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Negative Tax Incidence with Multiproduct Firms*

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Abstract

A fundamental result in the theory of commodity taxation is that taxes increase consumer prices and reduce supply, aggravating the distortions caused by market power. This result hinges on the assumption that each firm provides a single product. We study the effects of commodity taxes in presence of multiproduct firms that have market power. We consider a monopolist providing two goods and obtain simple conditions such that differentiated ad valorem tax reduce the prices and increases the supply of both goods, thereby increasing total surplus. We show that these conditions can hold in a variety of settings, including add-on pricing, multiproduct retailing with price advertising, intertemporal models with switching costs and two-sided markets. Differentiated unit taxes can induce prices to decrease (as the Edgeworth's paradox states), but the quantity of the taxed good always decreases.

JEL Classification: D42, H21, H22

Keywords: Commodity taxation, tax incidence, multi-product firms

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1 Introduction

Almost every firm sells more than one product. Transport companies, such as airlines and train operators, sell passages, baggage allowance and onboard meals. Supermarkets and online stores distribute multiple brands and product categories. Two-sided platforms, that sell different goods to different groups of users, are multiproduct firms as well. For instance, websites, newspapers and TV stations provide content to consumers and ads to firms seeking consumers' attention. A key aspect of multiproduct firms is that, the profitabilities of their goods are interrelated, because the demand for each good depends on the price of the others. As a result, multiproduct firms adopt pricing strategies differing from conventional, single-product, ones (Rhodes, 2015; Armstrong and Vickers, 2018).

Multiproduct suppliers are subject to indirect taxes, often with different tax rates on various goods. For instance, different VAT rates can apply to goods sold by the same retailer (e.g., alcohol and food in a supermarket). Since their pricing strategies differ from single-product firms, it is natural to expect the way multiproduct firms respond to taxation to be different as well. However, the effects of taxes on multiproduct suppliers are largely unexplored. This is the topic we study in this paper. We focus on multiproduct firms with market power and characterize conditions such that (ad valorem) taxation reduces prices, increases supply and expands total surplus. We also provide several simple applications where such conditions hold.

Our analysis considers a monopolist supplying two goods. We assume these goods have separable cost functions, but their demands are interdependent, in the sense that changes in the price of one good affect the demand for the other. These interdependencies may stem from the goods being substitutes or complements, but also from search costs, or externalities across markets as in the case of a two-sided platform. Our model can also accommodate the case where the firm sells a single good, but in two successive periods. In this context, the interdependencies between demand in the two periods may arise from switching costs.

We focus on the effects of ad valorem taxes, which have so far largely been ignored in multiproduct settings. We show that a change in the tax rate on one good has a *direct* effect, which captures how the tax affects the price of a good given the price of the other. Furthermore, there is an *indirect* effect, which captures the change in the price of a good mediated by the tax-induced adjustment in the price of the other. Our first key result is that an ad valorem tax can have a *negative* (i.e., reducing) direct effect on the price of the good it is imposed on. To see why, consider that the tax targets the revenue (price times quantity) the supplier earns from this good. Hence, the supplier has an incentive to reduce such revenue when the tax goes up. The revenue decreases with the price of the taxed good if and only if the equilibrium quantity lies on the elastic part of the demand. A fundamental observation we make is that, unlike a single-product supplier, a multi-product one operates on the *inelastic* part of demand when lowering the price of the taxed good stimulates demand for the other (i.e., the goods are complements), and its marginal cost is small enough. Under these conditions, the direct effect is negative.

When demands are interdependent, a tax imposed on a good also has a direct effect on the price of the other good. This effect is negative if and only if the goods are substitutes: given the price of the taxed good, the supplier can reduce the burden of the tax only by reducing its supply. That is, the supplier wants to reduce the price of the other good if and only if doing so reduces the demand for the taxed good. With interdependent demands, the indirect effect of the tax on a good also matters: this is determined by how the other price is affected (i.e., on the direct effect of the tax on the other price), and by the cross-price derivative of the profit function, which determines whether the two prices move in the same or in opposite directions.

Unit and ad valorem taxes affect prices differently. Although the indirect effects of unit and ad valorem taxes are similar, their direct effects are very different. While the direct effect of an ad valorem tax can be negative as explained above, the direct effect of a unit tax on the taxed good must be positive (i.e., it tends to increase the price). This is because the burden of a unit tax is proportional to the quantity of the good, rather than to its revenue, so the tax induces the supplier to reduce such quantity. Given the price of the other good, this can only be achieved by raising the price of the taxed good. Overall, although both unit and ad valorem taxes can reduce the price of the taxed good, with a unit tax this is only possible if the indirect effect is negative and stronger than the direct effect. Instead, a negative direct effect can reduce the price of both goods with an ad valorem tax.

The discussion above highlights another key difference between unit and ad valorem taxes regarding their effect on output. As mentioned, the supplier reduces the output of the good subject to a unit tax. By contrast, an ad valorem tax can increase the output of both goods when it reduces their price. Again, this difference is due to the tax targeting the good's revenue, rather than its quantity.

The interdependence between demands for multiple goods is the key ingredient driving the novel effects of taxation that we explore. Indeed, with a single-product firm, or if the demands for the goods are independent, the standard effects of taxation apply: the price of the good increases, while supply goes down. This applies to both unit and ad valorem taxes, suggesting that, by ignoring the multi-product nature of firms, one may fail to fully appreciate the differences between these two instruments.

We then focus on the implications of the above findings for tax policy. If the goods are undersupplied in equilibrium (which is often the case when firms have market power), the government should aim to increase the supply by decreasing their prices. As argued above, only differentiated ad valorem taxes may increase the supply of all goods, with unambiguous effects on welfare. When taxation reduces the price of both goods, the (second-best) optimal tax on a single good is strictly positive. This finding is in contrast to the standard prescription - derived in models with single-product suppliers - that the restrictive effects of market power on output can only be addressed with subsidies.

In the course of the analysis, we show that the conditions such that ad valorem taxation results in lower prices and higher supply can hold in several applications including add-on pricing (Ellison, 2005), multiproduct retailing with advertising (Rhodes, 2015), intertemporal markets with switching costs (Klemperer, 1995), and two-sided markets Armstrong (2006). Overall, the results indicate that imposing an ad valorem tax rate on goods sold at a discount is likely to reduce prices and increase supply of both goods. Our applications suggest that goods fitting this description include loss leaders in supermarkets, "base" goods that firms advertise the price of (e.g., low-cost flight tickets) and new customer deals by providers of subscription services (e.g., mobile or landline internet service providers). In two-sided markets, the above description fits the goods on the "discounted" side of the market, e.g., pay-per-view TV carrying advertising.

The remainder of the paper is organized as follows. Section 2 provides a review of the literature. Section 3 describes the model and derives the equilibrium. Section 4 gradually introduces the direct and indirect effects of ad valorem taxes in simplified settings, while Section 5 characterizes these effects in a general environment. We briefly present the effects of other tax instruments (unit taxes and a uniform ad valorem tax) in Section 6.Section 7 studies the welfare-optimal taxes. Section 8 concludes. The parts of the analysis not shown in the main text are relegated to the Appendix.

2 Literature review

As one of the oldest subjects in economics, the incidence of indirect taxes on consumer prices has received much attention in the literature (see, e.g., Fullerton and Metcalf, 2002). Many previous studies on commodity taxation have looked at imperfectly competitive markets (Delipalla and Keen, 1992; Anderson et al., 2001; Auerbach and Hines, 2002), focusing on single-product firms. A fundamental result in this literature is that (unit and ad valorem) taxes raise prices and reduce supply, aggravating the distortions caused by market power. We show that the differences between ad valorem and unit taxes are significantly more pronounced than models with single-product suppliers would suggest. Weyl and Fabinger (2013) provide general principles for the pass-through of production costs (akin to unit taxes) with single-product suppliers. Their analysis points to the role of market competitiveness and curvature of demand as key determinants of pass-through. We consider a multi-product supplier and focus on the role of the interdependency of demands for its products, showing that in this context the pass-through can be negative.

Within the literature on taxation in imperfectly competitive markets, only few papers have shown, in specific settings, that taxation can result in lower prices and higher supply. Cremer and Thisse (1994) show this result in a vertically differentiated oligopoly with endogenous quality, while Carbonnier (2014) considered nonlinear, price-dependent tax schedules. D'Annunzio et al. (2020) show that ad valorem taxes can correct underprovision if differentiated tax rates are applied on to the usage and access parts of a multi-part tariff.

The first author to study taxation with multi-product firms was Edgeworth (1925). He provided an example where a monopolist supplying two substitute goods responds to a unit tax on one good by reducing the price of both. This finding is known as Edgeworth's paradox of taxation, and was later re-elaborated by other authors, including Hotelling (1932), Coase (1946) and Salinger (1991), who focused on unit taxes exclusively. In an analysis developed concurrently and independently to ours, Armstrong and Vickers (2022) provide general conditions for the Edgworth's paradox to occur focusing on unit taxes. Our analysis mainly focuses on ad valorem taxes, showing that in many realistic settings ad valorem taxation can not only reduce prices, but also increase supply and total surplus. Moreover, we show that the goods do not need to be substitutes for this result to occur, unlike with unit taxes (Armstrong and Vickers, 2022).

Although the observation that firms provide multiple products is compelling, only a handful of other studies have investigated the effects of taxation in multiproduct settings. Agrawal and Hoyt (2019) consider tax incidence in a setting with multiple products and perfectly competitive firms. The authors show that taxation (on at least two goods) can result in lower prices if the goods are complements. In their model, suppliers do not internalize the interdependencies between demands for different products (indeed, they have no pricing power at all). Rather, the unconventional effect of taxation stems from the feedback effect that taxes on one good have on the demand for its complements or substitutes. Hamilton (2009) considers an oligopoly with endogenous entry and product breadth. He shows that an ad valorem tax on all commodities raises prices and reduces product breadth, but stimulates output per product and entry in the long run. However, the effects of taxation on welfare are negative. We consider a different setup and focus on the short-run effects of taxation (i.e., given the market structure and product breadth).

Our paper is also related to the literature on taxation of two-sided platforms, a particular kind of multiproduct firms. Kind et al. (2008) show that an ad valorem tax can reduce the prices and stimulate supply by a two-sided platform, due to the externalities across markets. We generalize their result and show that the efficiency-enhancing effect of ad valorem taxes can arise whenever a firm provides multiple goods with interdependent demands, even in absence of externalities across markets.[1]

Recently, industrial economists have looked with renewed interest at the behavior of multiproduct firms, focusing primarily on pricing and the effects of mergers (see, e.g., Chen and Rey, 2012; Rhodes, 2015; Armstrong and Vickers, 2018; Johnson and Rhodes, 2021). Unlike single-product firms, multi-product ones care not only for the price of a good, but also for the structure of their prices across markets. Alexandrov and Bedre-Defolie (2017) extend the LeChatelier-Samuelson principle to multiproduct settings, showing that the (short-run) pass-through of unit taxes when only the directly affected product's price is adjusted can be smaller than the (long-run) pass-through after accounting for adjustments of all the products. We focus on ad valorem taxes and characterize conditions such that pass-through is negative.

