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Loss Aversion, Risk Aversion, and the Shape of the Probability Weighting Function*

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Abstract

Loss aversion, risk aversion, and the probability weighting function (PWF) are three central concepts in explaining decisionmaking under risk. I examine interlinkages between these concepts in a model of decisionmaking that allows for loss averse/tolerant stochastic reference dependence and optimism/pessimism over probability distributions. I give a preference interpretation to commonly observed shapes of PWF and to risk aversion. In particular, I establish a connection between loss aversion and both risk aversion and the inverse-S PWF: loss aversion is a necessary condition to observe each of these phenomena. The results extend to distinct PWFs in the gain and loss domains, as under prospect theory.

JEL Classification: D91; D81; D01.
Keywords: probability weighting; rank dependent expected utility; loss aversion; risk aversion; reference dependence; optimism; pessimism; prospect theory.

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1 Introduction

Loss aversion and the probability weighting function (PWF), introduced to decision theory in original prospect theory (Kahneman and Tversky, 1979), are – alongside the much older concept of risk aversion, introduced in Bernoulli (1738/1954) – three central concepts in decision theory. The PWF, in particular, plays a central role in two of the most prominent alternatives to expected utility: rank-dependent expected utility (RDEU), introduced in Quiggin (1982), and prospect theory (PT), introduced in (Tversky and Kahneman, 1992). RDEU, for instance, preserves standard axioms of rationality, such as monotonicity, transitivity, and stochastic dominance, yet offers an account of many of the well-known violations of expected utility theory, including Allais' paradoxes, the common ratio effect, and the preference reversal effect (see, e.g., Quiggin, 1985; Karni and Safra, 1987; Segal, 1987).[1]

How to interpret the shape of the PWF? In themselves, RDEU and PT offer no guide to this question, for the PWF is taken as a primitive. Different from this interpretation, this paper shall explore an idea, implicit in some recent models, that the PWF is not itself a primitive, but is instead a concept that derives from a set more basic and interpretable primitives.[2] Barseghyan *et al.* (2018), for instance, "build" the PWF from (i) gain-loss emotions arising from stochastic reference dependence, à la Kőszegi and Rabin (2006, 2007); and (ii) preferences over probability distributions, à la Quiggin (1982). An important limitation of this approach, however, is that these two building blocks cannot be identified separately (Barseghyan *et al.*, 2018).

To (at least partially) overcome this problem of identification, and gain further insights into

[1] For axiomatizations of RDEU see Abdellaoui (2002) and the references therein. For further motivations for RDEU see Weber and Kirsner (1997).

[2] The case for the elements of models of decisionmaking to have plausible psychological interpretations is made in Wakker (2010). Dekel and Lipman (2010) argue that the "story" of a model is relevant and may provide a reason for preferring one model to another, even if two models predict the same choices. Absent of a plausible psychological interpretation, in the words of Diecidue and Wakker (2001, p. 281), the PWF risks being merely a "technical tool" with "no intuitive or empirical content."

how the PWF may derive from more basic and interpretable primitives, this paper proposes introducing additional insights from the economics and psychology literature on preferences over probability distributions. The two insights we propose are:

1. *Optimism/pessimism*: I characterize preferences over probability distributions in terms of an individual dispositional preference for either optimism or pessimism. An optimist places greater focus on the realization of favorable outcomes – the more favorable the outcome, the greater the subjective weight placed on it – whereas a pessimist does the opposite. As well as appearing in decision theory, the role of a dispositional preference for optimism/pessimism features importantly in the literature on health economics (e.g., Kubzansky *et al.*, 2004; Rozanski *et al.*, 2019), life satisfaction (Piper, 2022), social comparisons (Menon *et al.*, 2009), and stock market participation (Grevenbrock, 2020).

   A dispositional preference for optimism/pessimism can be further motivated by a series of experiments, Lopes (1984, 1987, 1990), in which participants choose between pairs of lotteries with very similar expected values, but in which the distribution of outcomes around the mean differs. Consistent with the notions of optimism and pessimism, Lopez accounts for her findings by arguing that individuals possess a stable dispositional tendency towards placing greater emphasis on different ends of the distribution of payoffs. Pessimists consistently place greater emphasis on outcomes at the low ("security") end of the distribution, while optimists consistently place greater emphasis at the high ("potential") end of the distribution. The maximin and maximax models of decisionmaking under ignorance (Arrow and Hurwicz, 1972) are extreme cases of optimism/pessimism, where all the weight goes to the worst or best outcome, respectively. Dillenberger *et al.* (2017) propose a way to integrate optimism/pessimism into expected utility theory. Further contributions that allow for optimism and pessimism in the broad sense defined here include Hey (1984), the security/potential level model of Cohen (1992), and the neo–additive expected utility model of Chateauneuf *et al.* (2007). While the latter two models emphasize the attention paid to extreme outcomes, optimism and pessimism have also been observed for non-extreme outcomes (see, e.g., Wu and Gonzalez, 1996). The piecewise linear RDEU model of Webb (2017) allows for this more general possibility.

2. *Perception of small probabilities*: People encounter difficulty perceiving small probabilities correctly, and different decisionmakers typically treat small probabilities in one of two very different ways: either they massively overweight small probabilities, or they all but ignore them. This bimodal pattern of behavior is observed across subjects in a range of insurance experiments (see, e.g., Slovic *et al.*, 1977; McClelland *et al.*, 1993; Botzen and van den Bergh, 2012), with many people willing to pay large (actuarially unfair) amounts to avoid rare risky events, but many others wishing to pay nothing for decreasing the level of risk. Ungemach *et al.* (2009) also report experimental evidence consistent with both under- and over-weighting of small probabilities. This dichotomy in the perception of small probabilities was, to our knowledge, first brought to the attention of economists in original prospect theory (Kahneman and Tversky, 1979), wherein the authors decline to specify the shape of the PWF over extreme probabilities for this reason. The incumbent challenges posed by low-probability events for economic policymaking are explored in, e.g., Camerer and Kunreuther (1989) and Kunreuther *et al.* (2001).

Utilizing the these two insights, I characterize the underlying preference profiles of optimism/pessimism and loss aversion/tolerance that may be inferred from particular shapes of the PWF, and that imply particular shapes of the PWF. I also characterize the underlying preference profile that – under concave utility – is equivalent to risk aversion, in the sense of aversion to mean preserving spreads. Abundant experimental evidence is consistent with an inverse-S shaped PWF (see, e.g., Table 1 in Booij *et al.* (2010) for a summary).[3] A striking finding in this context is that loss aversion is necessary to observe an inverse-S PWF, and also necessary to observe risk aversion. Specifically, an inverse-S PWF maps uniquely to loss aversion and optimism, while risk aversion is equivalent to loss aversion and pessimism (assuming concave utility). In this way I relate three of the most central features of decisionmaking: the loss aversion, risk aversion, and the inverse-S PWF. In particular, loss aversion emerges as a "primitive" concept that underlies more "complex" concepts, such as risk aversion and the inverse-S PWF.

