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# The Central Influencer Theorem: Spatial Voting Contests with Endogenous Coalition Formation *

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## Abstract

We introduce a spatial voting contest without the 'one person, one vote' restriction. Players exert costly effort to influence the policy and the outcome is obtained through an adjustment function. Players are heterogeneous in terms of the position in the policy line, disutility function, and the effort cost. In equilibrium, two groups endogenously emerge: players in one group try to implement more leftist policy, while those in the other group try more rightist one. Since the larger group suffers a more severe free-riding problem, the equilibrium policy converges to the center only when the larger group has a cost advantage. We demonstrate how the location of the center (i.e., the steady-state point) can be either median, or a mean of all points, or a mean of the extreme points, depending on the convexities of the utility and cost functions. This reflects some well-known results as special cases. We extend the model to an infinite horizon setting and show that the median outcome can be reached only under certain conditions.

JEL Classifications: C72; D72; D74; D78
Keywords: Spatial Competition; Contest; Lobbying; Median Voter Theorem

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## 1. Introduction

Policies within the government, organizations, committees, and similar entities are often selected and put into action through the process of (majoritarian) voting. Typically, a prevailing policy is in place, and the players engaged in the process possess individual preferences regarding the nature of a new policy while participating in the vote for policy alteration. This fundamental framework can be adapted to other communal decision-making procedures such as elections, lobbying, collective bargaining etc., and has been employed extensively in both theoretical and applied contexts within the realms of Economics and Political Science.

In the context of a policy being evaluated along a single dimension, this framework yields the well-known 'median voter' outcome (Black, 1948; Downs, 1957). To elaborate, players are positioned along a linear policy spectrum based on their favored policy position. These players experience disutility if the policy deviates from their preferred point, and they cast a single possible vote accordingly. Within this framework, numerous studies have demonstrated that the policy outcome ultimately mirrors the preference of the voter positioned at the median.

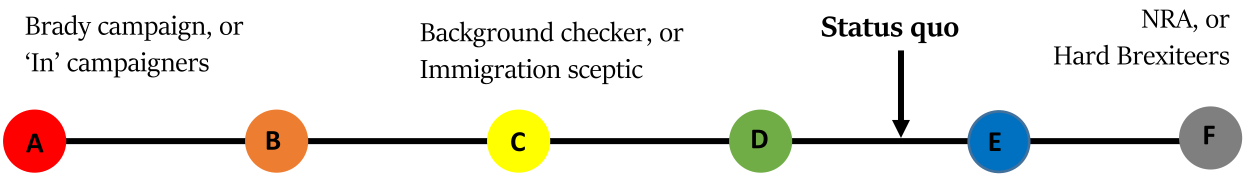
It's important, though, to acknowledge that numerous real-world scenarios surpass such straightforward frameworks, even within the context of a policy space characterized by a single dimension. To begin with, committees or members of the public frequently partake in policy contests, expending valuable resources to impact policies. In such instances, the conventional principle of one person, one vote does not hold. Take, for instance, scenarios involving rent-seeking, electoral expenditures, or committee lobbying, where the process of influence encompasses broader dynamics and incurs associated influence costs.

Furthermore, when it comes to implementing one's preferred policy or exerting influence on the existing status quo, players often form alliances or coalitions endogenously with other 'like-minded' participants. Lastly, within these coalitions, some members may opportunistically rely on the efforts of others to advance their shared interests in shaping more favorable policies.

In this study, we introduce a game-theoretic model that integrates the aforementioned elements within both static and dynamic frameworks and derive the resulting equilibria. As we demonstrate later, the model turns out to be an amalgamation of a spatial voting model and a collective contest model. Within this setting, players hold optimal preferences along a given line, characterized by a disutility function: the greater the divergence of the implemented policy from their preference, the higher their dissatisfaction.

The initial status quo policy is exogenously determined (rooted in historical context), yet players have the ability to exert efforts to influence and realign the policy. These efforts come at a cost and adhere to a known cost function. If players on the left (right) side of the initial policy collectively expend more effort, the ultimate policy outcome is more likely to move toward the left (right) side of the initial policy. Consequently, this process prompts players to naturally form coalitions and potentially exploit the efforts of fellow coalition members. An illustrative example of this scenario is provided in Figure 1 below.

**Figure 1.** Illustrative example




Suppose the line above illustrates the policy dimension. To illustrate the model structure, we offer two examples: first, consider the policy domain as gun control within the USA. In this context, a player situated at the far right extreme (designated as circle F) would symbolize the position of the National Rifle Association (NRA), making costly investment in advocating for the promotion and advancement of rifle shooting. Conversely, a player positioned at the far left extreme (circle A) corresponds to the Brady campaign, campaigning for stringent gun control measures to curb firearm-related violence. A player positioned in the middle (e.g., circle C) represents someone favoring gun ownership while advocating for rigorous background checks.

Likewise, let us apply this notion to the scenario of Brexit in the UK. In this case, circles F, A, and C would respectively represent the positions of the 'hard Brexiteers,' proponents of a firm break from the European Union; the 'Better Together' campaigners, advocating for continued EU membership; and the immigration skeptical 'In' campaigners who support a softer Brexit stance. Naturally, there are additional players (situated at B, D, E, etc.) in both examples, each holding distinct preferences and stances.

Continuing with the illustration, let us consider the black arrow as denoting the existing status quo. Since it is on the right side of Player C, in this case, player C aims for the policy to shift towards the left. It's important to note several intriguing aspects that set this structure apart from conventional voting models. First, each of the players can be different from each other not only in terms of the position in the policy line, but also in terms of how eager they are in shifting the status quo, and how cost effective they are in their effort to influence the policy. Second, although player C seeks a leftward policy shift, there exists a constraint on the extent of this shift; in other words, they do not want the policy to shift too far to the left (for instance, to point A). Third, player C forms an implicit alliance with players A, B, and D, as these players also share the preference for a leftward policy movement relative to the status quo. Finally, an interesting dynamic emerges: if players A, B, and D invest their efforts to facilitate the policy shift, it becomes feasible for player C to capitalize on their endeavors and abstain from expending effort, which is inherently costly. This interplay of preferences, constraints, and strategic considerations in contest distinguishes this model from standard voting frameworks.

Players decide on the level and direction of costly effort to expend to shift the status quo. To model this, we employ an adjustment function that incorporates everyone’s efforts. Within this framework, players may exhibit heterogeneity in three distinct aspects: position in the policy line, effort cost function, and disutility function. We find that in equilibrium, two groups endogenously emerge: players in one group try to implement more leftist policy (i.e., they direct their effort to shift the policy to the left side), while those in the other group more rightist one. In general, the equilibrium policy converges to the ‘center’ if the larger of the groups has a sufficient cost advantage. However, the equilibrium policy may not converge towards the center if there is no such an advantage since the larger group suffers from a more severe free rider problem. Notably, the 'center' need not correspond to the median, as typically seen in conventional voting literature. Depending on the attributes of the effort cost function and the disutility function, alternative measures of centrality can be obtained.