Although we concentrate on taxes, we note that they have a similar effect on the behavior of a firm to the fees charged by an upstream provider. Specifically, unit taxes are similar to wholesale prices, whereas ad valorem ones are similar to revenue-sharing arrangements. The empirical literature has provided evidence of negative pass-through of such fees (and costs more generally). Besanko et al. (2005) provide examples of negative pass-through of own- and cross-brand wholesale prices. Also, Froot and Klemperer (1989) find that firms may either increase or decrease their export price in response to an increase in the exchange rate. Luco and Marshall (2020) provide evidence supporting the conjecture that a merger may result in higher prices by a multiproduct supplier (Salinger, 1991), by eliminating double-

[1] More recent contributions include Wang and Wright (2017), who show that ad valorem taxes allow efficient price discrimination across goods with different costs and values on a large marketplace platform, Belleflamme and Toulemonde (2018), who show that ad valorem taxes can result in competing two-sided platforms making higher profits, and Tremblay (2018), who considers taxation at the access and the transaction level.

marginalization. Studying the US carbonated-beverage industry, they conclude that vertical integration increased the price of some products sold by a multiproduct supplier.

3 The model

We consider two goods, 1 and 2, and a numeraire. Let $Q_i(p_1, p_2)$ be the demand function for good $i = 1, 2$, where p_i is the price of good i. Each demand function is non-increasing in p_i, i.e. $\frac{\partial Q_i}{\partial p_i} \leq 0$. Furthermore, if good i is a substitute (resp. complement) to j, with $i \neq j$, then $\frac{\partial Q_i}{\partial p_j} > 0$ (resp. $\frac{\partial Q_j}{\partial p_i} < 0$). We assume the demand functions are twice continuously differentiable.[2] To avoid clutter in the formulas, we omit the argument of the demand functions from now on.

A monopolist supplier, M, provides goods 1 and 2 at constant unit cost c_i. Both goods are subject to indirect taxes that, without loss of generality, we assume fall on the supplier. Therefore, the profit function is

$$\pi(p, T) = \sum_{i=1,2} \left(p_i(1 - t_i) - c_i - \tau_i\right) Q_i, \tag{3.1}$$

where $t_i \leq 1$ is the ad valorem tax rate and τ_i is the unit tax rate on good i. We denote by p the vector of prices, (p_1, p_2), and by T the vector of tax rates, $(t_1, t_2, \tau_1, \tau_2)$. We assume the profit function is concave in p. Note also that, to focus on the implications of demand interdependencies, we assume a linear cost function and ignore interactions in the cost of producing the two goods.[3]

The assumption that the demand for one good depends on the price of the other plays a key role in our analysis. Such demand interdependencies can originate from consumer preferences, but also from search costs. Interdependency can occur, for instance, if the advertised price of a good drives consumers' decision to search (Lal and Matutes, 1994; Ellison, 2005; Rhodes, 2015). Below, we also provide an application with a single good provided in two successive periods, $i = 1, 2$. In this interpretation, demand for the good in one period depends on the price in the previous one (e.g., due to switching costs). Another instance where demands are interdependent is when M is a two-sided platform bringing together markets connected by externalities, such as media content and advertising (see Section 5.1.1).

[2] The cross-price derivatives of the demand functions are not necessarily symmetric. That is, we allow for $\frac{\partial Q_i}{\partial p_j} \neq \frac{\partial Q_j}{\partial p_i}$. We discuss some conditions under which one can expect these derivatives to be asymmetric below.

[3] As Hotelling (1932) explains, the cost function does not play a key role in identifying the Edgeworth's paradox under monopoly.

For the moment, we do not specify consumer utility since we start from characterising the effects of taxation on prices and output. We introduce the utility function when studying the effects of taxation on welfare in Section 7.

3.1 Equilibrium

In the following, we use superscript e to denote variables in equilibrium. Furthermore, we use superscript 0 to denote variables in the "laissez-faire" equilibrium without taxes, i.e. where $t_i = \tau_i = 0, \quad \forall i$. The vector of equilibrium prices, p^e, that maximize M's profit satisfies the following system of first-order conditions:

$$\frac{\partial \pi}{\partial p_i} = (1 - t_i) Q_i + (p_i (1 - t_i) - c_i - \tau_i) \frac{\partial Q_i}{\partial p_i} + (p_j (1 - t_j) - c_j - \tau_j) \frac{\partial Q_j}{\partial p_i} = 0, \; i, j = 1, 2, j \neq i. \tag{3.2}$$

Although p_i^e is a function of the tax rates T, in the following we omit the argument of the price function to avoid clutter in the formulas. Rearranging (3.2), we obtain

$$p_i^e = \frac{c_i + \tau_i}{1 - t_i} - \frac{Q_i^e}{\frac{\partial Q_i}{\partial p_i}} - \frac{\left(p_j^e (1 - t_j) - c_j - \tau_j\right) \frac{\partial Q_j}{\partial p_i}}{(1 - t_i) \frac{\partial Q_i}{\partial p_i}}, \quad i, j = 1, 2, \quad j \neq i. \tag{3.3}$$

To interpret the above expression, we first focus on the laissez-faire equilibrium:

$$p_i^0 = c_i - \frac{Q_i^0}{\frac{\partial Q_i}{\partial p_i}} - \frac{\left(p_j^0 - c_j\right) \frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}}, \quad i, j = 1, 2, \quad j \neq i. \tag{3.4}$$

The first two terms on the right-hand side of equation (3.4) coincide with the terms in the single-product monopoly price formula. The last term captures the effect of a change in the price of good i on the profitability of good j, and is therefore distinctive of a multiproduct firm. Clearly, with independent demands (i.e., $\frac{\partial Q_j}{\partial p_i} = 0$) the latter effect disappears. Now, suppose that $p_j^0 > c_j$ (this condition must hold for at least one of the two goods in equilibrium). If $\frac{\partial Q_j}{\partial p_i} > 0$ (e.g., if i is a substitute to j), p_i^0 tends to exceed the price level that M would set if it supplied good i only, because part of the loss in sales when raising p_i is compensated by a higher demand for good j. By contrast, if $\frac{\partial Q_j}{\partial p_i} < 0$ (e.g., if i is a complement to j), p_i^0 tends to be below the level that M would set if it supplied good i only. This is because the monopolist is willing to sell good 1 at a lower price in order to boost demand for the other good. In fact, if $p_j^0 - c_j$ is large enough, good 1 is a loss leader, i.e. $p_i^0 < c_i$ holds.

Observe that, if $\frac{\partial Q_j}{\partial p_i} < 0$ and $p_j^0 > c_j$ hold, the supplier may set p_i^0 low enough that the equilibrium quantity Q_i^0 lies on the *inelastic* part of the demand curve, i.e., $\frac{p_i^0}{Q_i^0}\frac{\partial Q_i}{\partial p_i} > -1$. More precisely, this condition holds whenever $c_i < (p_j^0 - c_j)\frac{\partial Q_j}{\partial p_i} / \frac{\partial Q_i}{\partial p_i}$. This outcome is peculiar to multiproduct pricing with interdependent demands: if demands were independent, the supplier would always operate on the *elastic* part of demand in equilibrium (as would a single-product supplier). This observation is relevant for the analysis of the effects of ad valorem taxation that we present below.

4 Simplified settings

In this section we introduce the effect of ad valorem taxes on prices and quantities. To focus on the most novel results, we allow for different tax rates on each good and momentarily ignore unit taxes, setting $\tau_i = 0$ for $i = 1, 2$. We postpone the analysis of the effects of unit taxes and of a uniform ad valorem tax rate on both goods to Section 6.

4.1 Introducing the direct effect

To gradually introduce the effects of taxation on a multiproduct supplier, we begin our analysis by assuming that the price of good 2 is given. Under this assumption, to be relaxed below, we can focus on the *direct* effect of the tax on good 1, i.e. its effect on p_1 given the price of the other good. Furthermore, this assumption allows us to concentrate on a simple and novel mechanism that will play an important role throughout the analysis.. In Section 4.1.1 we present an application where the price of good 2 is practically given (because consumers have identical, unit demands for such good).

Given p_2, the equilibrium price of good 1 satisfies the first-order condition in (3.2) with $i = 1$. Consider the effects of the ad valorem tax t_1. Differentiating (3.2), we find

$$\frac{\partial p_1^e}{\partial t_1} = -\frac{\frac{\partial^2 \pi}{\partial p_1 \partial t_1}}{\frac{\partial^2 \pi}{\partial p_1^2}}, \tag{4.1}$$

where

$$\frac{\partial^2 \pi}{\partial p_1 \partial t_1} = -Q_1^e \left(\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} + 1 \right), \quad \text{and } \frac{\partial^2 \pi}{\partial p_i^2} < 0.$$

The denominator in 4.1 is negative by the second-order conditions of the profit maximization problem. Hence, $\frac{\partial p_1^e}{\partial t_1}$ is negative if and only if the numerator is negative, that is, if and only if $Q_1^e \left(\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} + 1 \right) > 0$. To understand this condition, consider that t_1 gives the supplier an incentive to change p_1 in a way that reduces the revenue from good 1, $p_1 Q_1$. Given p_2, this objective can be achieved by reducing p_1 if and only if Q_1^e is on the *inelastic* part of the demand curve. With a multiproduct supplier, this condition depends crucially on the effect of p_1 on the demand for good 2. As argued above, in the benchmark scenario of independent demands (i.e. $\frac{\partial Q_2}{\partial p_1} = 0$) we find $p_1^e = \frac{c_1}{1-t_1} - \frac{Q_1^e}{\frac{\partial Q_1}{\partial p_1}}$, so Q_1^e lies on the elastic part of the demand curve, i.e., $\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} < -1$. Thus, as one would expect, the price of good 1 increases in t_1. If demands are interdependent (i.e. $\frac{\partial Q_2}{\partial p_1} \neq 0$), though, Q_1^e may lie on the inelastic part of the demand curve. Rearranging (3.3), we find that

$$\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} + 1 > 0 \Longleftrightarrow c_1 < \frac{\left(p_2^e\left(1-t_2\right) - c_2\right)\frac{\partial Q_2}{\partial p_1}}{\frac{\partial Q_1}{\partial p_1}}. \tag{4.2}$$

Assuming the profit margin on good 2 is positive, the above inequality is satisfied if and only if $\frac{\partial Q_2}{\partial p_1} < 0$ (e.g., good 1 is a complement to 2) and the marginal cost c_1 is small enough. Under these conditions, $\frac{\partial p_1^e}{\partial t_1} < 0$ holds. Furthermore, both Q_1^e and Q_2^e *increase* with t_1.