As well as explaining an inverse-S PWF, the model also maps preference profiles to a range of other shapes of PWF observed commonly in the experimental literature. Decisionmakers exhibiting evidence of a strictly convex PWF are relatively plentiful: for instance, 83 out of 124 subjects in the experiment of Qiu and Steiger (2011, p. 195) exhibit this shape of PWF, which is also observed in the individual level results of Gonzalez and Wu (1999, Fig. 6). A strictly convex PWF is also estimated as the median outcome of a range of experiments (e.g., Goeree *et al.*, 2002; Harrison *et al.*, 2010; van de Kuilen and Wakker, 2011; Krawczyk, 2015) and, albeit under strong structural identifying assumptions, in the field study of Jullien and Salanié (2000). Moreover, in a typical experiment, at least a minority of subjects exhibit choices consistent with an S-shaped or strictly concave PWF (see, e.g., Hey and Orme, 1994; Birnbaum and Chavez, 1997; Humphrey and Verschoor, 2004; Blavatskyy, 2010).

The plan of the paper is as follows. Section 3 develops the model, and Section 4 examines the preference inferences that can be drawn from the model in respect of risk aversion and the PWF, including for a simple parameterization of preferences for optimism/pessimism.

[3] For complementary evidence from non-laboratory environments see, e.g., Polkovnichenko and Zhao (2013), Rieger *et al.* (2017), and the references therein.

In Section 5 I show that the results from the basic model extend to a PT representation featuring differential probability weighting over (absolute) gain and loss outcomes in mixed lotteries. In particular, the unique preference inference required for the PWF to be inverse-S in both the gain and loss domains is loss aversion coupled with optimism. Section 6 concludes. All proofs are contained in Appendix B, and Figure 1 appears at the very rear.

2 Related Literature

In this section I briefly discuss some of the literature that is broadly related to the present contribution. The paper adds to a literature on the representation of risk preferences under RDEU (Chew *et al.*, 1987; Röell, 1987). In particular, whereas risk preferences are a property solely of the second derivative of utility under expected utility theory, they also depend on the second derivative of the PWF under RDEU. As I do not take the PWF to be a primitive, risk preferences arise in the present framework according to how the underlying primitives of the model map into the second derivative of the PWF.

The paper also connects to an existing literature seeking to understand the shape of the PWF. Tversky and Kahneman (1992) propose that people become less sensitive to changes in probability the further it lies from a reference point (diminishing sensitivity). If, in the probability domain, the two end points $\{0, 1\}$ serve as reference points this principle of diminishing sensitivity makes increments near the end points of the probability scale loom larger than increments near the middle of the scale. Relative to this explanation, an advantage of the present contribution is that it can explain many widely observed shapes of PWF in addition to the inverse-S PWF. Gonzalez and Wu (1999) characterize PWFs by their level (elevation) and curvature. They interpret the curvature of the PWF as reflecting the ability of an individual to discriminate between probabilities, and the elevation as reflecting how attracted an individual is to the chance domain of prospects. Elevation is, however, typically intertwined with curvature, except for some special forms of the PWF (Abdellaoui *et al.*, 2010).

The role of gain-loss preferences in probability weighting is explored in the prior contributions of Brandstätter *et al.* (2002) and Walther (2003). Brandstätter *et al.* (2002) propose a form of PWF in which, e.g., overweighting arises when expected elation from gains outweighs expected disappointment from losses. Walther (2003) modifies expected utility theory to allow for temporary emotional reactions (elation or disappointment) in response to resolution of uncertainty, resulting in a form of probability weighting similar, but not identical, to RDEU. In a related approach, Wang (2021) replaces objective probabilities with attention weights that reflect the salience of outcomes. The author obtains an RDEU representation with a flexible form of PWF, but the representation is only for the special case of binary lotteries. Relatedly, the disappointment aversion models of Bell (1985) and Gul (1991) also admit an RDEU representation, but only for binary lotteries (see, e.g., Abdellaoui and Bleichrodt, 2007). Moreover, in these two studies, the resulting PWF cannot take an inverse-S shape.

Kilka and Weber (2001) argue that probability weighting may be explained partly as a response to ambiguity over the likelihood of events. While important, this consideration does not explain evidence of probability weighting in settings where the objective probabilities of events are known (the case we consider here). Last, different from other approaches, decision by sampling (Stewart *et al.*, 2006, 2015) proposes that the shape of the PWF derives from a tendency toward ordinal comparisons of probabilities. Thus, the shape of the PWF reflects the shape of the cumulative distribution of the probabilities contained within the lotteries under consideration. Recent evidence casts doubt on the decision by sampling explanation, however (Alempaki *et al.*, 2019).

3 Model

I consider a model of decisionmaking under risk that admits a RDEU representation in which both preferences over probability distributions and the effects of stochastic reference dependence are nested within a single composite PWF. In particular, the model exploits the equivalence between Kőszegi and Rabin's (2007) notion of choice-acclimating personal equilibrium and a restricted version of RDEU, first documented by Masatlioglu and Raymond (2016).

3.1 Preferences over Probability Distributions

Let X be a lottery taking values on a finite subset of the real numbers, $\{x_1, \ldots, x_n\}$, ordered such that $x_1 < x_2 < \ldots < x_n$, $n \geq 2$. Each outcome x_i occurs with probability $p_i \in [0,1]$, $\sum_{j=1}^{n} p_j = 1$. In the tradition of rank-dependent probability weighting, as proposed in Quiggin (1982), the decisionmaker weighs each outcome x_i by the decision weight ϕ_i, given by

$$\phi_i = \pi\left(\sum_{j=i}^{n} p_j\right) - \pi\left(\sum_{j=i+1}^{n} p_j\right) \qquad i \in \{1, \ldots, n\}. \tag{1}$$

where $\pi : [0,1] \mapsto [0,1]$ is a continuous and strictly increasing weighting function satisfying $\pi(0) = 0$ and $\pi(1) = 1$. I suppose $\pi(p)$ is twice differentiable everywhere, except possibly at the endpoints $p \in \{0,1\}$.

As discussed at the outset, the shape of $\pi(p)$ plausibly relates to the psychological notions of *optimism/pessimism* over probability distributions (Quiggin, 1982). The idea is described in full in Yaari (1987) and Diecidue and Wakker (2001), among other studies, and is axiomatized in Wakker (2001). In essence, an optimistic decisionmaker tends to believe that the risk in a gamble will resolve favorably for them. According, if a given outcome is increased, so as to become more favorable to the decisionmaker, they place greater focus upon it.