We define steady-state equilibrium as the situation in which the status quo and the implemented policy coincide. Our findings reveal interesting outcomes based on the characteristics of the disutility function and that of the effort cost function. We find that if both the disutility of non-optimal policy and the effort cost function are linear, any policy can be a steady state policy. This is the case where there is no cost (dis)advantage. However, if the disutility of non-optimal policy is linear and the cost function is convex, then a steady-state policy is the optimal policy for a median player. On the contrary, if the disutility of non-optimal policy is concave (that is, the distance measure is convex) and the cost function is linear, then the steady-state policy is the mean of the two extreme players' optimal policies. Although no heterogeneity in terms of cost exists, all players except the two extreme ones engage in free riding in the equilibrium. Finally, if the distance measure and the cost function are equally convex, the steady-state policy is the (weighted) average of all players’ ideal points.

We then extend these analyses to a three-player dynamic setting with a logit-type adjustment function (a la Tullock (1980) contest success function). In an infinite horizon model, there exists an equilibrium in which the policy outcome converges again to the median player. Players expend more effort in each period as they become more patient (or forward-looking), but the convergence speed does not depend on the discount factor.

Our results contribute to the spatial voting literature, revealing an intriguing nuance: although a central measure turns out to be the optimal policy, it does not necessarily have to be the median. This aligns with the research by Krasa and Polborn (2010), albeit in a distinct context involving competition between diverse candidates across two policy areas. Their work also highlights a central tendency that diverges from the conventional median voter outcome. Expanding on this trajectory, our study introduces the novel concept of contest into the spatial voting literature.

In contrast to the assumptions of Downs (1957) and subsequent studies (e.g., Palfrey and Rosenthal, 1983; Becker, 1983; Sengupta and Sengupta, 2008, etc.), which posit equal influence for each player (one vote), our study embraces a more generic framework. We allow individuals to allocate varying resources, leading to the endogenous determination of an individual's level of influence.[1] This approach supports and complements the results in Hanson and Stuart (1984) found in a different set up. In generalization, Baron (1996) implements a collective goods problem through dynamic voting. Our analysis shares similarities with this approach, albeit showcasing a more extensive array of outcomes and implications.

We also contribute to the contest literature (Konrad, 2009) – especially in the areas of collective (group) contests and endogenous coalition formation. Traditionally, group contests entail pre-defined player groupings with fixed prize values, and an externally specified group impact (or production) function. In contrast, we consider an additive impact function (akin to Katz et al., 1990) but endogenize both the group size and the prize value. It's important to note that the contest literature generally lacks a specific focus on policy implementation directions. Moreover, the rules governing coalition formation, which dictate how prizes are distributed among group members, are often imposed arbitrarily (see the references in Balart et al., 2017). In contrast, our work tackles both these aspects by focusing on a linear policy dimension and offering an endogenous determination of each group member's potential share. Epstein and

[1] Lalley and Weyl (2018) propose quadratic voting as a method for binary collective decision making. Our model differs in that it doesn't presuppose the existence of two predetermined alternatives. Additionally, Xefteris and Ziros (2017) introduce vote trading under incomplete information, whereas we consider complete information.

Nitzan (2004) introduce a two-stage game. In the first stage the "groups decide which policy to lobby for and then, in a second stage, engage in a contest over the proposed policies". Hence, they endogenize the 'target' but not the prize. We extend it by also endogenizing the groups, but the preference points in our model remain fixed. Similarly, Hirata and Kamada (2020) introduce a two-stage game of policy choice and donation and find that only the extreme donors donate. We consider a single stage model and find the centrality of extreme players as an equilibrium in a special case. Finally, Baik (2017) considers a multi-players contest on a prize that is public good for some and public bad for some others, but only the prize spread matters in equilibrium that involves free riding. In contrast to this model, we endogenize the prize as well as the group formation.

Lastly, we also contribute to the emerging literature on contests with networks (Franke and Öztürk, 2015).[2] For examples, the Tug-of-war games, as characterized by Konrad and Kovenock (2005, 2009) and Agastya and McAfee (2006), involve a two-player scenario where policy shifts occur sequentially until a predetermined point is reached. Our model takes this concept further by accommodating multiple players and introducing an endogenously determined final point. Moreover, Duggan and Gao (2020) analyze a multi-dimensional tug-of-war model in which risk averseness results in Rawlsian equilibrium and risk loving equilibrium is the arithmetic mean of players' ideal points. In contrast, we implement a single dimension policy and risk neutrality but consider groups of contestants.

In conclusion, our present study bridges the spatial voting model and the collective contests model, making contributions to both areas of literature. The subsequent sections of this paper outline the details of our model and the ensuing analyses. In Section 2, we present the model itself, while Section 3 and Section 4 report the results from the static and the dynamic analyses, respectively. Section 5 concludes.

## 2. Model

Consider $N \geq 2$ players who spend effort to implement an individually more desirable policy. The policy space is bounded, continuous and one-dimensional. Such a space can be represented as a unit interval $[0,1]$. Let $x_i$ be the effort exerted by player $i$, $d_i \in \{-1,1\}$ be the direction toward which the effort is put, and $y_i \in [0,1]$ be the ideal point of player $i$. For expositional

[2] Our study diverges from the concept of spatial contests (Konrad, 2000), where the focus is on firms competing for positions within a differentiated market, aiming to secure advantageous locations. In contrast, this study investigates the dynamics of policy formation and decision-making about factors such as player preferences, policy adjustments, and coalition formations.

simplicity, we assume that all players are distinct in terms of their ideal points, but the distance between them can be arbitrary.[3] More specifically and without loss of generality, we let $0 = y_1 < y_2 < \cdots < y_N = 1$. Player $i$ decides on the pair $(x_i, d_i)$ to maximize:

$$u_i(\delta, x_i) = -\alpha_i \| \delta - y_i \| - c_i(x_i) \tag{1}$$

where $\alpha_i$ is a parameter capturing the sensitivity of the player to a change in policy, and the cost of effort $c_i(x_i)$ is $\mu_i(x^\gamma/\gamma)$ where $\gamma \geq 1$. We further assume the distance measure $||.||$ to have the same functional form as the cost function, i.e., $\| \delta - y_i \| = (|\delta - y_i|)^\lambda / \lambda$ where $\lambda \geq 1$. We find these assumptions about functional forms highly valuable to sharpen theoretical predictions. We present the results that do not rely on these functional form assumptions in Subsection 4.2.

Also, we mostly restrict our focus on symmetric players, i.e., for all $i$ and $j$, $\alpha_i = \alpha_j = \alpha$ and $\mu_i = \mu_j = 1$. The consequences of relaxing this symmetry assumption are discussed in Subsection 3.3.

The implemented policy is determined according to an 'adjustment rule', which is defined as follows. Let $\boldsymbol{x}$ and $\boldsymbol{d}$ denote the vectors of effort and directions, respectively. Then, given $(\boldsymbol{x}, \boldsymbol{d})$, the implemented policy is:

$$\delta(\boldsymbol{x}, \boldsymbol{d}) = S + p(\boldsymbol{x}, \boldsymbol{d})$$

where $S$ is the status quo or the default policy, and $p(\boldsymbol{x}, \boldsymbol{d})$ is the adjustment. In other words, the newly implemented policy is the status quo policy adjusted by the aggregated efforts. Our model allows $S$ to be different from previously implemented policy. But when discussing the dynamics and steady state equilibrium, we assume that the status quo at period $t$ is the implemented policy at period $t - 1$.