Proposition 1. *Given p_2, p_1^e decreases with the ad valorem tax t_1, and the supply of both goods increases, if and only if (4.2) holds.*

Although we postpone the analysis of the effects of unit taxes to Section 6.1, it is useful to point out here that a unit tax would not produce similarly counterintuitive effects. Indeed, whereas the burden imposed on M by an ad valorem tax, t_i, is proportional to the *revenue* from good i, the burden imposed by an unit tax, τ_i, is proportional to the *quantity* supplied. Thus, M could reduce the burden of τ_1 only by cutting the output of good 1 and raising p_1^e. In other words, unlike the ad valorem tax, a unit tax has the same effect as an increase in the cost of production.

4.1.1 Application 1: the add-on pricing model (Ellison, 2005)

Suppose good 1 is a "base" good (e.g., a flight ticket), whereas good 2 is an "add-on" (e.g., baggage allowance). There is a unit mass of consumers, each buying at most one unit of each good and having valuation v_i for good $i = 1, 2$. The valuation v_1 is uniformly distributed with support $[0, 1]$. All consumers have the same valuation for good 2, i.e. $v_2 = v > 0$ if and only if they buy good 1, and $v_2 = 0$ otherwise (the add-on has no value without the base good).[4]

Consumers know their valuations v_i and observe p_1 without visiting M, but observe p_2 only if they visit. However, consumers form rational expectations about this price. Visiting

[4] The model can be generalised by assuming consumers have heterogeneous valuations (drawn from a common distribution) for the add-on, but only realise these valuations after purchasing the base good. Consumers thus make their purchase decision of the add-on based on their common expected valuation.

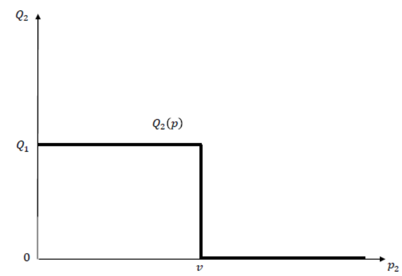

Figure 4.1: Demand for the add-on good.

M entails a small search cost s, with $s \to 0$. The timing is as follows: given t_1, M sets p_2 and p_1. Next, consumers observe p_1 and decide whether to visit M. Consumers who visit observe p_2 and decide whether to buy 1 and 2.

A consumer who visits M buys good 2 if and only if she/he buys 1 and $p_2 \leq v$. Figure 4.1 illustrates the demand function for good 2. Quite intuitively, the equilibrium must be such that $p_2^e = v$ (regardless of the tax rates).[5] Furthermore, as we show in Appendix A.1, we obtain

$$p_1^e = \frac{1}{2} + \frac{c_1 + c_2 - (1 - t_2)\, v}{2\,(1 - t_1)}. \tag{4.3}$$

Thus, M curtails the mark-up on the base good in order to boost demand for the add-on if $v\,(1 - t_2) \geq c_2$. Analyzing the effect of the tax, we obtain

$$\frac{\partial p_1^e}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^e}{Q_1^e} \frac{\partial Q_1}{\partial p_1} > -1 \Leftrightarrow c_1 < v\,(1 - t_2) - c_2,$$

which is consistent with (4.2). If the above inequality holds, the output of both goods increases with t_1.

4.2 Introducing the indirect effect

We now endogenize the price of good 2. Intuitively, with two endogenous prices and interdependent demands, a tax applied on one good can affect both prices. Consequently, as we shall see, the overall effect of the tax on the price of good i depends not only on the direct effect presented above (that is, the effect of the tax given the price of the other good). There is also an *indirect* effect, capturing the fact that a change in the price of good j in the response to the tax brings the supplier to adjust the price of good i.For the sake of exposition,

[5] As Figure 4.1 shows, the demand for good 2 is not everywhere continuously differentiable in p_2. This violates the assumptions made in Section 3, but it is of little importance for the results given we treat this price as given in this section.

we introduce this second effect in a simplified environment, where the demand for one of the goods (that we take to be good 1 without loss of generality) does not depend on the price of the other good. More precisely, we make the following

Assumption 1. $\frac{\partial Q_1}{\partial p_2} = 0$ *and* $\frac{\partial Q_2}{\partial p_1} \neq 0$.

This assumption holds in several settings, including two applications that we present below. In Section 4.2.1 we present an application where the supplier can advertise the price of good 1 only, so consumers must search (e.g., visit a store or website) to learn the price of the other good (Lal and Matutes, 1994; Rhodes, 2015). The demand for good 1 does not depend on p_2, but on the *expected price* that consumers form before searching. On the other hand, the demand for good 2 depends on the price of good 1, which drives the decision to visit the store. Similarly, Assumption 1 holds when consumers must decide whether to buy in one period before observing the price in the next period (see Section 4.2.2). While demand in the first period only depends on expected future prices, the demand in the next period is influenced by the previous price, e.g., due to switching costs (Klemperer, 1995).

Let us consider the effect of an ad valorem tax on good 1 on both prices. Totally differentiating the expressions in (3.2), under Assumption 1, we find

$$\frac{\partial p_1^e}{\partial t_1} = -\frac{\frac{\partial \pi}{\partial p_1 \partial t_1}\frac{\partial^2 \pi}{\partial p_2^2}}{H}, \quad \frac{\partial p_2^e}{\partial t_1} = \frac{\frac{\partial \pi}{\partial p_1 \partial t_1}\frac{\partial \pi}{\partial p_1 \partial p_2}}{H}. \tag{4.4}$$

where

$$\frac{\partial^2 \pi}{\partial p_i \partial t_i} = -Q_i^e\left(\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} + 1\right), \ \frac{\partial^2 \pi}{\partial p_i^2} < 0, \ H \equiv \frac{\partial^2 \pi}{\partial p_1^2}\frac{\partial^2 \pi}{\partial p_2^2} - \left(\frac{\partial \pi}{\partial p_1 \partial p_2}\right)^2 > 0.$$

The first derivative in (4.4) indicates that only the direct effect of the tax matters for $\frac{\partial p_1^e}{\partial t_1}$. Indeed, Assumption 1 implies that $\frac{\partial^2 \pi}{\partial p_2 \partial t_1} = 0$ (see (3.2)), so there is no indirect effect of t_1 on the price of good 1. The direct effect is as described in Section 4.1. By the second-order conditions of the supplier's problem, $\frac{\partial^2 \pi}{\partial p_2^2} \leq 0$ and $H \geq 0$ hold, so condition (4.2) is necessary and sufficient for p_1^e to decrease with t_1.[6]

Consider now the effect of t_1 on p_2. Because $\frac{\partial \pi}{\partial p_2 \partial t_1} = 0$ under Assumption 1, the tax has no direct effect on p_2. However, t_1 induces a change in the price of good 1, which, in turn, affects the price of good 2 because the demand for good 2 depends on p_1. This is the *indirect* effect of the tax, which depends on two factors. First, on how t_1 affects p_1^e, given p_2^e. This, recall, is the direct of effect of t_1 on p_1^e, and its sign is determined by the local elasticity of

[6] Given $\frac{\partial Q_1}{\partial p_2} = 0$, (3.2) implies that $p_2^e > c_2$.

demand, as discussed above. Secondly, the indirect effect depends on the sign of the cross derivative $\frac{\partial^2\pi}{\partial p_1 \partial p_2}$, i.e., on how changing one price affects the marginal profitability of the other. This sign indicates whether a change in p_1 induces p_2 to move in the same (if $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$) or in the opposite direction (if $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$). Note that, if (4.2) and $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$ hold, both p_1^e and p_2^e decrease in t_1. If $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$ holds but (4.2) does not, p_2^e decreases in t_1 while p_1^e increases in it.

Finally, consider the effect of t_1 on the equilibrium quantities. Since Q_1^e decreases with p_1 and does not depend on p_2 by assumption, (4.2) is necessary and sufficient for Q_1^e to increase with the tax. Notice also that (4.2) can hold only if good 1 is a complement to 2, i.e. $\frac{\partial Q_2}{\partial p_1} < 0$. Hence, if this condition holds and $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$, p_2^e decreases and, thus, Q_2^e increases with t_1 as well.[7]

Summing up, under Assumption 1, we can establish simple necessary and sufficient conditions for an ad valorem tax on one of the goods to *reduce the price* of both goods and *increase their supply*. As in the basic model of Section 4.1, these conditions can hold only if good 2 is a complement to good 1.

Proposition 2. *Given Assumption 1, p_1^e decreases with the ad valorem tax t_1 if and only if (4.2) holds. Furthermore, p_2^e decreases with t_1 if and only if (4.2) and $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$ hold. Under these conditions, the supply of both goods increases with t_1.*

4.2.1 Application 2: Multiproduct retailing with price advertising (Rhodes, 2015)

We consider a unit mass of consumers with valuation v_i for good i, distributed according to a c.d.f. $F(v_i)$ with support $[a, b] \subset \mathbb{R}$. This distribution has strictly positive, continuously differentiable, and log-concave density f. The parameter v_i is i.i.d. across products and consumers, who know their individual valuations for each product and buy at most one unit of each. The unit cost of each product is c, with $0 \leq c < b$. We assume M advertises the price of good 1. Hence, consumers observe p_1 at no cost, but must visit the store to know p_2, incurring a small search cost s. We assume that search costs are small enough that a positive mass of consumers searches the firm in equilibrium. For simplicity, we only consider an ad

[7] Assumption 1 states that the demand for good 1 is independent of p_2. As explained above, however, in many applications (including the ones we consider below) this demand may depend on consumers' rational *expectation* of p_2. Accounting for this aspect would not change these results in a fundamental way. Recall that complementarity is necessary for (4.2) to hold. Provided the goods are complements (so that Q_1 decreases with the expected p_2), conditions (4.2) and $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$ are still sufficient for both prices to decrease with the tax, and for both quantities to increase.

valorem tax on good 1, t_1.

The timing is as follows. At stage one, the government sets t_1. M then chooses p_1 and advertises this price to each consumer, choosing p_2 next. Consumers then learn p_1 and form expectations about p_2. Each consumer decides whether to visit at that point. Finally, consumers who visit learn p_2 and make their purchase decisions.

As we show in Appendix A.2, the demand Q_1 depends on the *expected* price of good 2, but not on p_2, because consumers observe this price only after visiting the supplier. Hence, Assumption 1 holds. Consequently, the supplier sets $p_2^e = -\frac{Q_2^e}{\frac{\partial Q_2}{\partial p_2}} + c$ in equilibrium. However, when choosing p_1, the supplier considers the effect of this price on Q_2. Hence, we get

$$p_1^e = -\frac{Q_1^e}{\frac{dQ_1}{dp_1}} + \frac{c}{1-t_1} + \frac{p_2^e - c}{1-t_1}\frac{dQ_2}{dp_1}, \tag{4.5}$$

where $\frac{dQ_1}{dp_1}$ and $\frac{dQ_2}{dp_1}$ are negative (see Appendix A.2). The necessary and sufficient condition for p_1^e to decrease with t_1, as stated in Proposition 2, is

$$\frac{\partial p_1^e}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^e}{Q_1^e}\frac{dQ_1}{dp_1} > -1 \Leftrightarrow c < \frac{(p_2^e - c)\frac{dQ_2}{dp_1}}{\frac{dQ_1}{dp_1}}. \tag{4.6}$$

Lemma 2 in Rhodes (2015) shows that, when M raises the price of the advertised good, the price of the other good increases as well, i.e. $\frac{\partial p_2^e}{\partial p_1} > 0$. Thus, $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ holds. Hence, if (4.6) holds, both prices decrease with t_1, while the output of both goods increases.