This necessarily entails – as the decision weights in (1) sum to unity – that the remaining outcomes receive less focus. By contrast, a pessimist tends to believe that randomness will realize against them, the opposite of an optimist. Wakker (1994) demonstrates that these notions of optimism/pessimism imply the behavioral notions of mixture aversion/tolerance. A decisionmaker is mixture averse if, given two lotteries over which they are indifferent, they prefer these lotteries to any mixture of them (believing that the extra risk entailed in the mixed lottery will not be favorable to them). Chew *et al.* (1987) establish that, alongside the second derivative of the utility function, the notion of optimism (pessimism) is required to ensure risk seekingness (risk aversion) in the rank-dependent framework.

An interpretation of $\pi(p)$ in terms of optimism/pessimism necessarily places restrictions on its possible shape. Formally, optimism corresponds to the situation in which an improvement in the ranking position of outcome x_i (by lowering the probability $\sum_{j=i+1}^{n} p_j$ of receiving a better outcome) increases its decision weight ϕ_i. It is equivalent to requiring $\pi(p)$ to be concave. Similarly, pessimism is equivalent to requiring $\pi(p)$ to be convex. I therefore assume the following:

A0. For all $p \in (0,1)$, either (i) $\pi''(p) \leq 0$ ("optimism"); or (ii) $\pi''(p) \geq 0$ ("pessimism").

A second feature I capture in $\pi(p)$ is the dichotomous nature of preferences over small probabilities. As discussed in the Introduction, some decisionmakers massively overweight small probabilities, implying $\lim_{p\downarrow 0} w(p)/p = \lim_{p\downarrow 0} w'(p) = +\infty$, while others all but ignore them, implying $\lim_{p\downarrow 0} w'(p) = 0$. Consistent with this point, all the widely-employed functionals in the literature on probability weighting (e.g., Goldstein and Einhorn, 1987; Tversky and Kahneman, 1992; Prelec, 1998) are consistent with either $\lim_{p\downarrow 0} w'(p) = +\infty$ or $\lim_{p\downarrow 0} w'(p) = 0$, depending on the values taken by auxiliary parameters.[4] I therefore assume the following:

[4]For example, Goldstein and Einhorn (1987) propose $w(p) = \delta p^{\gamma}/[\delta p^{\gamma} + [1-p]^{\gamma}]$, $\delta > 0$; Tver-

A1. $\lim_{p\downarrow 0} \pi'(p) \in \{0, +\infty\}$.

3.2 Stochastic Reference Dependence

Following Kőszegi and Rabin (2006), the utility of an outcome x is judged both in absolute and relative to a reference lottery R:

$$u(x|R) = v(x) + \Delta(x|R). \tag{2}$$

The first term in (2), $v(x)$, is absolute utility. As is conventional, v is an increasing and concave function, unique up to a positive affine transformation. The second term, $\Delta(x|R)$, measures "gain-loss utility."[5]

Gain-loss utility $\Delta(x|R)$ is specified to reflect the idea that an outcome x is compared to every outcome that might occur in the reference lottery. Let the reference lottery be a random variable taking values $\{r_1, \ldots, r_m\}$, ordered such that $r_1 < r_2 < \ldots < r_m$. Each outcome r_i occurs with probability $q_i \in [0, 1]$, $\sum_{j=1}^{n} q_j = 1$. As preferences over optimism/pessimism, as described by $\pi(p)$, apply to both the choice and reference lotteries, $\Delta(x|R)$ writes as

$$\Delta(x|R) = \sum_{j=1}^{m} \phi_j \mu(v(x) - v(r_j)), \tag{3}$$

where the function μ is Kőszegi-Rabin's "universal gain-loss function." As in many contexts, I adopt a piecewise-linear specification for μ:

sky and Kahneman (1992) propose $w(p) = p^\gamma / [p^\gamma + [1-p]^\gamma]^{1/\gamma}$; and Prelec (1998) proposes $w(p) = \exp(-[-\log p]^\gamma)$. Each of these functionals satisfies $\lim_{p\downarrow 0} w'(p) = +\infty$ for $\gamma < 1$ and satisfies $\lim_{p\downarrow 0} w'(p) = 0$ for $\gamma > 1$.

[5] Kőszegi and Rabin (2006) include in $u(x|R)$ a parameter $\eta > 0$ that weighs the strength of the gain-loss utility component, so $u(x|R) = v(x) + \eta\Delta(x|R)$. In this case, results must be discussed in the quantity $\Lambda \equiv \eta\lambda$, variously referred to as the *behavioral* or *de facto* coefficient of loss aversion. To discuss results in terms of λ (rather than Λ), as is common in the literature, the specification in (2) may be thought of as adopting the identifying assumption $\eta = 1$.

$$\mu(z) = \begin{cases} z & \text{if } z \geq 0; \\ [1+\lambda] z & \text{otherwise;} \end{cases} \qquad \lambda \in (-1, 1). \tag{4}$$

When $\lambda > 0$ the specification of μ in (4) captures the psychological concept of loss aversion, according to which losses loom larger than equivalent gains (Kahneman and Tversky, 1979). Loss neutrality is implied when $\lambda = 0$, while $\lambda < 0$ indicates loss tolerant preferences. The restriction $\lambda \in (-1, 1)$ ensures that choices satisfy first-order stochastic dominance (Masatlioglu and Raymond, 2016). Note that the Kőszegi-Rabin (expectation-based) formulation of loss aversion differs from the (status-quo-based) formulation of Kahneman and Tversky (1979) in that it is defined relative to an endogenous reference lottery, rather than to a fixed reference outcome. Although attempts to measure expectation-based loss aversion in the framework of (2)-(4) are in their infancy, Goette *et al.* (2018) report two such estimates of λ, both of which are positive. The wider literature on characterizing gain-loss preferences, broadly defined, also finds much evidence of loss averse decisionmakers, but also of heterogeneity, with some decisionmakers showing evidence of loss tolerant preferences (Brown *et al.*, forthcoming).

A key insight of Kőszegi and Rabin (2006) is to interpret the reference lottery R as the decisionmaker's expectation over the lottery X. In their *choice-acclimating personal equilibrium* (Kőszegi and Rabin, 2007) they argue that if a decisionmaker commits to a lottery X well in advance of the resolution of uncertainty, then by the time the uncertainty is resolved the decisionmaker will have come to expect the lottery X. Hence, R coincides with X. Under this interpretation of R, (2) becomes

$$u(x_i|X) = v(x_i) + \sum_{j=1}^{i} \phi_j [v(x_i) - v(x_j)] - [1+\lambda] \sum_{j=i}^{n} \phi_j [v(x_j) - v(x_i)]. \tag{5}$$

3.3 Rank-Dependent Representation

Evaluating $u(x_i|X)$ in (5) over all outcomes in the choice lottery X gives the ex ante evaluation of the lottery as

$$U(X|X) = \sum_{i=1}^{n} \phi_i u(x_i|X).$$

To admit a RDEU representation, there must be a valid weighting function $w : [0,1] \mapsto [0,1]$ such that $U(X|X)$ writes in the form

$$U(X|X) = \sum_{i=1}^{n} \omega_i v(x_i),$$

where

$$\omega_i = w\left(\sum_{j=i}^{n} p_j\right) - w\left(\sum_{j=i+1}^{n} p_j\right).$$