The adjustment function $p(\boldsymbol{x}, \boldsymbol{d}) \in [-1/2, 1/2]$ has the following form:

$$p(\boldsymbol{x}, \boldsymbol{d}) = p\left(\sum\nolimits_{j \in \hat{L}} x_j, \sum\nolimits_{j \in \hat{R}} x_j\right)$$

[3] If two or more players share the same ideal point (i.e., $y_i = y_j$ for some $i \neq j$), there may exist multiple equilibria in which those players free ride on each other's effort in various ways. We ignore these cases because they would make the exposition significantly messier without adding any interesting insights.

where $\hat{L}$ denotes the set of players who push the policy to the left, i.e., $\hat{L} = \{j|d_j = -1\}$ and similarly, $\hat{R} = \{j|d_j = 1\}$, and has the following properties:

i. $sgn[\partial p(\boldsymbol{x}, \boldsymbol{d})/\partial x_i] = sgn[d_i]$ and $sgn[\partial^2 p(\boldsymbol{x}, \boldsymbol{d})/\partial x_i^2] = -sgn[d_i]$.

ii. If $\sum_{j=1}^{N} d_j x_j = 0$, then $p(\boldsymbol{x}, \boldsymbol{d}) = 0$.

iii. If $\sum_{j=1}^{N} d_j x_j = 0$, then $\frac{\partial p(\boldsymbol{x},\boldsymbol{d})}{\partial x_i} = -\partial p(\boldsymbol{x}, \boldsymbol{d})/\partial x_j$ for $i \in \hat{L}$ and $j \in \hat{R}$.

The first is the usual assumption that the function is increasing and concave in effort. It looks different from the usual one because the objective of the players in $\hat{L}$ is to reduce $p(\boldsymbol{x}, \boldsymbol{d})$. The second assumption states that if the level of efforts put forward the opposite directions are identical (i.e., $\sum_{j\in\hat{L}} x_j = \sum_{j\in\hat{R}} x_j$)), then the default policy is implemented. It also implies that if nobody exerts a positive effort, then $\delta(\boldsymbol{x}, \boldsymbol{d}) = \mathrm{S}$. The last assumption is that $p(\boldsymbol{x}, \boldsymbol{d})$ is symmetric.[4] More specifically, at a symmetric point (i.e., when $\sum_{j\in\hat{L}} x_j = \sum_{j\in\hat{R}} x_j$), the marginal change of the policy is also symmetric. Our leading example for $p(\boldsymbol{x}, \boldsymbol{d})$ is a 'Contest Success Function (CSF)' in the spirit of Tullock (1980):

$$p(\boldsymbol{x}, \boldsymbol{d}) = \begin{cases} \frac{\sum_{j=1}^{N} d_j x_j}{2\sum_{j=1}^{N} x_j} & if\ \sum_{j=1}^{N} x_j > 0 \\ 0 & \text{Otherwise} \end{cases} \tag{2}$$

From this, it is clear that out model is a collective rent-seeking game (similar to Katz et al., 1990), which is played on a single-dimensional policy space. Another example is a linear function: $p(\boldsymbol{x}, \boldsymbol{d}) = \eta\left(\sum_{j=1}^{N} d_j x_j\right)$ for some positive but small $\eta$.

The ideal points of all players $\{y_i\}_{i=1}^{N}$ and the status quo policy $S$ are common knowledge. All players decide the effort level and the direction $\{x_i, d_i\}_{i=1}^{N}$ independently and simultaneously. An equilibrium is vectors of efforts and directions $(\boldsymbol{x}^*, \boldsymbol{d}^*)$ such that for all $i$, given $(\boldsymbol{x}_{-i}^*, \boldsymbol{d}_{-i}^*)$ and $S$, player $i$ maximizes (1).

## 3. Static Analysis

We first characterize the condition under which given $\boldsymbol{x}$, nobody has an incentive to change the direction of effort, then we explore how the equilibrium efforts determine the implemented policy. In Section 3.2, steady-state equilibria in which the status quo and the implemented

[4] Note that the last assumption does not directly imply that the marginal benefit t of exerting additional effort would be the same for those players, because in principle, both $\alpha_i$ and $|\delta - y_i|$ can influence the marginal utility of having $\delta$ closer to $y_i$.

policy coincide are characterized. To summarize the findings, if $\gamma = \lambda = 1$, i.e., both the distance and the effort cost functions are linear, then any policy $\delta \in [0,1]$ can be a steady-state equilibrium. In contrast, if the cost function is convex while the distance function is linear, i.e., $\gamma > 1$ and $\lambda = 1$, a steady state equilibrium policy must be in $[y_{m-}, y_{m+}]$ where $y_{m-}$ is the ideal point of the left median player and $y_{m+}$ is the idea point of the right median player. Of course, when $N$ is an odd number then $y_{m-} = y_{m+} = y_m$. If $\gamma = 1$ and $\lambda > 1$, the mean of the two extreme players' ideal points, ½, emerges as the steady state point. Finally, if $\gamma > 1$ and $\lambda > 1$, then the mean of all players' ideal points is the steady state.

## 3.1 Group formation

In this subsection, we consider how groups are formed, that is, given the vector of efforts $\boldsymbol{x}$, how that of the directions $\boldsymbol{d}$ is determined. The following lemma describes what the groups $\hat{L} = \{j | d_j = -1\}$ and $\hat{R} = \{j | d_j = 1\}$ look like in equilibrium.

**Lemma 1.** *In equilibrium, there exists a threshold (or grouping rule)* $\theta \in [0,1]$ *such that the players whose ideal policy is in the left of* $\theta$ *are in group* $\hat{L}$*, and those who are in the right are in group* $\hat{R}$*. The player whose ideal policy is* $\theta$*, if exists, can be in either group.*

**Proof:** Obvious.

We define $L(\theta)$ as the set of players who are at the left side of $\theta$, i.e., $L(\theta) = \{i | y_i < \theta\}$, and $R(\theta)$ as $R(\theta) = \{i | y_i \geq \theta\}$. Since now we can infer the vector of directions $\boldsymbol{d}$ from $(\boldsymbol{y}, \theta)$, below we discuss how to determine an equilibrium threshold $\theta^*$ instead of the vector of directions $\boldsymbol{d}^*$.

For the sake of concreteness of the discussion, let $p(\boldsymbol{x}, \boldsymbol{d})$ be the Tullock-type CSF defined in (2) for a moment.[5] Note that since $y_1 = 0$ and $y_N = 1$, $\sum_j x_j$ is never zero in equilibrium. Therefore, the implemented policy as a function of $\theta$ is given by:

$$\delta(\theta; \boldsymbol{x}, \boldsymbol{y}, S) = S + \frac{\sum_{j \in R(\theta)} x_j - \sum_{j \in L(\theta)} x_j}{2 \sum_{j \in L(\theta) \cup R(\theta)} x_j}$$

$$= S - \frac{1}{2} + \frac{\sum_{j \in R(\theta)} x_j}{2 \sum_{j \in L(\theta) \cup R(\theta)} x_j} \qquad (3).$$

[5] Except for the ones presented in Section 4.2, our results do not require any specific functional form assumption on the adjustment function $p(\boldsymbol{x}, \boldsymbol{d})$.