4.2.2 Application 3: An intertemporal model with switching costs

In Appendix A.3, we provide an application to an intertemporal setting with switching costs. The supplier provides a single product in two periods, $i = 1, 2$. In each period, a consumer decides whether to buy one unit of the good. If a consumer buys (resp. does not buy) in period 1, but does not buy (resp. buys) in period 2, she/he sustains a switching cost, s. The switching cost makes the demands in the two periods interdependent. However, Assumption 1 applies because in period 1 consumers do not observe the future price. In line with the literature, we find that in equilibrium the supplier sells the good at a discount in period 1, to expand the set of "locked in" consumers, but charges them extra in period 2 exploiting the switching cost.[8] As a result, we find that if $c < s$, Q_1^e lies on the inelastic part of demand and the conditions in Proposition 2 hold. Therefore, an ad valorem tax in period 1 has a

[8] See, e.g., Tirole (1988) and Belleflamme and Peitz (2015) for an overview the literature on switching costs.

price-reducing and supply-expanding effect in both periods.[9]

5 Fully interdependent demands

To complete the analysis of the effects of taxation, we now relax Assumption 1 and allow the demands for goods 1 and 2 to be fully interdependent. That is, we let $\frac{\partial Q_i}{\partial p_j} \neq 0$ for both goods.

5.1 Effects of taxation on prices and supply

Differentiating the expressions in (3.2) with respect to t_i, we find

$$\frac{\partial p_i^e}{\partial t_i} = -\frac{\frac{\partial^2 \pi_i}{\partial p_i \partial t_i}\frac{\partial^2 \pi}{\partial p_j^2} - \frac{\partial^2 \pi}{\partial p_j \partial t_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}{H}, \quad \frac{\partial p_j^e}{\partial t_i} = -\frac{\frac{\partial^2 \pi_i}{\partial p_j \partial t_i}\frac{\partial^2 \pi}{\partial p_i^2} - \frac{\partial^2 \pi_i}{\partial p_i \partial t_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}{H}, \; i, j = 1, 2, \quad j \neq i, \tag{5.1}$$

where

$$\frac{\partial^2 \pi}{\partial p_i \partial t_i} = -Q_i^e \left(\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} + 1 \right), \; \frac{\partial^2 \pi}{\partial p_i \partial t_j} = -p_j^e \frac{\partial Q_j}{\partial p_i}, \; \frac{\partial^2 \pi}{\partial p_i^2} < 0, \; H \equiv \frac{\partial^2 \pi}{\partial p_1^2}\frac{\partial^2 \pi}{\partial p_2^2} - \left(\frac{\partial \pi}{\partial p_1 \partial p_2} \right)^2 > 0.$$

As a preliminary step, observe that, in the benchmark case of independent demands (i.e. $\frac{\partial Q_2}{\partial p_1} = \frac{\partial Q_1}{\partial p_2} = 0$), the effects of taxation are standard. The first derivative in (5.1) boils down to $\frac{\partial p_i^e}{\partial t_i} = -\frac{\partial^2 \pi}{\partial p_i \partial t_i} / \frac{\partial^2 \pi}{\partial p_i^2}$, which is positive since, as pointed out above, the equilibrium quantity Q_i^e would lie on the elastic part of the demand curve for good i. Furthermore, the tax on good i does not affect the price of the other good, $\frac{\partial p_j^e}{\partial t_i} = 0$.

Return now to the case where demands are interdependent. The denominator of the expressions in (5.1) is the determinant of the Hessian matrix, which is positive by the second-order conditions of the profit maximisation problem. Hence, we have

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial t_i} \right) = \operatorname{sgn}\left(-\underbrace{\frac{\partial \pi_i}{\partial p_i \partial t_i}\frac{\partial^2 \pi}{\partial p_j^2}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi}{\partial p_j \partial t_i}\frac{\partial \pi}{\partial p_1 \partial p_2}}_{\text{indirect effect}} \right), \; i = 1, 2, \quad j \neq i. \tag{5.2}$$

$$\operatorname{sgn}\left(\frac{\partial p_j^e}{\partial t_i} \right) = \operatorname{sgn}\left(-\underbrace{\frac{\partial \pi_i}{\partial p_j \partial t_i}\frac{\partial^2 \pi}{\partial p_i^2}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi_i}{\partial p_i \partial t_i}\frac{\partial \pi}{\partial p_1 \partial p_2}}_{\text{indirect effect}} \right), \; i = 1, 2, \quad j \neq i. \tag{5.3}$$

[9] Different tax rates in period 1 and 2 can be interpreted as the government taxing the good at different rates for new and returning customers.

Effect on p_i^e	Effect on p_j^e
• $DE < 0$ iff $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$. • $IE < 0$ if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{\partial Q_i}{\partial p_j} > 0$, or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{\partial Q_i}{\partial p_j} < 0$. • Overall: see (5.4).	• $DE < 0$ iff $\frac{\partial Q_i}{\partial p_j} > 0$ • $IE < 0$ if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$, or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} < -1$. • Overall: see (5.5)

Table 1: Effects of tax t_i on equilibrium prices. DE stands for "direct effect" and IE for "indirect effect". An effect is negative whenever it tends to reduce the price.

The signs of $\frac{\partial p_i^e}{\partial t_i}$ and $\frac{\partial p_j^e}{\partial t_i}$ are determined by the sum of the direct and the indirect effects presented above. The direct effect on p_i^e is as described in Section 4.1. This effect is negative (i.e., it tends to reduce p_i^e) if and only if (4.2) holds. With fully interdependent demands, the tax on good i also has a direct effect on the price of the other good, p_j^e. This effect captures the change induced by the tax on the price of j, given p_i. Since $\frac{\partial^2 \pi}{\partial p_i^2} < 0$, the direct effect of t_i on p_j^e is negative if and only if $\frac{\partial Q_i}{\partial p_j} > 0$, i.e., when good j is a substitute to good i. The intuition is that the tax gives the supplier an incentive to reduce the revenue from good i, $p_i Q_i$. Given p_i, the supplier can achieve this objective by reducing p_j if and only if $\frac{\partial Q_i}{\partial p_j} > 0$.

Let us now turn to the indirect effect of the tax. For each price, this effect depends on two factors: first, the direct effect of the tax on the price of the other good and, second, the cross derivative $\frac{\partial^2 \pi}{\partial p_1 \partial p_2}$. As explained above, the direct of effect of t_i on p_j^e is negative if and only if $\frac{\partial Q_i}{\partial p_j} > 0$. Consequently, the indirect effect of t_i on p_i^e is negative if and only if either (i) $\frac{\partial Q_i}{\partial p_j} > 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$, or (ii) $\frac{\partial Q_i}{\partial p_j} < 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ hold. In words, when taxing good i determines an increase in the price of good j, given the price of the other good, (e.g., because j is a substitute for i) and p_i moves in the same direction as p_j, then the indirect effect pushes p_i upwards. The mechanism works in the opposite direction when good i is a complement to good j.

Consider now the indirect effect of t_i on p_j. As shown above, the direct effect of t_i on p_i is negative if and only if (4.2) holds. Therefore, the indirect effect of t_i tends to reduce p_j^e if either (i) $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and Q_i^e lies on the inelastic part of demand for good i or if (ii) $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and Q_i^e is on the elastic part of demand.

Table 1 summarizes the effects of taxation we just presented. Based on these effects, we can study the sign of the derivatives in equations (5.2) and (5.3). Rearranging (5.2), we obtain that $\frac{\partial p_i^e}{\partial t_i} < 0$ if and only if

$$c_i < \max\left(\frac{\left(p_j^e\left(1-t_j\right)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{\partial^2 \pi}{\partial p_1 \partial p_2}\frac{\partial Q_i}{\partial p_j}\frac{p_i^e\left(1-t_i\right)}{\frac{\partial^2 \pi}{\partial p_i^2}\frac{\partial Q_i}{\partial p_i}}, 0\right),\ i = 1,2,\ j \neq i. \tag{5.4}$$

The first term in brackets on the right hand side of this expression is the same as the right hand side of inequality (4.2): the direct effect of t_i is negative if and only if the unit cost c_i is below this threshold. Recall that this term can be positive only if good i is a complement to j ($\frac{\partial Q_j}{\partial p_i} < 0$). That is, complementarity is necessary for the direct effect to be negative. The second term in brackets is positive if and only if the indirect effect is negative. If this latter effect is negative, therefore, the condition in (5.4) is weaker than (4.2). Consequently, a negative direct effect is not necessary for $\frac{\partial p_i^e}{\partial t_i} < 0$ to hold and, hence, neither is product complementarity (see the application in Section 5.2). On the other hand, if the indirect effect is positive, a negative direct effect may not be enough for the price to decrease with the tax.

Rearranging (5.3), we obtain that $\frac{\partial p_j^e}{\partial t_i} < 0$ if and only if

$$\begin{aligned} c_i &< \max\left(\frac{\left(p_j^e(1-t_j)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{p_i^e(1-t_i)\frac{\partial Q_i}{\partial p_j}}{Q_i\frac{\partial^2\pi}{\partial p_i^2}\frac{\partial Q_i}{\partial p_i}\frac{\partial^2\pi}{\partial p_1\partial p_2}}, 0\right) \quad &\text{if } \frac{\partial^2\pi}{\partial p_1\partial p_2} > 0, \\ c_i &> \max\left(\frac{\left(p_j^e(1-t_j)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{p_i^e(1-t_i)\frac{\partial Q_i}{\partial p_j}}{Q_i\frac{\partial^2\pi}{\partial p_i^2}\frac{\partial Q_i}{\partial p_i}\frac{\partial^2\pi}{\partial p_1\partial p_2}}, 0\right) \quad &\text{if } \frac{\partial^2\pi}{\partial p_1\partial p_2} < 0. \end{aligned} \tag{5.5}$$

Suppose that $\frac{\partial^2\pi}{\partial p_1\partial p_2} > 0$. Similarly to (5.4), the direct effect of the tax on the price of good j can be negative if and only if $\frac{\partial Q_j}{\partial p_i} < 0$ (assuming $p_j^e > c_j$). Then, if and only if the direct effect of t_i on p_i is negative, the second term in brackets is positive, which makes the threshold on the marginal cost less stringent.

Finally, consider the effects of taxation on supply. With fully interdependent demands, the effect of t_i on the equilibrium quantity Q_j^e is

$$\frac{\partial Q_j^e}{\partial t_i} = \frac{\partial Q_j}{\partial p_1}\frac{\partial p_1^e}{\partial t_i} + \frac{\partial Q_j}{\partial p_2}\frac{\partial p_2^e}{\partial t_i},\ i, j = 1, 2. \tag{5.6}$$

When both prices decrease and the goods are not substitutes, the ad valorem tax results in higher supply of both.

