., Hentschel, F., & Hodler, R. (2019). Wars of conquest and independence. *Journal of Institutional and Theoretical Economics*, 175(4), 617-640.

# Appendix

## Proof of Proposition 1.

The first-order necessary conditions for an interior equilibrium are:

$$\frac{\partial u_A}{\partial x_A} = \frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} - c_A = 0, \tag{6}$$

$$\frac{\partial u_D}{\partial x_D} = \frac{x_A E_D - x_D c_D(2x_A + x_D)}{(x_A + x_D)^2} = 0. \tag{7}$$

Since the payoff functions are strictly concave, the above first-order necessary conditions are also sufficient. When $x_A > 0$, Eq. (7) implies that

$$x_A = \frac{c_D x_D^2}{E_D - 2c_D x_D}. \tag{8}$$

Eq. (6) further shows that an interior equilibrium exists only if $x_D < \frac{E_D}{c_A + c_D}$. Under that condition, $\frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} = c_A$. This, combined with Eq. (8), implies:

$$c_A\left(\frac{c_D x_D^2}{E_D - 2c_D x_D} + x_D\right)^2 = x_D(E_D - c_D x_D). \tag{9}$$

Eq. (9) has four solutions, $x_D = 0$, $x_D = \frac{E_D}{c_D}$, and $x_D = \frac{E_D}{2c_D}\left(1 \pm \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$. The first two are not consistent with an interior equilibrium (the second one does not satisfy $x_D < \frac{E_D}{c_A + c_D}$). Note that $x_D = \frac{E_D}{2c_D}\left(1 + \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$ implies $E_D - 2c_D x_D < 0$, and hence $x_A < 0$, which is impossible. Thus, the only candidate for an interior equilibrium is $x_D = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$, which is an equilibrium only if $\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right) < \frac{E_D}{c_A + c_D}$. It can be verified that this inequality is always satisfied.

Then $x_A = \frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right)$. One can verify that $x_A > 0$ for all $c_A, c_D > 0$.[2] We thus have a unique candidate for an interior equilibrium:

[2] Note: $c_A + 4c_D = 2c_D + (c_A + c_A + 4c_D)/2$ and the geometric mean of two distinct (positive) numbers (in our case, $\sqrt{c_A(c_A + 4c_D)}$ is the geometric mean of $c_A$ and $c_A + 4c_D$) is always strictly less than their arithmetic mean.

$$x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right) \tag{10}$$

$$x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right). \tag{11}$$

To ensure that it is indeed an interior equilibrium, we need to verify that $x_i^* < E_i/c_i$. Earlier we found that $x_D^* < E_D/(c_A + c_D)$, hence the defender's budget constraint is never binding in Eq. (11). Turning to the attacker's budget constraint, we need to ensure that

$$\frac{E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right) < \frac{E_A}{c_A}. \tag{12}$$

Equivalently,

$$\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}. \tag{13}$$

Hence, Eq. (10) – (13) fully characterize the interior equilibrium. ■

## Proof of Corollary 1.

**(i)** To investigate who exerts a higher effort - the attacker or the defender, note from Eq. (10) and Eq. (11) that

$$x_A^* - x_D^* = \frac{E_D}{4c_D\sqrt{c_A(c_A+4c_D)}}\left(3c_A + 4c_D - 3\sqrt{c_A(c_A + 4c_D)}\right).$$

This difference is positive if and only if $c_D > (3/4)c_A$, which tells us that the attacker exerts more effort than the defender if and only if her cost is sufficiently lower than that of the defender's. ■

**(ii)** $\frac{\partial x_A^*}{\partial c_A} = -\frac{E_D}{2c_A^{3/2}(c_A+4c_D)^{1/2}} < 0$ for all $c_A, c_D > 0$.

$\frac{\partial x_D^*}{\partial c_D} = \frac{E_D}{2c_D^2}\left(-1 + \sqrt{\frac{c_A}{c_A+4c_D}} \cdot \frac{c_A+6c_D}{c_A+4c_D}\right) < 0$ for all $c_A, c_D > 0$.

As for the cross effects:

$\frac{\partial x_A^*}{\partial c_D} = -\frac{E_D}{4c_D^2} \cdot \frac{(c_A+2c_D)}{\sqrt{c_A(c_A+4c_D)}} < 0$ for all $c_A, c_D > 0$.

$\frac{\partial x_D^*}{\partial c_A} = -\frac{E_D}{\sqrt{c_A}(c_A+4c_D)^{3/2}} < 0$ for all $c_A, c_D > 0$. ■

**(iii)** Plugging Eq. (10) and Eq. (11) into $p_A = x_A/(x_A + x_D)$ and simplifying yields:

$$p_A = \frac{\sqrt{c_A+4c_D}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}.$$

This implies $\frac{\partial p_A}{\partial c_A} = -\frac{4c_D}{(c_A+4c_D)\left(\sqrt{c_A+4c_D}+2\sqrt{c_A}\right)^2\sqrt{c_A}} < 0$ and $\frac{\partial p_A}{\partial c_D} = \frac{4\sqrt{c_A}}{\left(\sqrt{c_A+4c_D}+2\sqrt{c_A}\right)^2\sqrt{c_A+4c_D}} > 0$.