Note that given $\boldsymbol{x}, \boldsymbol{y}$ and $S$, the implemented policy $\delta(\theta)$ is a decreasing step function: as $\theta$ moves from 0 to 1, more and more players move from $R(\theta)$ to $L(\theta)$, so $\delta(\theta)$ decreases step by step.

If the threshold $\theta$ is too small, too many players are on the right side of it, so the implemented policy ends up being biased toward the right. In such a case, a player in $R(\theta)$ but located close to $\theta$ has an incentive to change the direction of the effort from the right ($d = 1$) to the left ($d = -1$). If too many players are in $L(\theta)$, similarly, the implemented policy is biased towards the left, and a player located close to $\theta$ is willing to change the sides, i.e., $\theta$ must be moving towards the left. In equilibrium, the threshold must be set in a way such that nobody gains by changing her direction. The following lemma states that an equilibrium policy $\delta(\boldsymbol{x}^*, \boldsymbol{d}^*;\, S)$ is such a threshold.

**Lemma 2.** *Given a vector of equilibrium effort $\boldsymbol{x}^*$, the corresponding equilibrium groups are $L(\theta^*)$ and $R(\theta^*)$ where $\theta^*$ satisfies:*

$$\theta^* = \delta(\theta^*; \boldsymbol{x}^*, \boldsymbol{y}, S). \tag{4}$$

**Proof:** Consider an arbitrary grouping rule $\theta_0 \in (0,1)$ according to which players on the left of $\theta_0$ are in $L$, and those on the right of or on $\theta_0$ are in $R$. Suppose that $\delta(\theta_0; \boldsymbol{x}^*, \boldsymbol{y}) > \theta_0$, and that when the threshold moves from $\theta_0 < y_i$ to $\theta_1 > y_i$, $\delta(\theta_1; \boldsymbol{x}^*, \boldsymbol{y})$ is still greater than $\theta_1$. Then, the change from $\theta_0$ to $\theta_1$ (equivalently, from $d_i = 1$ to $d_i = -1$) improves the utility of player $i$ because by the change, $||\delta - y_i||$ becomes smaller. This means that $\theta_0$ is not an equilibrium threshold, and furthermore, any $\theta < \theta_1$ is not an equilibrium threshold either. We can say the same thing about the case in which $\theta_0$ is greater than $\delta$, and moving $\theta_0$ toward the left does not change the rank of the two. Because in equilibrium, nobody has an incentive to change the direction of the effort, the equilibrium dividing rule $\hat{\theta}$ must satisfy –

$$\lim_{\theta \to \hat{\theta}_+} \delta(\theta; \boldsymbol{x}^*, \boldsymbol{y}, S) \leq \hat{\theta} \leq \lim_{\theta \to \hat{\theta}_-} \delta(\theta; \boldsymbol{x}^*, \boldsymbol{y}, S),$$

and the equilibrium implemented policy is either the right limit or the left limit.

To prove the lemma by contradiction, suppose that the above inequalities are strict, which implies that there exists a point $y_j = \hat{\theta}$ such that when threshold $\theta$ is on the left of $y_j$, $\theta < \delta(\theta; \boldsymbol{x}^*, \boldsymbol{y}, S)$, but $\theta > \delta(\theta; \boldsymbol{x}^*, \boldsymbol{y}, S)$ when $\theta$ is on the right of $y_j$. Graphically, $y_j$ is the threshold in which $\delta(\theta)$ jumps from the above of the 45-degree line to the beneath of it. In this case, player $j$ can pull the implemented policy $\delta(x_j, \boldsymbol{x}^*_{-j}; \boldsymbol{y})$ toward her ideal policy $y_j$ by reducing her effort $x_j$. This means $\boldsymbol{x}^*$ was not a vector of equilibrium efforts in the first place, which contradicts the assumption. Therefore, in equilibrium at least one of the inequalities must hold as equality, and the equilibrium policy must be the dividing rule which satisfies $\theta^* = \delta(\theta^*; \boldsymbol{x}^*, \boldsymbol{y}, S)$. ■

A few remarks follow immediately. First, the equilibrium grouping rule $\theta^*$ defined by Coate (2004) is unique if exists, because $\delta(\theta)$ is a decreasing step function. This, of course, does not mean that the equilibrium is unique. Second, even if the equilibrium groups are unique, there can be infinitely many thresholds $\theta$ that define the same groups. Third, it is $\delta^*$ not $S$ that determines the directions of efforts: even when S is on the left of $y_i$, player may prefer to push the policy to the left if the equilibrium policy $\delta^*$ ends up being on the right of $y_i$. Lastly, because $\delta(\theta; S)$ increases as $S$ gets larger, given $(x, \theta^*)$ is non-decreasing in $S$.

## 3.2. Steady State

In this subsection, we characterize equilibria where $\delta^* = S$, namely steady-state equilibria. In such an equilibrium, the equilibrium groups are simply defined as $L(S)$ and $R(S)$. Given that the directions are set in the way to maximize each individual's utility, the game boils down to a simple collective rent-seeking game or a group contest.

Again, for concreteness, let us consider the Tullock (1980) CSF and assume $\lambda = 1$. Taking the constants out of Eq. (1), the maximization problem of player $i$ in $Q(S)$ $(where\ Q = L, R)$ can be rewritten as:

$$\max_{x_i} \frac{\alpha \sum_{j \in Q} x_j}{\sum_{j \in L \cup R} x_j} - c(x_i).$$

Note that this objective function is identical to that in group contests with the value of the prize being $\alpha$ (see Katz et al., 1990). Since a player can "win a (public-good) prize" even if she exerts zero effort, players have an incentive to free ride on the efforts of the other players in the same group. The first-order condition for player $i$ in $Q(S)$ is:

$$\frac{\alpha \sum_{j \in Q^c} x_j^*}{\left(\sum_{j \in L \cup R} x_j^*\right)^2} - c'(x_i^*) \geq 0$$

where the inequality condition is for a player who would choose $x_i = 0$ because she is completely satisfied with the steady-state policy (i.e., $y_i = S = \delta^*$). In other words, in equilibrium, the FOCs hold as equality whenever $x_i^* > 0$. Recall that the implemented policy coincides with the status quo if and only if $\sum_{j\in L(S)} x_j^* = \sum_{j\in R(S)} x_j^*$. Using this, we derive the following conditions: for all $x_i^* > 0$,

$$\frac{\alpha}{4\sum_{j\in Q} x_j^*} = c'(x_i^*) = (x_i^*)^{\gamma-1}. \quad (5)$$

Equation (5) shows that the determination of the steady-state efforts and the corresponding policy crucially depend on the (non-)linearity of the cost function. If it is linear ($\gamma = 1$), all the FOCs are identical to each other, so there is a large indeterminacy. In contrast, if it is convex ($\gamma > 1$), regardless of how convex it is, everybody must expend the same amount of effort in a steady-state equilibrium.