The analysis with fully interdependent demands provides general conditions for the prices set by a multiproduct supplier to decrease with the ad valorem tax t_i. These findings extend the literature on the "taxation paradox", initiated by Edgeworth (1925), to the case of ad valorem taxes. Establishing simple conditions for this result to hold is however more difficult than in the simplified cases we presented in the previous sections. Nonetheless, some clear-cut results can be obtained by specifying the demand functions (see Section 5.2 below).

We emphasize that the result that supply can increase with taxation is specific to differentiated ad valorem taxes, and different to the effect of either unit taxes or a standard, uniform ad valorem tax on both goods. As we argue in Section 6, with these instruments the supply of the taxed good(s) tends to decrease, because, fundamentally, these taxes work as an increase in the cost of production.

5.1.1 Application 4: Two-sided markets

With fairly small adaptation, our analysis applies to the case where M is a two-sided platform. In a two-sided market, there are two groups of customers (e.g., viewers and advertisers), each buying one of the goods provided by M (e.g., content and ads). There are externalities across the two markets: the surplus of one group depends on the quantity supplied to the other group (e.g., viewers find ads a nuisance, while advertisers value reaching more viewers). Hence, the demand $Q_i(p_i, Q_j)$ for good $i = 1, 2$ does not depend directly on the price of the other good, but it depends on its quantity. Thus, we have that $\frac{\partial Q_i}{\partial p_j} = \frac{\frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial p_j}}{1-\frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial Q_i}}$. This derivative is generally not equal to zero, so demands for the two goods are interdependent. Assuming that $\frac{\partial Q_i}{\partial p_i} < 0$ and $1 - \frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial Q_i} > 0$, the condition $\frac{\partial Q_i}{\partial p_j} > 0$ holds if and only if $\frac{\partial Q_i}{\partial Q_j} < 0$, i.e. an increase in Q_j induces a drop in the demand for good i.

Assuming the same cost function as in the baseline model, the supplier's profit function is isomorphic to (3.1). Therefore, the first-order conditions that define the vector of equilibrium prices, p^e, are isomorphic to (3.2). It follows that the effects of taxation are as characterized above. Hence, our analysis generalizes previous findings by Kind et al. (2008), by showing that the effects of taxation that the authors characterized in a two-sided market apply more generally to markets served by a multiproduct firm, even if "one-sided", provided that the demands for the goods are interdependent.[10]

We provide an application based on Armstrong (2006). Consider a platform serving two

[10] As an illustration, consider the sufficient conditions for t_1 to decrease prices and increase supply that we provide in Proposition 2. These are equivalent to the sufficient conditions that Kind et al. (2008, p. 1535) provide in their main example. Specifically, their assumption (b) is tantamount to $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$. Furthermore, their assumption (a) is the same as Assumption 1 in our setting.

groups of users. The utility a user in group i gets by buying the good is

$$u_i = \alpha_i Q_j - p_i, \quad i, j = 1, 2, \quad j \neq i, \tag{5.7}$$

where p_i is the price set by the platform for users in group i and Q_j is the number of users in group $j \neq i$. A relevant case for our analysis is where users in one group, say 2, benefit from participation by users in the other, but not the other way round, i.e. $\alpha_1 \leq 0$ and $\alpha_2 > 0$. An example is a media platform (e.g., an online website or a TV station), where group 2 are advertisers and group 1 are viewers.

We assume the number of users that join the platform in each group is

$$Q_i = \phi_i\left(u_i\right), \quad i = 1, 2, \tag{5.8}$$

with $\phi_i' > 0$. Combining (5.7) and (5.8), one obtains the following own- and cross-price derivatives of demand:

$$\frac{\partial Q_i}{\partial p_i} = -\phi_i' < 0, \; \frac{\partial Q_j}{\partial p_i} = -\frac{\phi_1'\phi_2'\alpha_j}{1 - \phi_1'\phi_2'\alpha_1\alpha_2}, \quad i, j = 1, 2; j \neq i. \tag{5.9}$$

Assuming $1 > \phi_i'\alpha_i\phi_j'\alpha_j$, we have that $\frac{\partial Q_j}{\partial p_i} > 0$ if and only if $\alpha_j < 0$. That is, the effect of increasing p_i on the demand of the other side of the market depends on the externality that group i generates on group j: if greater participation on side i reduces the utility of users on side j ($\alpha_j < 0$), then a higher price on side i will increase demand on side j, and viceversa. The platform's profit is $\pi = Q_1\left(p_1\left(1 - t_1\right) - c_1\right) + Q_2\left(p_2\left(1 - t_2\right) - c_2\right)$. As we show in Appendix A.4.1, we have

$$p_i^e = \frac{c_i - \phi_j\alpha_j\left(1 - t_j\right)}{1 - t_i} + \frac{\phi_i}{\phi_i'} \; i, j = 1, 2, \; j \neq i. \tag{5.10}$$

Setting $t_1 = t_2 = 0$, expression (5.10) boils down to

$$p_i^0 = c_i + \frac{\phi_i}{\phi_i'} - \phi_j\alpha_j \; i, j = 1, 2, \; j \neq i. \tag{5.11}$$

This is a standard monopoly price formula (marginal cost plus mark up), except for the third term that accounts for the marginal external effect that users in group i produce on users in the other group. If $\alpha_j > 0$, the platform has an incentive to reduce the price of good i to raise the willingness to pay on the other side.

To illustrate the effects of taxation, let us focus on t_1. As we show in Appendix A.4.2, p_1

and p_2 decrease with this tax rate if the following conditions hold

$$\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} > -1 \Leftrightarrow c_1 < Q_2^e \alpha_2 (1-t_2), \ \alpha_1 \leq 0 \text{ and } \frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0. \tag{5.12}$$

The first condition states that Q_1^e lies on the inelastic part of the demand curve for good 1. This condition can hold only if participation by group 1 (e.g., viewers) produces a positive externality on group 2 (e.g. advertisers), i.e. $\alpha_2 > 0$. The second condition applies whenever users in group 2 produce either a negative or no externality on users in group 1. The third condition applies whenever a higher price of one good makes raising the other price more profitable to the platform. Given $\alpha_1 = 0$ and $\alpha_2 \neq 0$ (i.e., Assumption 1 holds), the conditions in (5.12) correspond to the sufficient conditions provided in Proposition 2. However, note that t_1 reduces both prices even if $\alpha_1 < 0$, as long as the other conditions in (5.12) hold. Hence, Assumption 1 is not necessary for both prices to decrease with the ad valorem tax.

5.2 Linear demands

Suppose now the demand functions are linear, i.e. $Q_i = \alpha_i - \beta_i p_i - \gamma p_j, \quad i, j = 1, 2, i \neq j$,where $\alpha_i > 0$ and $\beta_i < 0$. Furthermore, $\gamma > 0$ if the goods are complements ($\frac{\partial Q_i}{\partial p_j} < 0$), whereas $\gamma < 0$ if they are substitutes ($\frac{\partial Q_i}{\partial p_j} > 0$). It is straightforward to show that the cross-price derivative $\frac{\partial^2 \pi}{\partial p_1 \partial p_2}$ is positive if and only if the goods are substitutes, i.e., $\gamma < 0$. This observation helps in streamlining the sign of the price derivatives. Specifically, the indirect effect of t_i on p_i is negative (see Table (1)). Hence, a negative direct effect, determined by condition (4.2), is sufficient for the derivative $\frac{\partial p_i}{\partial t_i}$ to be negative. However, since (4.2) can only hold if the goods are complements ($\gamma > 0$), both the direct and the indirect effect of t_i on p_j are positive, so $\frac{\partial p_j}{\partial t_i} > 0$. Therefore, we get the following result:

Proposition 3. *With linear demands, if (4.2) holds, p_1^e decreases with the ad valorem tax t_1, whereas p_2^e increases.*

If condition (4.2) does not hold, however, the direct and indirect effects of t_i on both prices go in opposite directions. Therefore, the sign of $\frac{\partial p_i}{\partial t_i}$ and $\frac{\partial p_j}{\partial t_i}$ ultimately depends on which of the two effects is larger in magnitude (see (5.4) and (5.5)).

5.2.1 Application 5: an example from Hotelling (1932)

We consider now a simple example with linear demands, borrowed from Hotelling (1932), and show that the introduction of an ad valorem tax on one good may produce a reduction of both prices. Consider the following demand functions for substitute goods introduced by Hotelling (1932):[11]

$$Q_1 = 4 - 10p_1 + 7p_2, \quad Q_2 = 4,2 - 7p_2 + 9,8p_1.$$

Consider a tax t_1 on good 1 and set $t_2 = 0$. Solving the system of first-order conditions in (3.2) we find the following equilibrium prices

$$p_1 = \frac{-452 + 305t_1 + 35c_2\left(2 + 5t_1\right) - 5c_1\left(16 + 35t_1\right)}{8 + 5t_1\left(32 + 35t_1\right)},$$

$$p_2 = \frac{20\left(1 - t_1\right)\left(5t_1 - 27\right) + c_2\left(88 + 255t_1\right) - 50c_1\left(2 + 5t_1\right)}{8 + 5t_1\left(32 + 35t_1\right)}.$$

By deriving both equilibrium prices by t_1 and evaluating the derivatives at $t_1 = 0$, one can show that there exist values of c_1 and c_2 such that both prices decrease when a tax on good 1 is introduced. By equation (4.2), we know that the equilibrium quantity of good 1 never lies on the inelastic part of the demand because goods are substitutes, implying that the direct effect of the tax on p_1 is always positive. Instead, the indirect effect is negative because $\frac{\partial Q_i}{\partial p_j} > 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ hold. Also, looking at the effect of the tax on the price of good 2, we know that the direct effect is always negative (because $\frac{\partial Q_i}{\partial p_j} > 0$), while the indirect effect is always positive (because $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and the equilibrium quantity does not lie on the inelastic part of the demand). Hence, the effect of the tax on p_1 (resp. p_2) can be negative if and only if the indirect (resp. direct) effects is strong enough.

We now look for values of c_1 and c_2 such that both prices decrease when a tax on good 1 is introduced. Equations (5.4) and (5.5) (when $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ holds) indicate that c_1 has to be low enough for this to occur. Instead, a c_2 high enough favor the negative effects on both prices. For instance, if we set $c_1 = 0$ and $c_2 = 9$, it is easy to verify that both prices decrease when a tax on good 1 is introduced. Furthermore, the quantity of good 1 increases when a tax is introduced, while the quantity of good 2 decreases.

6 Other tax instruments

After focusing on ad valorem taxes with different rates for each good, we briefly present the effects of alternative tax instruments.

[11] Unlike the linear demands considered above, these functions cannot be derived from a standard utility function because the cross-price parameter is asymmetric.