Since $p_D = 1 - p_A$, we conclude that $\frac{\partial p_D}{\partial c_A} > 0$ and $\frac{\partial p_D}{\partial c_D} > 0$. ■

**(iv)** $u_A = (E_D - c_D x_D)\frac{x_A}{x_A+x_D} + (E_A - c_A x_A) = \left[E_D - c_D\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)\right]\frac{\sqrt{c_A+4c_D}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}} +$
$\left[E_A - \frac{c_A E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right)\right]$

Or, $u_A = \frac{E_D}{2}\left[\frac{\sqrt{c_A+4c_D}+\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}\right] + \frac{1}{4c_D}\left[4c_D E_A - c_A E_D\left(\frac{(c_A+4c_D)-\sqrt{c_A(c_A+4c_D)}}{\sqrt{c_A(c_A+4c_D)}}\right)\right]$

Let $\sqrt{c_A + 4c_D} = m$ and $\sqrt{c_A} = n$

then $u_A = \frac{E_D}{2}\left[1 - \frac{n}{m+2n}\right] + \left[E_A - E_D\frac{n}{(m+n)}\right]$

Hence, $\frac{\partial u_A}{\partial E_A} = 1 > 0$ (an increase in attacker resources always increases attacker's payoff).

And $\frac{\partial u_A}{\partial E_D} = \frac{1}{2} - \frac{n}{2(m+2n)} - \frac{n}{(m+n)} = \frac{c_A+4C_D-3C_A}{2(m+2n)(m+n)}$

Hence, $\frac{\partial u_A}{\partial E_D} > 0$ if $2c_D > c_A$. So, an increase in defender resources increases the payoff of attacker – only if the attacker cost is relatively low enough.

Similarly, $u_D = (E_D - c_D x_D)\frac{x_D}{x_A+x_D}$

Or, $u_D = \left(E_D - c_D\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)\right)\frac{2\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}$

Or, $u_D = \left(\frac{\sqrt{c_A+4c_D}+\sqrt{c_A}}{2\sqrt{c_A+4c_D}}\right)\frac{2E_D\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}$

As $\sqrt{c_A + 4c_D} = m$ and $\sqrt{c_A} = n$ then $u_D = \frac{n(m+n)}{m(m+2n)}E_D$

It is straightforward that $\frac{\partial u_D}{\partial E_D} > 0$ and $\frac{\partial u_D}{\partial E_A} = 0$. ■

## Proof of Proposition 2.

From Eq. (6) and Eq. (7), it is easy to see that the equilibria such that $(x_A = 0, x_D > 0)$ or $(x_A > 0, x_D = 0)$ do not exist, provided that $c_A, c_D > 0$. Thus, the only candidate is $x_A = x_D = 0$. In this case, the attack succeeds with probability 1/2. But then Player A improves the payoff by exerting a very small yet positive effort level. Thus, $x_A = x_D = 0$ is not an equilibrium either.

The only remaining candidates for corner equilibria are profiles with $x_A = E_A/c_A$ or $x_D = E_D/c_D$. Earlier we established that when $x_D = E_D/c_D$, the attacker's best response is $x_A^* = 0$. But this cannot be sustained in equilibrium since the defender is better-off reducing the defense effort.

Suppose that $x_A = E_A/c_A$, then we can solve for the defender's best response from Eq. (7) and obtain $x_D^* = min\left\{-\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}, \frac{E_D}{c_D}\right\}$.

The above formula reflects the fact that the defender's budget constraint may or may not be binding. It is straightforward to check that $-\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}} < \frac{E_D}{c_D}$. So, in fact, the defender's best response is always in the interior of the strategy space. Thus,

$$x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}. \tag{14}$$

It remains to check whether $x_A^* = E_A/c_A$ is the attacker's best response against $x_D^*$. This is the case if and only if $\frac{\partial u_A}{\partial x_A}(x_A^*, x_D^*) \geq 0$. Substituting $x_A^*$ and $x_D^*$ into Eq. (4), yields

$$E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A + 2c_D}{\sqrt{c_A + 4c_D}} - \sqrt{c_A}\right). \tag{15}$$

Eq. (15), in conjunction with $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$, the latter ruling out an interior equilibrium, represents the necessary and sufficient condition for the existence of a corner equilibrium. ■

## Proof of Corollary 2.

**(i)** Note from Proposition 2, and after some manipulation that: $x_D^* - x_A^* = \frac{E_A}{c_A}\left(\frac{E_D}{c_D} - 3\frac{E_A}{c_A}\right)$. Hence, $x_D^* > x_A^*$ if and only if $(E_D/c_D) > 3(E_A/c_A)$. ■

**(ii)** It is easy to see from Proposition 2 that in the corner equilibrium, provided that it exists, $\frac{\partial x_A^*}{\partial c_A} = -\frac{E_A}{c_A^2} < 0$ and $\frac{\partial x_D^*}{\partial c_D} = -\frac{E_A E_D}{c_A c_D^2} \cdot \frac{1}{2\left(\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}\right)^{0.5}} < 0$.