Now, consider the case with a convex distance measure, i.e., $\lambda > 1$. Given that $\delta^* = S$, the FOC of player $i$'s maximization problem is:

$$\alpha(|S - y_i|)^{\lambda-1} \frac{\alpha \sum_{j\in Q^c} x_j^*}{\left(\sum_{j\in L\cup R} x_j^*\right)^2} - (x_i^*)^{\gamma-1} \geq 0.$$

Let us first consider the case of the linear cost function ($\gamma = 1$). Since in a steady-state, $\alpha \sum_{j\in Q^c} x_j^* / \left(\sum_{j\in L\cup R} x_j^*\right)^2$ is common to every player, the players with the largest $\alpha(|S - y_i|)^{\lambda-1}$, that is, those farthest from $S$ expends a positive effort, while the others free ride. Because $S$ must be in between 0 and 1, the players farthest from $S$ in each group are those at the extremes, players 1 and $N$. In order for the FOCs of the extreme players to simultaneously hold as equality, $|S - y_1|$ must equal $|S - y_N|$. Therefore, $S = (y_1 + y_N)/2 = 1/2$.

Next, suppose that the cost function is also strictly convex ($\gamma > 1$). Notice that equilibrium effort $x_i^*$ is $(|S - y_i|)^{\frac{\lambda-1}{\gamma-1}}$ multiplied by $\left[\frac{\alpha \sum_{j\in Q^c} x_j^*}{\left(\sum_{j\in L\cup R} x_j^*\right)^2}\right]^{1/(\gamma-1)}$ which is a factor common to everybody. Thus, for $\sum_{j\in L} x_j^* = \sum_{j\in R} x_j^*$ to be the case, $\sum_{j\in L}\left(|S - y_j|\right)^{\frac{\lambda-1}{\gamma-1}}$ must equal $\sum_{j\in R}\left(|S - y_j|\right)^{\frac{\lambda-1}{\gamma-1}}$. Suppose the distance measure and the cost function are convex by the same degree, i.e., $\gamma = \lambda$. Then, by equating $\sum_{j\in L}(S - y_j)$ and $\sum_{j\in R}(S - y_j)$, we conclude that in such a case, $S = \sum_{j=1}^{N} y_j / N$. The above logic is valid for a more general adjustment function $p(\boldsymbol{x}, \boldsymbol{d})$, and the discussion thus far is summarized in the following proposition.

**Proposition 1.** Suppose that a steady-state equilibrium exists.

**(i)** If $\gamma = \lambda = 1$, any point in $[0,1]$ can be a steady-state equilibrium policy.

**(ii)** If $\gamma > 1$ and $\lambda = 1$, a steady-state policy is the median player's ideal policy. That is, if $N$ is an odd number, the steady-state policy must be $y_m$, and for $N$ an even number, any point in $[y_{m-}, y_{m+}]$ can be a steady-state policy.

**(iii)** If $\gamma = 1$ and $\lambda > 1$, the steady-state policy is $1/2$.

**(iv)** If $\gamma = \lambda > 0$, the steady-state policy is the average of all ideal policies, $\sum_{j=1}^{N} y_j / N$.

**Proof.** Note first that for a more general $p(\boldsymbol{x}, \boldsymbol{d})$, which can also be written as $p(\boldsymbol{x}, \theta)$, for a proper $\theta$ by Lemma 1, the FOC is:

$$\alpha(|S - y_i|)^{\lambda-1} \left| \frac{\partial p(x_i, \boldsymbol{x}_{-i}^*, \theta^*)}{\partial x_i} |_{x_i = x_i^*} \right| - (x_i^*)^{\lambda-1} \geq 0.$$

**(i)** Suppose $\gamma = \lambda = 1$. Since a steady-state equilibrium exists by assumption, there exists a tuple $(\boldsymbol{x}^*, \theta^*)$ that satisfies the FOCs. Now, pick an arbitrary status quo $S \in [0,1]$, let the groups be $L(S)$ and $R(S)$. Pick a vector $\boldsymbol{x}^{**}$ such that,

$$\textstyle\sum_{j \in R(S)} x_j^{**} = \sum_{j \in R(\theta^*)} x_j^* \text{ and } \sum_{j \in L(S)} x_j^{**} = \sum_{j \in L(\theta^*)} x_j^*.$$

Then, since $p(\boldsymbol{x}, \theta) = p\left(\sum_{j \in L(\theta)} x_j, \sum_{j \in R(\theta)} x_j\right)$, in other words, since the adjustment function depends only on the sums of efforts, $\left|\frac{\partial p(x_i, \boldsymbol{x}_{-i}^*, \theta^*)}{\partial x_i} |_{x_i = x_i^*}\right| = \left|\frac{\partial p(x_i, \boldsymbol{x}_{-i}^{**}, S)}{\partial x_i} |_{x_i = x_i^{**}}\right|$. Since the marginal cost of exerting effort is independent of the size (i.e., the RHS of the FOC is constant), the vector of efforts $x^{**}$ also satisfies the FOCs, and $\sum_{j \in R(S)} x_j^{**} = \sum_{j \in L(S)} x_j^{**}$ by definition. Hence, $x^{**}$ together with the status quo $S$ also constitutes a steady-state equilibrium. ■

**(ii)** Now suppose $\gamma > 1$ and $\lambda = 1$. Because at a symmetric point (when $\sum_{j \in R} x_j = \sum_{j \in L} x_j$) the marginal change $\frac{\partial p(x, \theta)}{\partial x_i}$ is identical to everybody, the first-order condition implies that the equilibrium effort, too, must be identical, unless $x_i^* = 0$. Note that since $\lim_{x \to 0} c'(x) = 0$, $\lim_{x \to 0} u_i'(x, \delta) > 0$, which implies everybody but the player with $y_i = S = \delta^*$ prefers to exert a positive amount of effort. Since the equilibrium effort level is identical across the players, in order for the sums of efforts to be equal to each other, there should be an equal number of players in each group. This implies that as long as $S$ divides the players into two symmetric groups, $S$ can be a steady-state policy. ■

**(iii)** Next, consider the case where $\gamma = 1$ and $\lambda > 1$. Since in a steady-state, $\frac{\partial p(x_i, x_{-i}^*, \theta^*)}{\partial x_i}|_{x_i = x_i^*}$ is common to every player, the players with the largest $\alpha(|S - y_i|)^{\lambda-1}$, that is, those farthest from $S$ expends a positive effort, while the others free ride. Because $S$ must be in between 0 and 1, the players farthest from $S$ in each group are those at the extremes, players 1 and $N$. In order for the FOCs of the extreme players to simultaneously hold as equality, $|S - y_1|$ must equal $|S - y_N|$. Therefore, $S = (y_1 + y_N)/2 = 1/2$. ■

**(iv)** Lastly, suppose that $\gamma = \lambda > 0$. From the FOC, we derive the following: if $x_i^* > 0$,

$$x_i^* = |S - y_N|. \left[\alpha \left|\frac{\partial p(x_i, \boldsymbol{x}_{-i}^*, \theta^*)}{\partial x_i}|_{x_i = x_i^*}\right|\right]^{1/(\gamma-1)}$$

Because $\frac{\partial p(x_i, x_{-i}^*, \theta^*)}{\partial x_i}|_{x_i = x_i^*}$ is common to every player, for $\sum_{j \in R} x_j^* = \sum_{j \in L} x_j^*$ to be the case, $\sum_{j \in L} |S - y_j|$ must equal $\sum_{j \in R} |S - y_j|$. By equating $\sum_{j \in L}(S - y_j)$ and $\sum_{j \in R}(y_j - S)$, we conclude that $S = \sum_{j=1}^{N} y_j / N$. ■

From the objective function of player $i$, one can see that as the number of players in a group increases, the incentive to free ride on the others' efforts increases. When both the distance and the cost functions are linear so that the smaller group has no disadvantage in terms of the cost or the utility, any policy can be a stable outcome of the game because the larger group suffers more with free rider problem than the smaller group does. In contrast, when the cost function is convex, while the asymmetry in the severity of free rider problem still exists, the cost disadvantage breaks the balance between the groups. The balance can be recovered only when the relative powers between the groups are equalized.