6.1 Unit taxes

To illustrate the differences with the effects of ad valorem taxes, we begin by focusing on the direct effect of a unit tax on good 1, τ_1, on the price of that good. Holding p_2 constant for the moment (as in Section 4.1), we have

$$\frac{\partial p_1^e}{\partial \tau_1} = -\frac{\frac{\partial^2 \pi}{\partial p_1 \partial \tau_1}}{\frac{\partial^2 \pi}{\partial p_1^2}} = \frac{\frac{\partial Q_1}{\partial p_1}}{\frac{\partial^2 \pi}{\partial p_1^2}} > 0. \tag{6.1}$$

Given p_2, the unit tax on good 1 can only increase the good's price and, consequently, reduce Q_1^e. The reason is that, whereas the burden imposed on M by an ad valorem tax t_1 is proportional to the *revenue* from good i, the burden imposed by an unit tax τ_1 is proportional to the *quantity* supplied. Thus, given p_2, M can reduce the burden of a unit tax on good 1 only by reducing the output of good 1 and raising p_1^e. A unit tax has the same effect as an increase in the cost of production, unlike the ad valorem tax.

The intuition just provided applies also to the case where p_2 is endogenous. As we show in Appendix A.5, a unit tax on i can reduce p_i^e, but only if the indirect effect is negative and larger in magnitude than the (positive) direct effect. Furthermore, as previously shown in related literature (Edgeworth, 1925; Armstrong and Vickers, 2022), a unit tax can reduce the price of both goods only if they are substitutes, because only in this case the indirect effect can exceed the direct one. Most importantly, the supply of the taxed good always decreases, because the firm can reduce the burden from the unit tax only in this way. This is another fundamental difference with ad valorem taxes.

Proposition 4. *Differently from an ad valorem tax, a unit tax (i) has a positive direct effect on the price of the taxed good, (ii) can decrease the price of both goods only if they are substitutes and (iii) reduces the supply of the taxed good.*

6.2 Uniform ad valorem tax

Suppose now the government sets the same ad valorem tax rate on both goods, i.e. $t_1 = t_2 = t$. The effects of this tax are similar to those of a unit tax, and, overall, quite different from the effects of differentiated ad valorem taxes targeting. To grasp the intuition, it is useful to write the profit function (ignoring unit taxes) as $\pi = (1-t)\sum_{i=1,2}\left(p_i - \frac{c_i}{1-t}\right)Q_i$. By extracting an equal share of the supplier's revenue from both goods, the tax t affects prices and supply in the same way as a simultaneous increase in the cost of both goods. As we show in Appendix A.6, this tax can in principle have a negative effect on prices, but only if the goods are substitutes and under quite peculiar conditions (e.g., the derivative of a good's demand function with respect to its own price should be smaller in absolute value than the derivative with respect to the price of the other good). To illustrate, in Appendix A.6, we consider the case where the demand functions are linear (as in Section 5.2) and symmetric. The tax increases both prices and reduces the supply of both goods if and only if $\beta > |\gamma|$.

7 Welfare effects of taxation and optimal policy

To set the stage for the analysis of the optimal policy, we now assume that a representative consumer buys both goods. The consumer has the following utility function

$$U(Q_1, Q_2) + y - p_1 Q_1 - p_2 Q_2,$$

where y is the exogenous income.[12] This function is continuously differentiable and concave. The demand functions $Q_i(p_1, p_2)$ are defined by the equilibrium conditions

$$\frac{\partial U}{\partial Q_i} = p_i, \ i = 1, 2. \tag{7.1}$$

Consumer surplus is

$$CS \equiv U(Q_1, Q_2) + y - p_1 Q_1 - p_2 Q_2. \tag{7.2}$$

Social welfare, denoted by W, is the sum of CS, π and tax revenue, $\sum_{i=1,2}(p_i t_i + \tau_i) Q_i$, that boils down to the total net surplus generated in this market, i.e.

$$W(p) = U(Q_1, Q_2) + y - c_1 Q_1 - c_2 Q_2. \tag{7.3}$$

We assume the government faces no revenue requirements and its objective is to maximize W. Note that, given the assumption of quasi-linear utility, there is no loss in considering a single representative consumer.[13]

[12] For concreteness, we focus on one-sided markets in this section. See Kind et al. (2008) for an analysis of optimal taxes in two-sided markets.

[13] With multiple consumers, aggregate demands would depend only on the vector of prices and not on the distribution of income.

7.1 Laissez-faire equilibrium vs. social optimum

The socially optimal quantities, denoted as Q_1^* and Q_2^*, satisfy the system of equations $\frac{\partial U}{\partial Q_i} = c_i$, $i = 1, 2$. It is straightforward to show that this optimal allocation is decentralized by the optimal prices $p_i^* = c_i$ for $i = 1, 2$. To compare the laissez-faire to the social optimum, we evaluate the first-order derivatives of the monopolist's problem in (3.2), conditional on zero taxes, at the vector of optimal prices, p^*. Given concavity of the profit function, we find that

$$p_i^0 \left(p_j^*\right) > p_i^* \quad i, j = 1, 2, \quad j \neq i, \tag{7.4}$$

where $p_i^0 \left(p_j^*\right)$ denotes the equilibrium price conditional on $p_j = p_j^*$. Because for a given p_j the demand for good i is a decreasing function of p_i, we say that the monopolist underprovides (and overprices) good i in the laissez-faire whenever $p_i^0 \left(p_j^*\right) > p_i^*$. This condition holds in this setting due to the supplier's market power.[14]

Generally, the allocation and prices in the no-tax equilibrium do not coincide with the welfare-maximizing ones, suggesting that intervention from the government is warranted. Whenever equilibrium prices are too high, the objective should be to reduce them. Quite interestingly, our previous analysis suggests that this objective can be achieved by appropriately designed taxes.

7.2 Optimal tax on a single good

Consider introducing a small tax on good i starting from the laissez-faire. We take the derivative of (7.3) with respect to t_1, conditional on $t_i = \tau_i = 0, \quad \forall i$. Using the first-order conditions of the monopolist's problem (3.2) and the equilibrium conditions of the consumer's problem in (7.1), we can write this derivative as

$$\left.\frac{\partial W}{\partial t_i}\right|_{\left(Q_1^0, Q_2^0\right)} = -Q_1^0 \frac{\partial p_1}{\partial t_i} - Q_2^0 \frac{\partial p_2}{\partial t_i}, \quad i = 1, 2, \tag{7.5}$$

which shows that a sufficient condition for the tax to increase welfare is that its introduction brings to a reduction in the price of both goods.

We now study the optimal (second-best) tax rate on good i, assuming no tax on the other good. Given the equilibrium conditions of the consumers' problem in (7.1), the optimal tax

[14] This finding does not imply that *both* equilibrium prices, $p^0 \equiv \left(p_1^0, p_2^0\right)$, exceed the first-best levels, $p^* \equiv (p_1^*, p_2^*)$. For example, as we argued above, if $\frac{\partial Q_i}{\partial p_j} < 0$ the firm may use good j as a loss-leader, setting $p_i^0 < c_i$. Obviously, at equilibrium, at least one price is set above the marginal cost.

on good i (conditional on $t_j = \tau_1 = \tau_2 = 0$) is such that

$$\begin{aligned} \frac{\partial W}{\partial t_i} &= (p_1 - c_1)\left(\frac{\partial Q_1}{\partial p_1}\frac{\partial p_1}{\partial t_i} + \frac{\partial Q_1}{\partial p_2}\frac{\partial p_2}{\partial t_i}\right) + \\ &\quad + (p_2 - c_2)\left(\frac{\partial Q_2}{\partial p_1}\frac{\partial p_1}{\partial t_i} + \frac{\partial Q_2}{\partial p_2}\frac{\partial p_2}{\partial t_i}\right) = 0, \quad i = 1, 2. \end{aligned} \tag{7.6}$$

Evaluating the above expression at the equilibrium prices (that satisfy (3.2)) and rearranging, we get the following expression for the (second-best) optimal ad valorem tax on good 1, that we denote by t_i^{SB}:

$$t_i^{SB} = \frac{Q_1 \frac{\partial p_1}{\partial t_i} + Q_2 \frac{\partial p_2}{\partial t_i}}{Q_i\left(1 + \frac{p_i}{Q_i}\frac{\partial Q_i}{\partial p_i}\right)\frac{\partial p_1}{\partial t_i} + p_i \frac{\partial Q_i}{\partial p_j}\frac{\partial p_j}{\partial t_i}}, \quad i, j = 1, 2,\ i \neq j. \tag{7.7}$$

To understand this expression, observe that the denominator captures the change in the tax base ($p_i Q_i$) induced by t_i, through the adjustment in the prices of both goods. Intuitively, the tax induces the supplier to adjust its equilibrium prices so that $p_i Q_i$ shrinks, to reduce the tax expenditure. Hence, the denominator of (7.7) must be negative. The numerator of A.6 is negative whenever both prices decrease with the tax rate. Therefore, $t_i^{SB} > 0$ (respectively, $t_i^{SB} < 0$) if the tax induces a reduction (respectively, an increase) in the equilibrium prices. Note that, if the demands for the two goods were independent ($\frac{\partial Q_1}{\partial p_2} = \frac{\partial Q_2}{\partial p_1} = 0$), the standard result $t_i^{SB} < 0$ would apply, since then $\frac{\partial p_i}{\partial t_i} > 0$, $\frac{\partial p_j}{\partial t_i} = 0$ and $1 + \frac{p_i}{Q_i}\frac{\partial Q_i}{\partial p_i} < 0$ would hold, as we argued previously.We have thus established that the optimal tax on one of the goods sold by a multiproduct supplier can be positive, even though the goods are underprovided in the laissez-faire, so long as the tax reduces the prices.