I term the function w the *composite PWF*. It combines information concerning both the decisionmaker's preferences over probability distributions and for loss aversion/tolerance:

Lemma 1 *(Barseghyan et al., 2018) $U(X|X)$ admits a RDEU representation with a composite PWF, $w(p)$, of the form*

$$w(p) = \pi(p) - \lambda[1 - \pi(p)]\pi(p). \tag{6}$$

Lemma 1 breaks down the composite PWF into two terms. The first, $\pi(p)$, captures preferences over probability distributions (risk). The second term, $\lambda[1 - \pi(p)]\pi(p)$, captures gain-loss preferences. When gain-loss preferences are symmetric (i.e., loss neutral) this term is zero, and otherwise takes the sign of λ. As such, this term captures the effect due to *asymmetric* gain-loss preferences. Under (6), greater salience of losses, as measured by λ, reduces the elevation of the composite PWF ($\partial w(p)/\partial\lambda < 0$ for all $p \in (0,1)$), as does an increase in pessimism, in the sense of a convex transformation of $\pi(p)$. A decrease in the salience of losses and an increase in optimism instead both increase the elevation of the composite PWF.[6] In respect of small probabilities, the following Lemma shows that the composite PWF $w(p)$ has identical properties to that of $\pi(p)$.

Lemma 2 *The composite PWF satisfies* $\lim_{p\downarrow 0} w'(p) \in \{0, +\infty\}$.

The proof of Lemma 2 is straightforward, relying on the exact proportionality between $w'(p)$ and $\pi'(p)$ for small probabilities.

Despite the formal equivalence between the model presented here and RDEU, there are two important distinctions in the underlying intuition. To facilitate a discussion of these points, let "classical" RDEU refer to the special case of (6) with $\lambda = 0$, such that $w(p) = \pi(p)$. First, in the classical case, the equivalence of $w(p)$ and $\pi(p)$ implies that the PWF is a primitive of the model. When $\lambda \neq 0$, however, $w(p)$ is a derived concept, distinct from the primitive $\pi(p)$. Second, in classical RDEU, preferences consist of two independent and clearly separate components: (i) sensitivity towards outcomes, modeled through utility, and (ii) sensitivity towards risk, modelled through a PWF. This generalizes expected utility theory, where preferences are captured solely through utility. To these two components of classical RDEU the present model adds a third component: sensitivity to gains and losses, modelled through stochastic reference dependence. This point is important for the way behavior relates to the underlying features of preferences: behavior must be understood as arising from the simultaneous interplay of multiple components within the model. For example, under expected utility, diminishing marginal utility is necessary and sufficient for aversion to mean preserving spreads (Rothschild and Stiglitz, 1970). This behavioral link is preserved in classical RDEU only when also $\pi'' \geq 0$, i.e., the decisionmaker is pessimistic

[6] The models of Masatlioglu and Raymond (2016), Delquié and Cillo (2006), and quadratic utility (Chew *et al.*, 1991) may be seen as special cases of (6) when $\pi(p) = p$. Expected utility is the special case in which $\pi(p) = p$ and $\lambda = 0$.

(Chew *et al.*, 1987). In Section 4 we shall show that, in the present framework, preferences towards losses also mediate this behavioral link.

4 Preference Inferences from the Composite PWF

In this section I explore the implications of the model developed in Section 3 for risk aversion and the shape of the PWF.

4.1 Risk Preferences

A connection between pessimism and risk aversion under RDEU has been understood since Chew *et al.* (1987): the former is necessary for the latter when utility is concave. Building on this finding, in this section I establish that, in the present framework, loss aversion also connects to risk aversion. A decisionmaker displays risk aversion if, given any two lotteries X and Y, where Y differs from X by a mean-preserving spread, the decisionmaker prefers X to Y.

Proposition 1 *Under the model of Section 3, the decisionmaker is risk averse if and only if they are loss averse and pessimistic.*

Proposition 1 clarifies that, in the present framework, loss aversion is necessary for risk aversion. To understand this result intuitively, recall that, under classical RDEU ($\lambda = 0$) with diminishing marginal utility, $w'' \geq 0$ is necessary and sufficient for risk aversion (Chew *et al.*, 1987). As it will also hold that the sign of w'' is the sign of π'', risk aversion manifests in classical RDEU if and only if the decisionmaker is pessimistic ($\pi'' \geq 0$). In the present model, however, the sign of w'' need not be the sign of π'' when $\lambda \neq 0$. Instead, the shape of the composite PWF will reflect not only preferences over probability distributions, but also preferences over gains and losses. Accordingly, $\pi'' \geq 0$ must be accompanied by $\lambda > 0$ to ensure $w'' \geq 0$ in this case.

4.2 Probability Weighting

I now analyze the possible shapes of the composite PWF in (6), and relate these shapes to the underlying preferences for optimism/pessimism and loss aversion/tolerance. The common forms of the PWF observed experimentally, i.e., inverse-S, S-shaped, concave, and convex, are distinguished in the model with respect to the properties of their first and second derivatives as in Table 1:

	$\lim_{p\downarrow 0} w'(p)$	$\lim_{p\downarrow 0} w''(p)$	$\lim_{p\downarrow 0} w''(1-p)$
Inverse-S	$+\infty$	< 0	> 0
S	0	> 0	< 0
Concave	$+\infty$	≤ 0	≤ 0
Convex	0	≥ 0	≥ 0

Table 1: Properties of $w'(p)$ and $w''(p)$ for different shapes of the composite PWF

In interpreting the first column of Table 1, note that, for small probabilities, $w'(p)$ measures the extent of probability under- or over-weighting. Specifically, by L'Hôpital's rule,

$$\lim_{p\downarrow 0} w'(p) = \lim_{p\downarrow 0} \frac{w(p)}{p}.$$

Accordingly, $\lim_{p\downarrow 0} w'(p) > 1$ ($\lim_{p\downarrow 0} w'(p) < 1$) should be interpreted as implying the over-weighting (under-weighting) of small probabilities. The second two columns in Table 1 relate to the second derivative. For the concave and convex shapes of PWF, $w''(p)$ cannot switch sign on $p \in (0, 1)$, whereas for the inverse-S and S-shapes $w''(p)$ must switch sign on this interval. In particular, for the inverse-S (S) shape, $w''(p) < 0$ ($w''(p) > 0$) at low p, switching to $w''(p) > 0$ ($w''(p) < 0$) at high p.