This proposition shows that if the policy converges, it does to a 'center', whose definition depends on the convexity of the distance measure ($\lambda$) and that of the cost function ($\gamma$). The convergence point can be the median, the mean of two extreme players, the mean of all players or some point between a mean and the median. An interesting question to ask is to which point the steady-state policy would converge (i) as $\lambda$ goes to infinity, or (ii) as $\gamma$ goes to infinity. First, if $\lambda$ is very high, the marginal utility of having a policy closer to the ideal point will be extremely high for the players at the extremes compared to the other players. Thus, as the distance measure becomes more convex (i.e., as $\left(|S - y_j|\right)^{\frac{\lambda-1}{\gamma-1}}$) becomes more sensitive to the distance), the steady-state policy would get closer to $1/2$, the middle point of the two extremes. It is not difficult to see that this result resonates with the case of $\gamma = 1$.

If the cost function is extremely convex, on the other hand, in the limit, the cost will be zero up to a certain point, then suddenly become infinite. Thus, it will be like everybody has one "vote" (observe that $\left(\left|S - y_j\right|\right)^{\frac{\lambda-1}{\gamma-1}}$ becomes 1 as $\gamma$ goes to infinity), therefore the policy that the median voter prefers will be implemented. Of course, this result is comparable to the case with $\lambda = 1$.

Proposition 1 essentially provides the *Central Influencer Theorem* result for the static case. These results highlight the contribution of our study to the existing research, offering insights into the dynamics of policy shift, coalition formation, and the intricate interplay of costs and preferences. These are summarized in Table 1 below.

**Table 1.** Result summary for the static case

| **Effort cost** | **Disutility** | **Steady state equilibrium** |
|---|---|---|
| Linear | Linear | Any policy |
| Convex | Linear | Median player's ideal policy |
| Linear | Convex | Mean of two extreme players' ideal policy |
| Convex | Convex | Mean of all players' ideal policies |

## 3.3 Heterogeneous Players

What will steady-state equilibria look like if players are heterogeneous? Examining all potential cases as comprehensively as Proposition 1 is challenging. Nonetheless, we can derive outcomes similar to those in Proposition 1.

Suppose first that $\gamma = 1$ and $\lambda = 1$. When the players are identical in their valuation, this free-riding incentive prevents the policy from converging to a center. In contrast, if the players are heterogeneous, everybody but those with the highest marginal utility free rides completely. Thus, the policy would converge to a point between the ideal points of two players with the highest marginal utilities. This is comparable to the model of Baik (2016) who analyzes a group contest where the players decide whether to support one or both of the two alternatives. In his model, the cost function is assumed to be linear, and the (asymmetric) marginal utilities are exogenously given.

Suppose that $\gamma > 1$ and $\lambda = 1$ and that players differ in their valuation (i.e., for some $i \neq j, \alpha_i \neq \alpha_j$) and in their power or resource (i.e., $\mu_i \neq \mu_j$).[6]

Define the power-adjusted valuation as $\bar{\alpha}_i = \left(\frac{\alpha_i}{\mu_i}\right)^{1/(\gamma-1)}$. If there exists a player i such that $\left|\sum_{j\in L(y_i)} \bar{\alpha}_j - \sum_{j\in R(y_i)} \bar{\alpha}_j\right| < \bar{\alpha}_i$, then it is straightforward to show that $y_i$ can be a steady-state equilibrium policy. That is, such player i is a median influencer. Notice that the inequality holds more easily when $\bar{\alpha}_i$ is greater. So, the steady-state outcome is likely to be the ideal policy of a strong player who is more or less in the middle. If there does not exist such a player, the steady-state outcome will be somewhere in between two median players. This type of "median voter theorem" (or some variations of it) has often been used by some political scientists to predict the outcome of a complicated political game (e.g., Bueno de Mesquita, 2000, 2002). Its performance has proven outstanding, but it is difficult to say that such practices have always been firmly micro-founded. The analysis in this subsection provides a micro-foundation of such forecasting exercises.

The third and fourth parts of Proposition 1 can easily be generalized to heterogeneous players. If $\gamma = 1$ and $\lambda$ is sufficiently large, then still only the two extreme players will be active. However, the steady-state policy will not be the exact mean of the two ideal policies, but a weighted average of which weight reflects the power-adjusted valuations of the two players.

Lastly, suppose that $\gamma = \lambda > 0$. Then, for $\sum_{j\in L} x_j^*$ to equal $\sum_{j\in R} x_j^*$, $\sum_{j\in L} \bar{\alpha}_j \left|S - y_j\right|$ must be equal to $\sum_{j\in R} \bar{\alpha}_j \left|S - y_j\right|$. Therefore, the steady-state outcome must be the weighted average of the players' ideal policies: $S = \frac{\sum_{j=1}^{N} \bar{\alpha}_j y_j}{\sum_{j=1}^{N} \bar{\alpha}_j}$.

## 4. Dynamic Analyses

In this Section we extend our model to the dynamics of an infinite-horizon structure to show that the incentive to free ride slows down the convergence to the median player. First, we characterize the dynamics under certain conditions. Then we run a robustness check.

### 4.1 Dynamic Equilibrium

In this subsection, we investigate the dynamics of the model, assuming that (i) the adjustment function is the Tullock-type CSF, (ii) $\lambda = 1$ and (iii) $N = 3$. Suppose that the game analyzed

[6] Recall that $\mu_i$ is a factor multiplied to the effort cost. So, a smaller $\mu_i$ represents a stronger power.

so far is repeated infinitely many times ($t = 1,2,..,\infty$) and that for any $i$ and $t$, player $i$ maximizes the discounted utility:

$$U_{it} = \sum_{\tau=t}^{\infty} \beta^{\tau-t}\, u_i(\delta_\tau, x_{i\tau}) = \sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[-\alpha|\delta_\tau - y_i| - x_{i\tau}^{\gamma}/\gamma\right]$$

where $\beta \in [0,1)$ is the common discount factor, and $x_{it}$ is the effort exerted at period $t$. The status quo at period $t+1$ is given by the implemented policy at $t$ (i.e., $S_{t+1} = \delta_t$). For $\beta$ sufficiently large, there exist infinitely many (collusive) subgame perfect equilibria which depend on the history of actions, which we do not intend to explore here. Instead, we focus on equilibria in which a strategy $x_{it}$ is a function of the status quo $S_t$. If $\gamma = 1$, any policy can be a steady-state policy, which means that in such a case, the dynamics is either trivial or arbitrary. Thus, in this subsection, the cost function is assumed to be strictly convex.