Proposition 5. *A sufficient condition for the optimal tax on a single good to be positive is that it induces a reduction in the prices of both goods.*

7.3 Optimal taxes on both goods

We now let the government set two tax rates, one for each good. Assume that $\tau_1 = \tau_2 = 0$. Intuitively, with two ad valorem tax rates the government can implement the first-best allocation, i.e. $Q_i^*, \quad i = 1, 2$. In Section 7.1 we show that the prices that decentralize this allocation are such that $p_i^* = c_i$ for $i = 1, 2$. Plugging these prices in (3.2) and rearranging, we find that the optimal ad valorem taxes satisfy the following system:

$$t_i^* = \frac{Q_i^* - p_j^* t_j^* \frac{\partial Q_j}{\partial p_i}}{Q_i^* \left(1 + \frac{p_i^*}{Q_i^*} \frac{\partial Q_i}{\partial p_i}\right)}, \ i = 1, 2, \quad j \neq i. \tag{7.8}$$

The denominator is positive if and only if Q_i^* lies on the inelastic part of demand for good i. The first term at the numerator is positive, while the second term depends on the tax rate on the other good. Solving the system in (7.8) above, we obtain

$$t_i^* = \frac{Q_i^* Q_j^* \left(1 + \frac{p_j^*}{Q_j^*} \frac{\partial Q_j}{\partial p_j}\right) - p_j^* \frac{\partial Q_j}{\partial p_i} Q_j^*}{Q_i^* \left(1 + \frac{p_i^*}{Q_i^*} \frac{\partial Q_i}{\partial p_i}\right) Q_j^* \left(1 + \frac{p_j^*}{Q_j^*} \frac{\partial Q_j}{\partial p_j}\right) - p_i^* p_j^* \frac{\partial Q_j}{\partial p_i} \frac{\partial Q_i}{\partial p_j}}, \ i = 1, 2, \quad j \neq i. \tag{7.9}$$

These expressions are quite hard to sign at this level of generality. To simplify, we use Assumption 1, and we obtain

$$t_1^* = \frac{Q_1^* - \frac{Q_2^* c_2 \frac{\partial Q_2}{\partial p_1}}{Q_2^* + c_2 \frac{\partial Q_2}{\partial p_2}}}{Q_1^* \left(1 + \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}\right)}, \quad t_2^* = \frac{1}{Q_2^* \left(1 + \frac{c_2}{Q_2^*} \frac{\partial Q_2}{\partial p_2}\right)}. \tag{7.10}$$

Suppose now the optimal (decentralized) allocation is such that Q_2^* lies on the elastic part of demand for good 2, so $t_2^* < 0$. If the conditions outlined in Proposition **??** hold, then $-1 < \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}$ and $\frac{\partial Q_2}{\partial p_1} < 0$ hold as well. Therefore, we get $t_1^* > 0$ as long as $Q_1^* > \frac{c_2 \frac{\partial Q_2}{\partial p_1}}{Q_2^* + c_2 \frac{\partial Q_2}{\partial p_2}}$. Furthermore, if $-1 < \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}$ and $-1 < \frac{c_2}{Q_2^*} \frac{\partial Q_2}{\partial p_2}$, both t_2^* and t_1^* are positive. In sum, it is possible that the optimal tax rates on one or both goods are strictly positive.

8 Concluding remarks

A fundamental result in the theory of commodity taxation is that taxes increase consumer prices and reduce supply, aggravating the distortions caused by market power. This result hinges on the assumption that each firm provides a single product. We have studied the effects of commodity taxation in presence of a multiproduct monopolist. We consider a firm providing two goods and obtain simple conditions such that an ad valorem tax reduces the prices and increases the supply of both goods. Whenever both goods are underprovided and imposing an ad valorem tax on one good increases both quantities, the tax has a positive effect on welfare. Differently from an ad valorem tax, a unit tax can reduce both prices, but can only reduce the supply of the taxed good.

This paper broadens previous findings on the Edgeworth's paradox by considering general demand functions and studying ad valorem taxes. We show that taxes can induce a price decrease in a variety of settings, including add-on pricing, multiproduct retailing with price advertising, and intertemporal models with switching costs. Moreover, we generalize previous findings on the effects of taxation in two-sided markets, showing that the effects found by Kind et al. (2008) apply more generally to markets served by a multiproduct firm, even if "one-sided", provided that the demands for the goods are (at least partially) interdependent.

As a final remark, we note that the effects of taxation that we characterized should apply more generally to other settings. In particular, when considering vertical relations, unit taxes are similar to wholesale prices, whereas ad valorem ones are similar to revenue-sharing arrangements. We plan to explore the implications of the mechanisms we identified for vertical relations among multiproduct firms in future research.

References

Agrawal, D. R. and Hoyt, W. H. (2019). Pass-through in a multiproduct world. Available at: https://ssrn.com/abstract=3173180.

Alexandrov, A. and Bedre-Defolie, O. (2017). Lechatelier samuelson principle in games and passthrough of shocks. *Journal of Economic Theory*, 168:44–54.

Anderson, S. P., de Palma, A., and Kreider, B. (2001). The efficiency of indirect taxes under imperfect competition. *Journal of Public Economics*, 81(2):231–251.

Armstrong, M. (2006). Competition in two-sided markets. *The RAND Journal of Economics*, 37(3):668–691.

Armstrong, M. and Vickers, J. (2018). Multiproduct pricing made simple. *Journal of Political Economy*, 126(4):1444–1471.

Armstrong, M. and Vickers, J. (2022). Multiproduct cost passthrough: Edgeworth's paradox revisited. *CEPR Discussion Paper*, (DP17202).

Auerbach, A. J. and Hines, J. (2002). Taxation and economic efficiency. In Auerbach, A. J. and Feldstein, M., editors, *Handbook of Public Economics*. Elsevier, Amsterdam.

Belleflamme, P. and Peitz, M. (2015). *Industrial Organization: Markets and Strategies*. Cambridge University Press, 2 edition.

Belleflamme, P. and Toulemonde, E. (2018). Tax incidence on competing two-sided platforms: Lucky break or double jeopardy. *Journal of Public Economic Theory*, 20(1):9–21.

Besanko, D., DubÃ©, J.-P., and Gupta, S. (2005). Own-brand and cross-brand retail pass-through. *Marketing Science*, 24(1):123–137.

Carbonnier, C. (2014). The incidence of non-linear price-dependent consumption taxes. *Journal of Public Economics*, 118:111–119.

Chen, Z. and Rey, P. (2012). Loss leading as an exploitative practice. *American Economic Review*, 102(7):3462–82.

Coase, R. H. (1946). Monopoly pricing with interrelated costs and demands. *Economica*, 13(52):278–294.

Cremer, H. and Thisse, J.-F. (1994). Commodity taxation in a differentiated oligopoly. *International Economic Review*, 35(3):613–633.

D'Annunzio, A., Mardan, M., and Russo, A. (2020). Multi-part tariffs and differentiated commodity taxation. *The RAND Journal of Economics*, 51(3):786–804.

Delipalla, S. and Keen, M. (1992). The comparison between ad valorem and specific taxation under imperfect competition. *Journal of Public Economics*, 49(3):351–367.

Edgeworth, F. Y. (1925). The pure theory of monopoly. In *Papers Relating to Political Economy*. Burt Franklin, New York.

Ellison, G. (2005). A model of add-on pricing. *The Quarterly Journal of Economics*, 120(2):585–637.

Froot, K. and Klemperer, P. (1989). Exchange rate pass-through when market share matters. *American Economic Review*, 79(4):637–654.

Fullerton, D. and Metcalf, G. E. (2002). Tax incidence. In Auerbach, A. J. and Feldstein, M., editors, *Handbook of Public Economics*, pages 1787–1872. Elsevier, Amsterdam.

Hamilton, S. F. (2009). Excise taxes with multiproduct transactions. *The American Economic Review*, 99(1):458–471.

Hotelling, H. (1932). Edgeworth's taxation paradox and the nature of demand and supply functions. *Journal of Political Economy*, 40(5):577–616.

Johnson, J. P. and Rhodes, A. (2021). Multiproduct mergers and quality competition. *The RAND Journal of Economics*, 52(3):633–661.

Kind, H. J., Koethenbuerger, M., and Schjelderup, G. (2008). Efficiency-enhancing taxation in two-sided markets. *Journal of Public Economics*, 92(5-6):1531–1539.

Klemperer, P. (1995). Competition when Consumers have Switching Costs: An Overview with Applications to Industrial Organization, Macroeconomics, and International Trade. *The Review of Economic Studies*, 62(4):515–539.

Lal, R. and Matutes, C. (1994). Retail pricing and advertising strategies. *The Journal of Business*, 67(3):345–370.

Luco, F. and Marshall, G. (2020). The competitive impact of vertical integration by multiproduct firms. *American Economic Review*, 110(7):2041–64.

Rhodes, A. (2015). Multiproduct retailing. *The Review of Economic Studies*, 82(1 (290)):360–390.

Salinger, M. A. (1991). Vertical mergers in multi-product industries and Edgeworth's paradox of taxation. *The Journal of Industrial Economics*, 39(5):545–556.

Tirole, J. (1988). *The Theory of Industrial Organization*, volume 1 of *MIT Press Books*. The MIT Press.

Tremblay, M. (2018). Taxing a platform: Transaction vs. access taxes. Available at: https://ssrn.com/abstract=2640248.

Wang, Z. and Wright, J. (2017). Ad valorem platform fees, indirect taxes, and efficient price discrimination. *The RAND Journal of Economics*, 48(2):467–484.

Weyl, E. G. and Fabinger, M. (2013). Pass through as an economic tool: Principles of incidence under imperfect competition. *Journal of Political Economy*, 121(3):528–583.

A Proofs and additional analysis not included in the text

A.1 Proof of the results in Section 4.1.1

No consumer buys good 2 if $p_2 > v$, so we can assume without loss that $p_2 \leq v$. Provided that $p_2 \leq v$, the demands for good 1 and 2 are identical, i.e. $Q_2(p) = Q_1(p)$. Consumers buy both goods if and only if

$$v_1 \geq p_1 - (v - p_2). \tag{A.1}$$

To characterize the demands, consider that, given a small search cost, a consumer visits M if and only if condition (A.1) holds (after replacing p_2 with the expected price, which coincides with p_2^e in equilibrium). Since $v_1 \sim U[0,1]$, we have

$$Q_1(p) = 1 - p_1 + (v - p_2^e). \tag{A.2}$$

Given $Q_1 = Q_2$, the profit of M can be written as

$$\pi = (p_2(1-t_2) - c_2 + p_1(1-t_1) - c_1)\, Q_1. \tag{A.3}$$

Replacing $p_2^e = v$ in (A.2) and maximizing (A.3) with respect to p_1, we obtain (4.3).

A.2 Proof of results in Section 4.2.1

A consumer that visits M purchases good i if and only if $v_i \geq p_i$. Thus, the consumer visits if and only if her expected surplus is higher than the search cost s, i.e. $\max(v_2 - p_2^e; 0) + v_1 - p_1 > s$ holds. The demands for good 1 and 2 respectively are

$$Q_1(p) = \int_{p_1}^{b} f(v_1)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right) dv_1, \tag{A.4}$$

$$Q_2(p) = \int_{p_2}^{b} f(v_2)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_2\right) dv_2. \tag{A.5}$$

Observe that Q_1 depends on p_2^e but not on p_2, so $\frac{\partial Q_1}{\partial p_2} = 0$. Furthermore, we obtain the following derivatives:

$$\frac{\partial Q_2}{\partial p_2} = -f(p_2)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s\right) < 0$$

and

$$\begin{aligned} \frac{dQ_1}{dp_1} &= -f(p_1)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s\right) + \\ &\quad + \int_{p_1}^{b} f(v_1)\, \frac{dPr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right)}{dp_1} dv_1 < 0, \end{aligned}$$

where

$$\begin{aligned} \frac{dPr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right)}{dp_1} &= \frac{\partial Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_1} + \\ &+ \frac{\partial Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_2^e} \frac{\partial p_2^e}{\partial p_1} \end{aligned}$$

Given that $\frac{\partial p_2^e}{\partial p_1} > 0$ (Rhodes, 2015, Lemma 2), both derivatives on the right hand side of the above expression must be nonpositive. However, note that each may take a different value depending on whether$v_2 \lessgtr p_2^e$.