As discussed in the Introduction, if one is unwilling to impose any further interpretation upon $\pi(p)$ other than as an arbitrary weighting function, then observing a decisionmaker's (composite) PWF does not, in general, permit the separate identification of λ and $\pi(p)$. When giving additional interpretation, the strongest set of restrictions one might lay on $\pi(p)$ would be those sufficient to generate full preference inference, i.e., an invertible mapping from every $w(p)$ to a pair $\{\lambda, \pi(p)\}$. This would, however, come at the price of greatly limiting the flexibility the model. Weaker sets of restrictions would retain greater flexibility, but allow only partial inference – for instance, relating each $w(p)$ to a set of possible $\{\lambda, \pi(p)\}$. The restrictions embodied in assumptions A0 and A1 are of this second type. They acknowledge the flexibility-inference trade-off in that the model is sufficiently flexible as to be able to predict all the the common shapes of PWF observed in empirical and experimental studies, and yet sufficiently specific as to permit some (albeit not full) preference inference.

Proposition 2 *The composite PWF $w(p)$ in (6) admits the following preference inferences: If $w(p)$ is*

(i) *inverse-S, then the decisionmaker is loss averse and optimistic;*

(ii) *S-shaped, then the decisionmaker is loss tolerant and pessimistic;*

(iii) *concave, then the decisionmaker is optimistic;*

(iv) *convex, then the decisionmaker is pessimistic.*

Part (i) of Proposition 2 associates the inverse-S PWF to a unique underlying preference profile: to observe this shape of PWF loss aversion is a prerequisite feature of preferences. Alongside loss aversion, an inverse-S decisionmaker must also be optimistic. No other underlying psychology is consistent with an inverse-S PWF. Together, Propositions 1 and 2 reveal that loss aversion lies behind both risk aversion and the inverse-S PWF. That is, loss aversion appears to unite two central, and apparently disjoint, phenomena.

Note that the ability of the model to generate an inverse-S PWF relies both on optimism/pessimism preferences *and* gain-loss preferences. On their own, the optimism/pessimism preferences in A0 cannot generate an inverse-S shape for $w(p)$. If one allows in (6) for optimism/pessimism without asymmetric gain-loss preferences (by setting $\lambda = 0$ as in classical RDEU) then $w(p) = \pi(p)$. An inverse-S shape for $w(p)$ is ruled out as $\pi(p)$ cannot take an inverse-S shape under A0.[7] Also, on their own, gain-loss preferences are also inconsistent with an inverse-S shape for $w(p)$. When allowing for gain-loss preferences without optimism/pessimism (by setting $\pi(p) = p$) the composite PWF reduces to the linear-quadratic form $w(p) = \{1 - \lambda[1 - p]\} p$. In this case, the sign of $w''(p)$ is the sign of λ, so again the composite PWF cannot be inverse-S.

The remaining parts of Proposition 2 associate other commonly observed shapes of PWF to preference profiles. An S-shaped PWF is associated uniquely with loss tolerance and pessimism, while concave and convex PWFs are consistent both with loss aversion and loss tolerance.

What inference is possible in the opposite direction to Proposition 2, i.e., from $\{\lambda, \pi(p)\}$ to the shape of $w(p)$?

Corollary 1 *The preference profile* $\{\lambda, \pi(p)\}$ *admits the following inferences for the composite PWF* $w(p)$ *in (6): If the decisionmaker is*

(i) *loss averse, then* $w(p)$ *cannot be S-shaped;*

(ii) *loss tolerant, then* $w(p)$ *cannot be inverse-S shaped;*

[7] Only under what Neilson (2003, p. 181) describes as a "rather strange pattern of optimism and pessimism" in which a decisionmaker may be at once both optimistic *and* pessimistic with respect to different outcomes in the same gamble, can the inverse-S PWF be explained in terms of optimism and pessimism alone. Such a pattern is inconsistent with optimism/pessimism being stable individual traits, as suggested by experimental evidence, and is therefore ruled out by assumption A0.

(iii) *optimistic, then* $w(p)$ *cannot be convex or S-shaped;*

(iv) *pessimistic, then* $w(p)$ *cannot be concave or inverse-S shaped.*

According to Corollary 1, certain shapes of the composite PWF are ruled out for particular preference profiles. In particular, pessimism is incompatible with the inverse-S PWF.

<Figure 1 here – see p. 29>

An illustration of the findings of Propositions 1 and 2, as well as Corollary 1, is given in Figure 1 (see p. 29) for a simple choice of optimism-pessimism function: $\pi(p) = p^{1+\gamma}$, $\gamma > -1$, $\gamma \neq 0$. The parameter γ may be considered as an index of pessimism: $\gamma < 0$ corresponds to optimism while $\gamma > 0$ corresponds to pessimism. Figure 1 shows the preference combinations that induce each shape of composite PWF in Table 1, while the set of preference combinations that induce risk aversion is the gray-shaded area. It is seen that an inverse-S PWF arises for loss aversion and optimism when the former preference is sufficiently strong and the latter preference is sufficiently weak. Preference combinations associated with risk aversion are distinct from those associated with an inverse-S PWF, for a decisionmaker with an inverse-S PWF will be amenable to some mean-preserving spreads. An S-shaped PWF is associated with sufficiently strong loss tolerance and sufficiently weak pessimism. A convex PWF, which is observed in experimental data relatively frequently, requires pessimism, so occurs only in the left-half of Figure 1. This shape of PWF, however, is compatible with sufficiently weak loss aversion or outright loss tolerance. Similarly, a concave PWF requires optimism, but is compatible with both loss aversion and loss tolerance. In this way, a preference interpretation may be given to the commonly observed shapes of PWF.

5 Probability Weighting of Absolute Losses

The widely utilized PT model of Tversky and Kahneman (1992) allows that decisionmakers may weight probabilities attached to negative outcomes in lotteries (absolute losses) differently from those attached to positive outcomes (absolute gains).[8] Thus, it posits separate PWFs for gains (w^+) and losses (w^-). The model of section 3 can readily be adapted to yield this feature: one merely duplicates the analysis across the separate gain and loss domains. PT, however, also posits that, different from the model of section 3, the optimism-pessimism function over absolute losses is formed with respect to the cumulative, rather than decumulative, distribution. In the terminology of Diecidue and Wakker (2001), use of the cumulative distribution is the *bad-news* transformation, in the sense that the decision-maker's optimism-pessimism function is formed with consideration to outcomes worse than the realized outcome. Similarly, the use of the decumulative distribution is referred to as the *good-news* transformation. This section explicates the implications of this difference between models and shows that the preference inferences of the previous section extend to the PT setting.

To permit separate composite PWFs for gains and losses, this section considers the case of a mixed lottery X with loss component X^- and gain component X^+. Relegating the details – which mostly entail duplicating each aspect of the model in section 3 across two domains – to Appendix A, I arrive at the following representation result:

Proposition 3 *$U(X|X)$ admits a representation of the form in PT, with composite probability weighting functions $\{w^-, w^+\}$ given by*

$$w^-(p) = \left[1 - \pi^-(1-p)\right]\left[1 - \lambda\pi^-(1-p)\right]; \qquad w^+(p) = \left\{1 - \lambda\left[1 - \pi^+(p)\right]\right\}\pi^+(p).$$

[8]The "absolute" terminology here is intended to draw a clear distinction with expectation-based gains and losses, where the benchmark for judging gains and losses is typically non-zero.