Furthermore, numerical methods show that $\frac{\partial x_D^*}{\partial c_A} < 0$. ■

**(iii)** Note that the probability of the attack's success is given by $p_A = \frac{E_A}{\sqrt{E_A^2 + \frac{c_A E_A E_D}{c_D}}}$.

Using numerical methods, one can show that $\frac{\partial p_A}{\partial c_A} < 0$ and $\frac{\partial p_A}{\partial c_D} > 0$. Using the identity $p_D = 1 - p_A$, we conclude that $\frac{\partial p_D}{\partial c_A} > 0$ and $\frac{\partial p_D}{\partial c_D} > 0$. ■

**(iv)** $x_A^* = \frac{E_A}{c_A}$ and $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$

Let $\frac{E_A}{c_A} = m$ and $\frac{E_D}{c_D} = n$, then $p_A = \frac{\sqrt{m}}{\sqrt{m+n}}$ and $p_D = \frac{\sqrt{m+n}-\sqrt{m}}{\sqrt{m+n}}$

$$u_A = (E_D - c_D x_D)\frac{x_A}{x_A + x_D} + (E_A - c_A x_A) = c_D\sqrt{m}\left(\sqrt{m+n} - \sqrt{m}\right)$$

$$\frac{\partial u_A}{\partial c_A} = c_D\left[\frac{1}{2\sqrt{m}}\left(-\frac{E_A}{c_A^2}\right)\left(\sqrt{m+n} - \sqrt{m}\right) + \sqrt{m}\left(\frac{1}{2\sqrt{m+n}}\left(-\frac{E_A}{c_A^2}\right) - \frac{1}{2\sqrt{m}}\left(-\frac{E_A}{c_A^2}\right)\right)\right]$$

$$= -\frac{E_A}{c_A^2}\frac{c_D}{2}\left[\frac{\sqrt{m+n} - \sqrt{m}}{\sqrt{m+n}} + \frac{\sqrt{m}}{\sqrt{m+n}} - 1\right]$$

$$\frac{\partial u_A}{\partial c_A} = -\frac{E_A c_D}{c_A^2}\left[\frac{\left(\frac{m}{2} + \frac{m+n}{2}\right) - \sqrt{m(m+n)}}{\sqrt{m(m+n)}}\right] < 0$$

The above inequality holds because the geometric mean of two distinct (positive) numbers (in our case, $\sqrt{m(m+n)}$ is the geometric mean of $m$ and $m+n$) is always strictly less than their arithmetic mean.

$$\frac{\partial u_A}{\partial c_D} = \sqrt{m}\left[\sqrt{m+n} - \sqrt{m} + \frac{c_D}{2\sqrt{m+n}}\left(-\frac{E_D}{c_D^2}\right)\right] = \sqrt{m}\left[\sqrt{m+n} - \sqrt{m} - \frac{n}{2\sqrt{m+n}}\right]$$

$$= \frac{\left(\frac{m}{2} + \frac{m+n}{2}\right) - \sqrt{m(m+n)}}{\sqrt{m+n}} > 0$$

The above inequality holds again, because arithmetic mean is greater than the geometric mean.

$$u_D = (E_D - c_D x_D)\frac{x_D}{x_A + x_D} = c_D\left(\sqrt{m+n} - \sqrt{m}\right)^2$$

$$\frac{\partial u_D}{\partial c_A} = 2c_D\left(\sqrt{m+n} - \sqrt{m}\right)\left(-\frac{E_A}{c_A^2}\right)\left[\frac{1}{2\sqrt{m+n}} - \frac{1}{2\sqrt{m}}\right] > 0$$

$$\frac{\partial u_D}{\partial c_D} = \left(\sqrt{m+n} - \sqrt{m}\right)^2 + c_D\left(\sqrt{m+n} - \sqrt{m}\right)\frac{1}{2\sqrt{m+n}}\left(-\frac{E_D}{c_D^2}\right)$$

$$= \left(\sqrt{m+n} - \sqrt{m}\right)\left[\sqrt{m+n} - \sqrt{m} - \frac{n}{\sqrt{m+n}}\right]$$

$\frac{\partial u_D}{\partial c_D} = \left(\sqrt{m+n} - \sqrt{m}\right)\frac{m - \sqrt{m(m+n)}}{\sqrt{m+n}} < 0.$ ∎