Since the dynamics from the left to the right and the other way around are symmetric, we only consider the case with $S_1 < y_m = y_2$. As will be shown more clearly in the proof of Proposition 2, given $S_1 < y_m$, $\delta_t \leq y_m$ for all $t$. In other words, the equilibrium policy never crosses the ideal point of the median player. This implies that $\hat{L}_t = \{1\}$ and $\hat{R}_t = \{2,3\}$ for all $t$.

Using the fact that $S_{t+1} = S_t + p_t = S_{t-1} + p_{t-1} + p_t = \ldots = S_{t-k} + \sum_{\tau=0}^{k} p_{t-\tau}$, we can rewrite the discounted utility as:

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[-\alpha|S_\tau + p_\tau - y_i| - x_{i\tau}^{\gamma}/\gamma\right]$$

$$= -\alpha\left|\frac{1}{1-\beta}(S_\tau + p_\tau - y_i) + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \sum_{\eta=t+1}^{\tau} p_\eta\right| - \frac{x_{it}^{\gamma}}{\gamma} - \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\frac{x_{i\tau}^{\gamma}}{\gamma}$$

One can easily see that because a change in the policy has a permanent effect, the marginal utility of having a more desirable policy is always $\alpha/(1-\beta)$, meaning that it does not depend on $S_t$. Then by the envelop theorem, we can ignore the decision in period $t+1$ and onward ($\{x_{i\tau}\}_{\tau+1}^{\infty}$) when considering the decision making at period $t$. Thus, the first-order condition with respect to $x_{it}$ is:

$$\frac{\alpha}{1-\beta}\left|\frac{\partial p(x_{it}, \boldsymbol{x}_{-it}^{*}, \theta_t^{*})}{\partial x_{it}}\Big|_{x_{it}=x_{it}^{*}}\right| - (x_{it}^{*})^{\gamma-1} \geq 0$$

where the inequality condition is for the median player facing $S_t$ close enough to $y_m$.

To proceed further, suppose the adjustment function is the Tullock-type CSF defined in (2). Let us suppose for a moment that all the FOCs hold as equality as $S_t$ is far enough from $y_m$. In this case, $x_{2t}^* = x_{3t}^*$ because players 2 and 3 are in the same group and because their FOCs are identical. Thus, the FOCs are:

$$\left(\frac{\alpha}{1-\beta}\right)\left(\frac{2x_{2t}^*}{\left(x_{1t}^*+2x_{2t}^*\right)^2}\right) = (x_{1t}^*)^\gamma - 1 \qquad (5)$$

$$\left(\frac{\alpha}{1-\beta}\right)\left(\frac{2x_{1t}^*}{\left(x_{1t}^*+2x_{1t}^*\right)^2}\right) = (x_{2t}^*)^\gamma - 1 \qquad (6)$$

from which we derive the following proposition.

**Proposition 2.** Suppose that the adjustment function is the Tullock-type CSF. For any initial policy $S_1 \in [0,1]$, there exists an equilibrium in which $\{\delta_t\}_{t=1}^{\infty}$ converges to the median player's ideal policy $y_m$. As $\alpha$ or $\beta$ gets larger, the equilibrium effort level $x_{it}^*$ also grows larger. However, the speed of convergence does not depend on $\alpha$ and $\beta$, but it increases in $\gamma$.

**Proof.** We prove the proposition by construction. Observe that according to (3), given the groups and the efforts $x$, $S$ and $\delta$ are one-to-one. Thus, we can recover $S$ from $\delta$. Using this idea, an equilibrium can be constructed as follows. First, pick a dividing rule $\bar{\theta}_t(< y_m)$, and calculate the optimal efforts using (6) and (7). This yields the equilibrium efforts:

$$x_{1t}^* = 2^{\frac{1}{\gamma}}\left(\frac{\alpha}{1-\beta}\frac{2^{\frac{1}{\gamma}}}{2+2^{\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma}}$$

$$x_{2t}^* = x_{3t}^* = \left(\frac{\alpha}{1-\beta}\frac{2^{\frac{1}{\gamma}}}{2+2^{\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma}}$$

and the adjustment $p(x_t^*, \bar{\theta}_t) = \left(\frac{2-2^{\frac{1}{\gamma}}}{2+2^{\frac{1}{\gamma}}}\right) = \bar{p}$.

Let $\delta_t = \bar{\theta}_t$, and solve (3) for $S_t$. Denote this calculated status quo by $\bar{S}_t$. Then, given $\bar{S}_t$, an equilibrium policy is $\bar{\theta}_t$. Note that when $\bar{\theta} = y_m$, the first-order condition for the median player does not have to hold as equality. Let us define $S_{m+} = y_m + \bar{p}$ and $S_{m-} = y_m - \bar{p}$. Then, for $S_t \in [S_{m-}, S_{m+}]$, instead of the equality FOC of the median player, equation $p(x_m, \boldsymbol{x_{-m}}, d^*) = y_m - S_t$ together with the FOCs for the other players characterizes the equilibrium.

In this equilibrium, $\{\delta_t\}_{t=1}^{\infty}$ converges to the median in each round as much as $\bar{p}$ when $S_t \notin [S_{m-}, S_{m+}]$, and once $S_t \in [S_{m-}, S_{m+}]$, $\delta_t$ is decided to be $y_m$, and stays there forever. Thus, as claimed above, $\{\delta_t\}_{t=1}^{\infty}$ does not oscillate around $y_m$, so the initial grouping remains valid until $\delta_t$ reaches $y_m$. From the above formulas, one can easily see that $x_{it}^*$ increases in $\alpha$ and $\beta$, and that the speed of convergence does not depend on $\alpha$ and $\beta$ but on $\gamma$. More specifically, $\bar{p}$ increases in $\gamma$. ■

We have reasonable belief that $\{\delta_t\}_{t=1}^{\infty}$ converges to the median under a set of more relaxed assumptions, and the result regarding the equilibrium effort level will remain valid. However, the speed of convergence may depend on $\alpha$ and $\beta$ if another type of adjustment function is used, if the players are asymmetric, or when the distance measure is non-linear.

It is also worth mentioning that the speed of convergence is determined by both the free-riding incentive and the cost advantage: the amount of efforts in the larger group $(x_{2t}^* + x_{3t}^*)$ is not twice as large as the effort level in the smaller group $(x_{1t}^*)$ because the players in the larger group have an incentive to free ride on each other's effort, which slows down the convergence. As $\gamma$ gets larger, on the other hand, the cost advantage of the larger group becomes more significant, so the policy converges faster to the median.

## 4.2 Robustness in dynamics

One may wonder whether our main results remain valid even if the functional form assumptions on the distance and the cost functions are relaxed. The answer is yes if we make a simplifying assumption on the adjustment function as follows:

$$p(\boldsymbol{x}, \boldsymbol{d}) = \eta \sum_{j=1}^{n} d_j x_j$$

for some $\eta$ positive but small. That is, the adjustment function is linear. Then, we can show the following.