The form of $w^+(p)$ in Proposition 3 is identical to that in (6). The form of $w^-(p)$ is, however, impacted by use of the bad-news transformation. Thus, preference consistency across the gain and loss domains, i.e., $w^-(p) = w^+(p)$, does not hold for $\lambda^- = \lambda^+$ and $\pi^-(p) = \pi^+(p)$. Instead, it holds for $\lambda^- = \lambda^+$ and $\pi^-(p) = 1 - \pi^+(1-p)$. Initially, this might seem consequential for preference inference, as, for instance, if $\pi^+(p)$ is concave, then $\pi^-(p)$ is convex. Note, however, that – as well as changing the form of $w^-(p)$ – use of the bad-news transformation also changes the relationship between optimism/pessimism (in the sense of whether the decisionmaker focuses on good or bad outcomes) and the shape of the optimism-pessimism function. As I show formally in Appendix A, under the bad-news transformation, optimism (pessimism) is equivalent to convexity (concavity) of $\pi^-(p)$. Thus, use of the bad-news transformation over absolute losses has two distinct effects. As these effects counteract perfectly, however, I have:

Proposition 4 *Under the model of Appendix A, the preference inferences in Proposition 1 (risk aversion) Proposition 2 (shape of PWF) hold for both $w^-(p)$ and $w^+(p)$ in Proposition 3.*

Proposition 4 demonstrates that the preference inferences possible in the framework of Section 3 carry over to a PT framework. In particular, when both $\{w^-, w^+\}$ are observed to be inverse-S, as found commonly in the experimental literature (e.g., Tversky and Kahneman, 1992; Abdellaoui, 2000), the unique preference inference – in both the gain and loss domains – is loss aversion coupled with optimism.

6 Conclusion

This paper explores the idea that the PWF is not itself a primitive, but is instead a concept that derives from a set more basic and interpretable primitives. I construct different shapes of PWF for decisionmakers who (i) exhibit stable optimistic/pessimistic preferences over probability distributions; (ii) either overweight or underweight small probabilities; and (iii) exhibit gain/loss preferences that can be either loss averse or loss tolerant. These three features of the model not only determine the shape of the PWF, but also the presence or absence of risk aversion under diminishing marginal utility.

The key finding is that loss aversion importantly mediates both risk attitudes and the shape of the PWF. In particular, loss aversion is necessary for risk aversion when marginal utility is diminishing, in the sense of aversion to mean preserving spreads, and also necessary to observe the inverse-S probability weighting function. Risk aversion is, under concave utility, equivalent to loss aversion and pessimism, while the inverse-S probability weighting function can arise only when the decisionmaker exhibits loss aversion and optimism. As such, I forge a link between loss aversion, risk aversion and the inverse-S probability weighting function. That loss aversion emerges as a primitive concept that underlies more complex concepts, such as risk aversion and the inverse-S PWF, is consistent with ideas advanced by several researchers that loss aversion may have evolved early in human evolution. In an environment where humans must forage for food, a loss may result in death if certain needs are no longer met, whereas a gain "merely" increases the chances of survival (e.g., Aktipis and Kurzban, 2004; McDermott *et al.*, 2008). Given these asymmetric fitness consequences of equivalent gains and losses, "organisms should be designed to be more concerned with avoiding losses than with realizing gains" (Aktipis and Kurzban, 2004, p. 143).

Although the present contribution is a theoretical one, the predicted consistency between, e.g., loss aversion and the inverse-S probability weighting function is one that can be tested with experimental choice data that permits simultaneous estimation of the PWF and the coefficient of loss aversion at the level of individual subjects. There also exist techniques for direct psychometric measurement of dispositional optimism and pessimism (Herzberg *et al.*, 2006), and these are beginning to be reported in large-scale surveys (e.g., the Health and

Retirement Study). Thus, there may be scope to collect both psychometric and choice data within an experimental study, thus permitting a full test of the theory.

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Appendix A: Probability Weighting of Losses

Here I develop a model of decisionmaking consistent with a PT representation. To permit separate composite PWFs for (absolute) gains and losses, I develop the model for the case of a mixed lottery X with loss component X^- and gain component X^+. The lottery X^- takes values $\{x_1^-, x_2^-, \ldots, x_{n^-}^-, 0\}$, $n^- \geq 2$, and the lottery X^+ takes values $\{0, x_1^+, x_2^+, \ldots, x_{n^+}^+\}$, $n^+ \geq 2$. The values taken by X^- are ordered such that $x_1^- < x_2^- < \ldots < x_{n^-}^- < 0$, and the values taken by X^+ are ordered such that $0 < x_1^+ < x_2^+ < \ldots < x_n^+$. Outcome x_i^z, $z \in \{-, +\}$, occurs with probability p_i^z and the outcome zero occurs with probability $1 - \sum_{j=1}^{n^z} p_j^z$.

Decision weights are generated separately for the gain and loss components with respect to the weighting functions $\pi^z(p)$, $z \in \{-,+\}$. Both $\pi^-(p)$ and $\pi^+(p)$ are strictly increasing, continuous everywhere, and twice differentiable, except possibly at $p \in \{0,1\}$. Following Tversky and Kahneman (1992), I suppose that, in the loss domain, each decision weight π_i^- is formed according to the so-called *bad-news* transformation (Diecidue and Wakker, 2001), which reflects the probability weight on outcomes worse than x_i^-. By contrast, in the gain domain I retain the *good-news* transformation given in (1), which reflects the probability weight on outcomes better than x_i^+. Thus,

$$\begin{aligned}
\phi_i^- &= \pi^-\left(\sum_{j=1}^{i} p_j^-\right) - \pi^-\left(\sum_{j=1}^{i-1} p_j^-\right) \qquad i \in \{1,2,\ldots,n^-\};\\
\phi_i^+ &= \pi^+\left(\sum_{j=i}^{n^+} p_j^+\right) - \pi^+\left(\sum_{j=i+1}^{n^+} p_j^+\right) \qquad i \in \{1,2,\ldots,n^+\}.
\end{aligned}$$

Note that, in the loss domain, the psychological notion of optimism requires that the decision weight ϕ_i^- increases as outcome i is improved, such that the weight on outcomes lower than i, $\sum_{j=1}^{i-1} p_j^-$, increases. Thus, different from in the gain domain, optimism is associated with $\phi_i^{-''} \geq 0$, while pessimism is associated with $\phi_i^{-''} \leq 0$.

Whereas the specification of reference-dependent preferences in Section 3 supposes that an outcome is compared to every alternative outcome in X, here losses are compared only to other losses, and gains compared only to other gains. Thus, the utility of an outcome $x_i^z \in X^z$, $z \in \{-,+\}$, is given by

$$u\left(x_i^z | X^z\right) = v\left(x_i^z\right) + \sum_{j=1}^{i} \phi_j^z \left[v\left(x_i^z\right) - v\left(x_j^z\right)\right] - [1+\lambda] \sum_{j=i}^{n^z} \phi_j^z \left[v\left(x_j^z\right) - v\left(x_i^z\right)\right].$$

The ex ante evaluation of lottery X is therefore

$$U(X|X) = \sum_{i=1}^{n^-} \phi_i^- u\left(x_i^- | X^-\right) + \sum_{i=1}^{n^+} \phi_i^+ u\left(x_i^+ | X^+\right). \tag{A.1}$$

To admit a representation of the form in PT, it must be possible to find valid weighting functions $\{w^-, w^+\}$ such that $U(X|X)$ writes as

$$U(X|X) = \sum_{i=1}^{n^-} w^-(p)\, v\left(x_i^-\right) + \sum_{i=1}^{n^+} w^+(p)\, v\left(x_i^+\right).$$

I demonstrate in Proposition 3 that $U(X|X)$ can indeed be written in this form. The restrictions placed on $\pi^-(.)$ and $\pi^+(.)$, analogous to A0 and A1 in section 3, are:

A0′. For all $p \in (0,1)$, either

(i) $\pi^{+\prime\prime}(p) \geq 0$ or $\pi^{+\prime\prime}(p) \leq 0$;

(ii) $\pi^{-\prime\prime}(p) \leq 0$ or $\pi^{+\prime\prime}(p) \geq 0$.

A1′. (i) $\lim_{p\downarrow 0} \pi^{+\prime}(p) \in \{0, +\infty\}$

(ii) $\lim_{p\downarrow 0} \pi^{-\prime}(1-p) \in \{0, +\infty\}$.

Assumption A0′ endows both π^- and π^+ with an interpretation as optimism-pessimism functions, analogous to Assumption A0. Assumption A1′ captures perceptions of extreme probabilities, analogous to Assumption A1.

Appendix B: Proofs

Proof of Lemma 1. See p. 13 of the online appendix to Barseghyan *et al.* (2018) (https://www.aeaweb.org/content/file?id=7267). ■

Proof of Lemma 2. The first derivative of (6)is given in (B.1) below. Thus, $\lim_{p\downarrow 0} w'(p) = [1-\lambda]\lim_{p\downarrow 0} \pi'(p)$. As $\lim_{p\downarrow 0} w'(p) \propto \lim_{p\downarrow 0} \pi'(p)$ it follows that $\lim_{p\downarrow 0} w'(p) \in \{0, +\infty\}$ if and only if $\lim_{p\downarrow 0} \pi'(p) \in \{0, +\infty\}$. ■

Proof of Proposition 1. By Corollary 2 of Chew *et al.* (1987), risk aversion holds if and only if $v'' \leq 0$ and $w'' \geq 0$. As $v'' \leq 0$ by assumption, proof of the proposition follows if $w'' \geq 0$ if and only if $\pi' \geq 0$ (pessimism) and $\lambda \geq 0$ (weak loss aversion). From (6), it holds that, where well-defined:

$$w'(p) = [1 - \lambda + 2\lambda\pi(p)]\, \pi'(p)\,; \tag{B.1}$$

$$w''(p) = \frac{w'(p)}{\pi'(p)}\pi''(p) + 2\lambda\,[\pi'(p)]^2\,. \tag{B.2}$$

As $w'(p)/\pi'(p) > 0$ and $2\,[\pi'(p)]^2 > 0$ it follows that $w''(p) \geq 0$ if $\pi' \geq 0$ and $\lambda \geq 0$. Going the other way, if $\lambda < 0$ then there exists an interval for $\pi''(p)$, $0 \leq \pi''(p) < -2\lambda\,[\pi'(p)]^3 / w'(p)$, such that risk aversion fails. If $\pi''(p) < 0$ then there exists an interval for λ, $-(1/2)\, w'(p)\,\pi''(p)/\,[\pi'(p)]^3 > \lambda \geq 0$, such that risk aversion fails. Thus, $w'' \geq 0$ if and only if $\pi' \geq 0$ and $\lambda \geq 0$. ■

Proof of Proposition 2. First, note that when the composite PWF is inverse-S or S-shaped it holds that $w''(p)$, given by (B.2), must switch sign at least once. Note that the first term in (B.2) takes the sign of π'' and the second term takes the sign of λ. Thus, if $\operatorname{sign}(\pi'') = \operatorname{sign}(\lambda)$ then (B.2) cannot switch sign. When $w(p)$ is inverse-S or S-shaped, therefore, it must be that $-\operatorname{sign}(\pi'') = \operatorname{sign}(\lambda)$. Second, from (B.1), we have $\lim_{p\downarrow 0} w'(p) = [1-\lambda]\lim_{p\downarrow 0} \pi'(p)$, so $\lim_{p\downarrow 0} w'(p) = 0 \Leftrightarrow \lim_{p\downarrow 0} \pi'(p) = 0$. $\lim_{p\downarrow 0} \pi'(p) = 0$ is incompatible with $\pi(p)$ being concave: $p = 0$ would then be a local maximum for $\pi(p)$, creating a contradiction as $\pi(1) = 1 > \pi(0) = 0$. Under A0, as $\pi(p)$ is not concave it must be convex. Moreover, as A1 precludes $\pi(p)$ being linear, $\pi(p)$ must be strictly convex. Hence $\lim_{p\downarrow 0} w'(p) = 0 \implies \pi''(p) > 0$. To go in the other direction, suppose $\pi''(p) > 0$. Were $\lim_{p\downarrow 0} w'(p) = +\infty$ then $w'(p)$ is infinite for all p, which is a contradiction, as the average gradient of a probability weighting function cannot exceed one. It follows from A1 that $\lim_{p\downarrow 0} w'(p) = 0$. Taken together, the last two arguments prove $\lim_{p\downarrow 0} w'(p) = 0 \Leftrightarrow \pi''(p) >$