**Proposition 3.** Suppose the adjustment function is linear. Let us further assume that the distance function $\|\delta - y\|$ and the cost function $c_i(x)$ are strictly convex and continuously differentiable and that the first derivative of the distance function is zero at $\delta = y$, i.e., $\frac{\partial\|\delta-y\|}{\partial\delta}|_{\delta=y} = 0$. Then, the steady state equilibrium exists and is unique. If the agents behave myopically (i.e., the static optimal behavior is repeated), the policy $\delta$ converges to the steady state equilibrium.

**Proof.** Given the status quo policy, an individual's maximization problem has an interior solution because the distance and the cost functions are strictly convex. And the first-order condition is:

$$\alpha_i \eta \frac{\partial \|\delta - y_i\|}{\partial \delta} |_{\delta=\delta^*} - c_i'(x^*) = 0$$

Let $x_i(\tilde{\delta})$ be the optimal effort given the policy $\tilde{\delta}$, i.e.,

$$x_i(\tilde{\delta}) = (c')^{-1} \left( \alpha_i \eta \frac{\partial \|\delta - y_i\|}{\partial \delta} |_{\delta=\tilde{\delta}} \right).$$

Since $c_i(x)$ is strictly convex and continously differentiable, the inverse function of the marginal cost is monotone increasing and continuous. Therefore, the optimal effort increases as $\tilde{\delta}$ moves away from $y_i$. Define

$$f(\tilde{\delta}) = \sum_{i \in L(\tilde{\delta})} x_i(\tilde{\delta}) - \sum_{i \in R(\tilde{\delta})} x_i(\tilde{\delta}).$$

Note that $f(0) \leq 0 \leq f(1)$ because $y_i \in [0,1]$ for all $i$ and that $f(\tilde{\delta})$ is increasing. $f(\tilde{\delta})$ continous in $\tilde{\delta}$ because $\frac{\partial \|\delta - y\|}{\partial \delta} |_{\delta=y} = 0$. Thus, $f(\tilde{\delta}) = 0$ for a $\tilde{\delta}$, and such $\tilde{\delta}$ is unique.

To see the myopic dynamics, suppose that the equilibrium policy $\delta^*$ is smaller than the steady state policy, which means that $f(\delta^*) < 0$ or equivalently $p(\boldsymbol{x}, \boldsymbol{d}) > 0$. Since the adjustment is made toward the right extreme, the status quo s must be even smaller than $\delta^*$. In other words, $S$ was farther from the steady-state than $\delta^*$ is. Thus, the static equilibrium policy converges toward the steady state policy. The analysis for $\delta^*$ is smaller than the steady-state policy is analogous, and thus omitted. ■

## 5. Discussion

We construct a spatial voting model that operates without the constraint of the ‘one person, one vote’ principle. Within this framework, players in a policy line can exert costly effort to shift the status quo towards their favorable position. We demonstrate the endogenous formation of two distinct groups in equilibrium, each actively seeking to implement opposing policies. Furthermore, we show that the characteristics of effort costs and preferences play crucial roles, ultimately leading to the determination of a central policy as the steady-state equilibrium. This centrality can take various forms, such as the median, mean, or the average of extreme policies. The comprehensive outcomes of our analysis from Proposition 1 are stated below.

1. If both the disutility stemming from non-optimal policies and the effort cost function are linear, any policy point can potentially serve as a steady-state policy. This scenario reflects a lack of cost (dis)advantage across the policy spectrum.
2. When the disutility due to non-optimal policies is linear, but the effort cost function is convex, the optimal policy for a median player becomes a steady-state policy. This mirrors the conventional median voter theorem, a milestone in the voting literature.
3. If the disutility of non-optimal policy is convex but the effort cost function is linear, then our model is close Baik (2017). However, there are two important distinctions: whereas Baik's model resembles a representative democracy, ours is comparable to a direct democracy. Also, unlike Baik (2017), the marginal utility of exerting effort is endogenous in our model, and hence we find that the steady-state policy is the mean of the two extreme players' optimal policies.
4. When both the disutility stemming from non-optimal policies and the effort cost function exhibit equal convexity, the steady-state policy converges to the (weighted) average of all players' optimal policies. This concurs with the 'risk loving equilibrium' identified by Duggan and Gao (2020), in a different setting. Despite risk-neutrality in our setting, the players form groups endogenously, and we reach similar conclusions.

We extend the static model to an infinite horizon model to study dynamics but restrict our attention to a 3-player case. In such a structure, the equilibrium policy converges to the median player only under certain conditions. Players also expend more effort in each period as they become more patient. The convergence speed does not depend on the discount factor but does depend on the effort cost. These results are in contrast with the dynamic model of Baron (1996) who finds that in collective goods programs equilibrium ultimately converges to the median.

Our findings exhibit broad similarities with Gerber and Lewis (2004), who empirically demonstrate that legislative decisions align with the median voter's preference under specific conditions. Moreover, our results can be aligned with a series of findings from various contexts, which highlight instances where a median voter equilibrium might not be achieved. These collective outcomes lend credence to the notion advanced by Hinich (1977), suggesting that median voter results should be viewed as context-dependent rather than universally applicable.

Note that our results extend beyond their immediate domain and carry implications for various fields, including spatial competition models and political analysis. They offer connections to research exploring lobbying, politics, and interest group dynamics. While the spatial competition model has been extensively employed in general interest politics, our study bridges the gap by considering its applicability in the realm of special interest politics. Whereas Coate (2004) covers special interest politics through a game involving advertising campaigns and contributions, we examine direct competition among players in a single-dimensional policy space. Moreover, our model covers a wider spectrum of political influences, including those not necessarily mediated by or directed toward public elections, such as international politics.

Furthermore, it's important to note that a strand of literature within political science has often predicted the outcomes of intricate political games by employing variations of the median voter theorem (see, e.g., Bueno de Mesquita (2000, 2002)). However, one challenge has been the difficulty in conclusively establishing the micro-foundations underlying such practices. In this regard, our study serves a crucial role by offering a micro-foundation for these predictions. This enhancement in micro-foundation lends credibility to the utilization of the variations of median voter theorem-based analyses.

The current study lays a foundation for further exploration, and there are several intriguing directions in which the research can be extended. We discuss three such avenues.

Generalizing the model: one can generalize the model by incorporating more generic cost and disutility functions. This extension could also involve exploring policy spaces beyond linearity, enabling a more comprehensive analysis of policy dynamics. Moreover, extending the dynamic version to include more than three players could lead to richer and more complex outcomes.

Incorporating different collective action and influence structures: The assumption of an additive collective action function can be expanded to encompass other network structures such as weakest link (Lee, 2012), best shot (Chowdhury et al., 2014), or a combination thereof (Chowdhury and Topolyan, 2016). Similarly, other contest success functions can be included.

Empirical and experimental investigations: Translating the theoretical findings into empirical or experimental investigations could provide real-world validation and insights into the practical implications of the model. This step could help bridge the gap between theoretical predictions and actual decision-making scenarios.

It is important to emphasize that while these extensions hold potential for further enriching the understanding of collective decision-making processes, they do not replace the fundamental insights provided by this study. Hence, we leave these ideas for future research.

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