0. Similar arguments establish that $\lim_{p\downarrow 0} w'(p) = +\infty \Leftrightarrow \pi''(p) < 0$. For the last argument we establish that when $w(p)$ is inverse-S it must be that $\lim_{p\downarrow 0} w'(p) = +\infty$. Were this not the case then, by A1, it would hold that $\lim_{p\downarrow 0} w'(p) = 0$. But then $\lim_{p\downarrow 0} w''(p) < 0$, as required for an inverse-S shape, results in a local violation of monotonicity. By similar arguments, when $w(p)$ is S-shaped it must be that $\lim_{p\downarrow 0} w'(p) = 0$. Putting these arguments together, for an inverse-S composite PWF it must be that $\lim_{p\downarrow 0} \pi'(p) = +\infty$, which requires $\pi'' < 0$ (optimism), and also that w'' switches signs, which requires that $\operatorname{sign}(\lambda) = -\operatorname{sign}(\pi'') > 0$. Thus, an inverse-S composite PWF associates uniquely with loss aversion and optimism (part *i* of the proposition). For an S-shaped composite PWF it must be that $\lim_{p\downarrow 0} \pi'(p) = 0$, which requires $\pi'' > 0$ (pessimism), and also that w'' switches signs, which requires that $\operatorname{sign}(\lambda) = -\operatorname{sign}(\pi'') < 0$. Hence it associates uniquely with pessimism and loss tolerance (part *ii*). If $w(p)$ is concave then it is straightforward to use prior arguments to show that $\lim_{p\downarrow 0} w'(p) = +\infty$. As we have established that $\lim_{p\downarrow 0} w'(p) = +\infty \Leftrightarrow \pi''(p) < 0$ it follows that a concave composite PWF is associated uniquely with optimism (part iii). A similar argument relates a convex composite PWF uniquely with pessimism (part iv). ■

Proof of Proposition 3. The form of $w^+(p)$ is identical to that in (6). Hence the proof of Lemma 1 applies. From (6), if, counterfactually, the good-news transformation applied in the loss domain then $U(X|X)$ would admit a PT representation in the loss domain with composite weighting function

$$w^-(p) = \left\{1 - \lambda^- \left[1 - \pi^-(p)\right]\right\} \pi^-(p). \tag{B.3}$$

Note that probability weighting according to the good-news weighting function $\pi^-(p)$ is equivalent to probability weighting according to the bad-news weighting function $1 - \pi^-(1-p)$. As the bad-news transformation holds in the loss domain, substituting $\pi^-(p) = 1 - \pi^-(1-p)$ in (B.3) yields the expression for $w^-(p)$ in the proposition. ■

Proof of Proposition 4. The proof of the properties of $w^+(p)$ is identical to the proof of Proposition 2. The proof of the properties of $w^-(p)$ is also analogous to the proof of Proposition 2. In particular, note that

$$w^{-\prime\prime}(p) = -\frac{w^{-\prime}(p)}{\pi^{-\prime}(1-p)} \pi^{-\prime\prime}(1-p) + 2\lambda^- \left[\pi^{-\prime}(1-p)\right]^2, \tag{B.4}$$

such that, for $w^{-\prime\prime}(p)$ to switch sign on $p \in (0,1)$ it must be that $\operatorname{sign}(\pi^{-\prime\prime}) = \operatorname{sign}(\lambda)$. Also, from (B.1), we have $\lim_{p\downarrow 0} w^{-\prime}(p) = \left[1 - \lambda^-\right] \lim_{p\downarrow 0} \pi^{-\prime}(1-p)$, so $\lim_{p\downarrow 0} w^{-\prime}(p) = 0 \Leftrightarrow \lim_{p\downarrow 0} \pi^{-\prime}(1-p) = 0$. $\lim_{p\downarrow 0} \pi^{-\prime}(1-p) = 0$ is incompatible with $\pi^-(p)$ being convex: $p = 1$ would then be a local minimum for $\pi^-(p)$, creating a contradiction as $\pi^-(0) = 0 < \pi^-(1) = 1$. Under A0′(*ii*), as $\pi^-(p)$ is not convex it must be concave (and, by A1′(*ii*), strictly concave), hence $\lim_{p\downarrow 0} w^{-\prime}(p) = 0 \Longrightarrow \pi^{-\prime\prime}(p) < 0$. One can then go in the other direction, to establish $\lim_{p\downarrow 0} w'(p) = 0 \Leftrightarrow \pi''(p) < 0$. The remainder of the proof then follows from the arguments in the proof of Proposition 2. ■

Figures

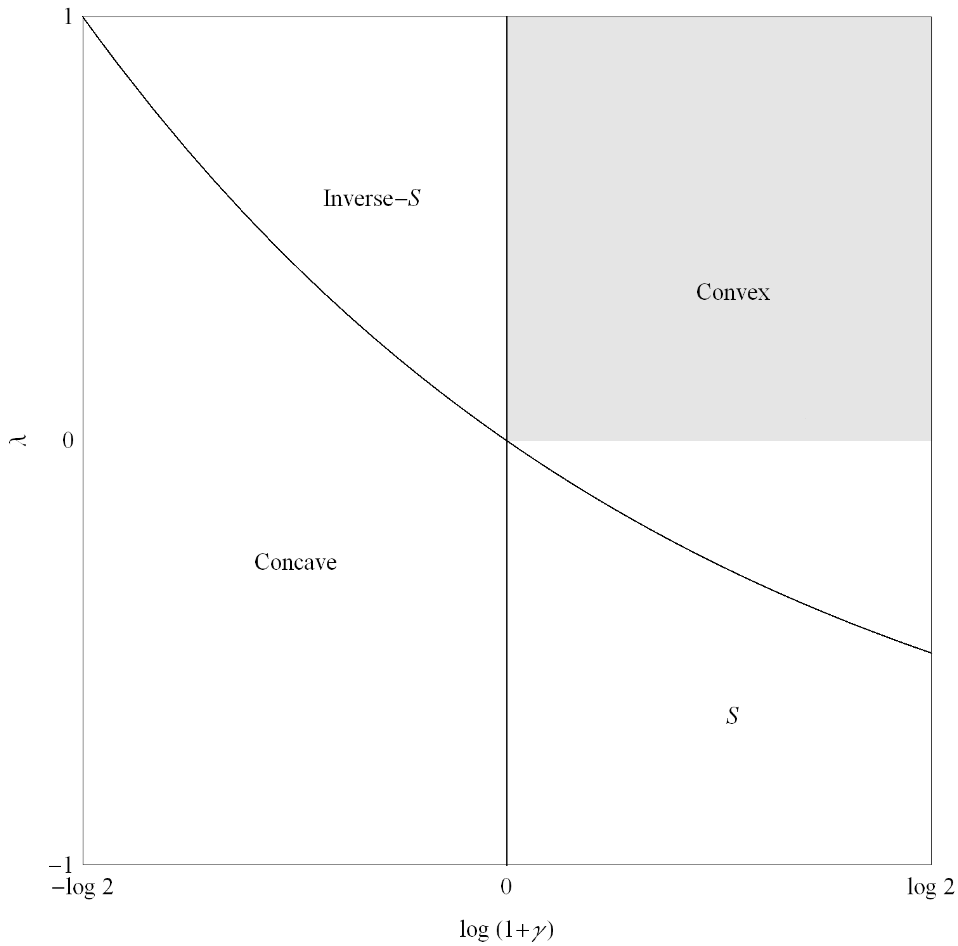

Figure 1: Shapes of the composite PWF in $(\log(1+\gamma), \lambda)$-space for $\pi(p) = p^{1+\gamma}$. Optimism corresponds to $\log(1+\gamma) < 0$ and pessimism to $\log(1+\gamma) > 0$. The shaded region is the region in which risk aversion holds.