Finally, we have

$$\begin{aligned}\frac{dQ_2}{dp_1} &= -f(p_2)\,Pr\left(\max\left(v_2 - p_2^e;0\right) + v_1 - p_1 \geq s\right)\frac{\partial p_2^e}{\partial p_1} + \\ &\quad + \int_{p_2}^{b} f(v_2)\,\frac{dPr\left(\max\left(v_2 - p_2^e;0\right) + v_1 - p_1 \geq s \mid v_2\right)}{dp_1}dv_2 < 0.\end{aligned}$$

where

$$\begin{aligned}\frac{dPr\left(\max\left(v_2 - p_2^e;0\right) + v_1 - p_1 \geq s \mid v_2\right)}{dp_1} &= \frac{\partial Pr\left(\max\left(v_2 - p_2^e;0\right) + v_1 - p_1 \geq s \mid v_2\right)}{\partial p_1} + \\ &+ \frac{\partial Pr\left(\max\left(v_2 - p_2^e;0\right) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_2^e}\frac{\partial p_2^e}{\partial p_1}\end{aligned}$$

Given that $\frac{\partial p_2^e}{\partial p_1} > 0$ (Rhodes, 2015, Lemma 2), both derivatives on the right hand side of the above expression must be nonpositive. However, note that each may take a different value depending on whether $v_2 \lesseqgtr p_2^e$.

The vector of equilibrium prices, p^e, maximizes $\pi = (p_1(1-t_1) - c)\,Q_1 + (p_2 - c)\,Q_2$. In this equilibrium, p_2^e satisfies the following

$$p_2^e = -\frac{Q_2^e}{\frac{\partial Q_2}{\partial p_2}} + c,$$

where $\frac{\partial Q_2}{\partial p_2} < 0$. Hence, p_2 is set according to the standard "cost plus mark-up" formula and is strictly above marginal cost. As shown in Rhodes (2015), this price exceeds the "typical" monopoly price without search costs, because consumers observe p_2 only after searching. Finally, maximising π with respect to p_1, we obtain (4.5).

A.3 Proofs of the results in Section 4.2.2

M provides a single product in two periods, $i = 1, 2$, at a constant unit cost $c < 1$. There is a unit mass of consumers. In each period, a consumer decides whether to buy either one unit of the good or none. If a consumer buys in a given period, she/he gets utility $1 - x$, where x is uniformly distributed on the $[0, 1]$ interval and time-invariant, but if she/he buys (resp. does not buy) in period 1 and not (resp. buys) in period 2, she/he sustains a small switching cost,

<table>
<tr><td rowspan="2">$i = 1$</td><td colspan="2">Buy</td><td colspan="2">Not buy</td></tr>
<tr><td colspan="2">$1 - x - p_1$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">$i = 2$</td><td>Buy</td><td>Not buy</td><td>Buy</td><td>Not buy</td></tr>
<tr><td>$1 - x - p_2$</td><td>$-s$</td><td>$1 - x - p_2 - s$</td><td>0</td></tr>
</table>

Table 2: Consumer payoffs in the switching cost model.

s.[15] Table 2 summarizes a consumer's payoff in period i.[16]

In period 1, consumers observe p_1 and form rational expectations about p_2. Furthermore, they choose whether to buy the M's product anticipating their payoff at the following stage. M's intertemporal profit is $\pi = \sum_{i=1,2} (p_i (1 - t_i) - c) Q_i$, where t_i is the ad valorem tax rate in period i.[17] We ignore intertemporal discounting.

We solve the model by backward induction. In period 2, consumers who bought previously sustain the cost s if not buying anymore (all else given), which increases their willingness to pay. Similarly, the switching cost decreases the willingness to pay by consumers who did not buy from M in period 1. Therefore, the demand function $Q_2(p)$ is kinked, as represented in the left panel of Figure A.1 (see Section A.3.1 for the derivation of this function):

$$Q_2(p) = \begin{cases} 1 - s - p_2 & \text{if } p_2 < 1 - s - Q_1, \\ Q_1 & \text{if } p_2 \in [1 - s - Q_1; 1 + s - Q_1], \\ 1 + s - p_2 & \text{if } p_2 > 1 + s - Q_1. \end{cases} \tag{A.6}$$

Note that $Q_2(p)$ is flat over an interval of values of p_2 such that all old customers buy again, but the price is too high to attract any new customer.

To find p_2^e, we maximize the profit in period 2, $(p_2 (1 - t_2) - c) Q_2(p)$, with respect to p_2. As we show in Section A.3.1, the solution is such that

$$p_2^e = 1 + s - Q_1. \tag{A.7}$$

Hence we find that $Q_2^e = Q_1$. The price p_2^e coincides with the rightmost kink in the demand function $Q_2(p)$. Exploiting the switching cost, M imposes the largest possible markup

[15] See, e.g., Tirole, 1988, and Belleflamme and Peitz, 2015, for a an overview of the literature on switching costs.

[16] For example, suppose M sells a certain software in period 1, and the update in period 2. If the consumer does not buy the software, she/he can use an alternative one available for free. The cost s when switching in period 2 can capture, e.g., the extra effort of adapting to a different software after learning how to use one in the first period.

[17] Different tax rates in period 1 and 2 can be interpreted as the government taxing the good at different rates for new and returning customers.

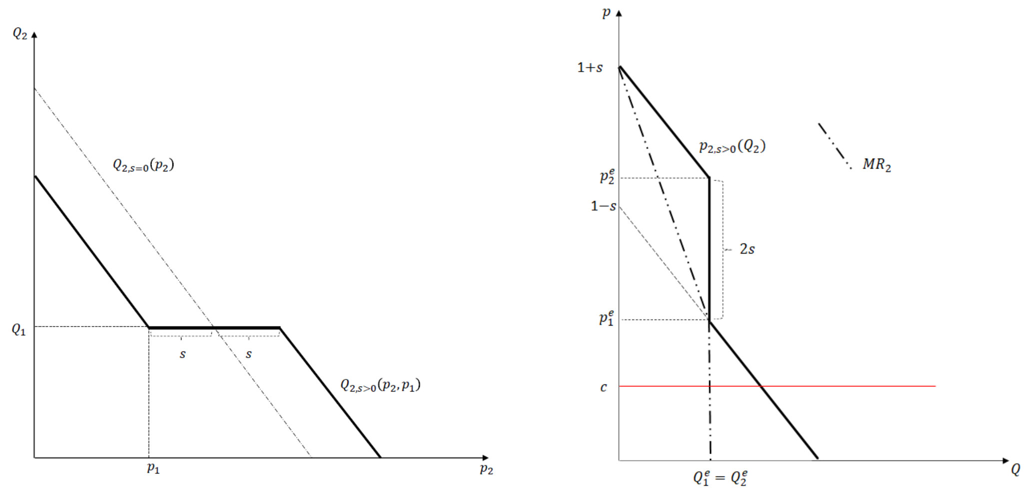

Figure A.1: Left panel: demand in period 2 with and without the switching cost. Right panel: inverse demand, marginal revenue and equilibrium price in period 2.

conditional on maintaining the previous customer base (see Figure A.1, right panel).

In period 1 a consumer buys M's good if and only if she/he anticipates she/he will buy again in period 2. That is, consumers correctly anticipate that they will be locked-in. Therefore, the marginal consumer is indifferent between buying in both periods or not buying at all, i.e. $1 - \bar{x}_1 - p_1 + 1 - \bar{x}_1 - p_2^e = 0$ holds. Given $Q_1 = \bar{x}_1$ and (A.7), we get

$$Q_1(p_1) = 1 - p_1 - s.$$

Replacing the latter expression in (A.7), we obtain that $p_2^e = p_1 + 2s$. Consumers expect M to exploit the switching cost in period 2, and already incorporate this cost when determining their willingness to pay in period 1. Note that the demand in period 1 does not depend on p_2, but on the expected equilibrium price, p_2^e. Hence, Assumption 1 holds in this setting.

Given $p_2^e = p_1 + 2s$ and $Q_2^e = Q_1$, we can write M's intertemporal profit as

$$\pi = (p_1(1 - t_1) - c + (p_1 + 2s)(1 - t_2) - c) Q_1.$$

Maximizing the above expression with respect to p_1 we find

$$p_1^e = \frac{1}{2} + \frac{2c - s(4 - t_1 - 3t_2)}{2(2 - t_1 - t_2)}, \tag{A.8}$$

which boils down to $p_1^0 = \frac{1}{2} + \frac{c}{2} - s$ when all taxes are zero. Focus now on the effects of taxation. Starting from the equilibrium without taxes, the necessary and sufficient condition for the price of good 1 to decrease with t_1 (as stated in Proposition 2) is:

$$\frac{\partial p_1^0}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^0}{Q_1^0}\frac{dQ_1}{dp_1} > -1 \Leftrightarrow c < s. \tag{A.9}$$

Note that, while p_2^e is not directly affected by t_1, it does depend on p_1^e, since $p_2^e = p_1^e + 2s$. Thus, (A.9) is necessary and sufficient for the price to decrease with t_1 in *both* periods (starting from the no-tax equilibrium). This condition is sufficient for Q_1^e and Q_2^e to increase with the tax.

A.3.1 Characterizing expression (A.6)

Let $\bar{x}_1$ denote the marginal consumer in period 1. All consumers such that $x \in [0, \bar{x}_1]$ bought M's product in period 1 and thus incur s if they do not buy again. Within this set of consumers, the marginal consumer in period 2, denoted $\bar{x}_2$, is such that $1 - \bar{x}_2 r - p_2 = -s \Rightarrow \bar{x}_2 = \frac{1+s-p_2}{r}$ holds. Clearly, if and only if $\bar{x}_2 \geq \bar{x}_1$, all consumers who bought in period 1 buy in the next period as well. The consumers who did not buy in period 1 are such that $x \in [\bar{x}_1, 1]$. These consumers thus incur s if they buy in period 2. Hence, the marginal consumer within this group, denoted $\tilde{x}_2$, is such that $1 - \tilde{x}_2 r - s - p_2 = 0 \Rightarrow \tilde{x}_2 = \frac{1-s-p_2}{r}$. Clearly, if and only if $\tilde{x}_2 \leq \bar{x}_1$, no consumer that did not buy previously buys in period 2. Note that $\bar{x}_2 > \tilde{x}_2$ for $s > 0$. We can therefore write the demand for M's product in period 2 as

$$Q_2 = \min\left(\bar{x}_1, \bar{x}_2\right) + \max\left(\tilde{x}_2 - \bar{x}_1, 0\right).$$

Recalling that $Q_1 = \bar{x}_1$, we can rewrite the above expression as in (A.6).

A.3.2 Establishing the equilibrium price of good 2

We first show that the subgame perfect equilibrium cannot be such that $p_2^e > 1 + s - Q_1^e r$, i.e. $Q_2^e < Q_1^e$. If $p_2 > 1 + s - Q_1^e r$ holds, we have $Q_2 = \frac{1+s-p_2}{r}$ given (A.6). The profit in period $i = 2$ is thus $\pi_2 = \left(\frac{1+s-p_2}{r}\right)\left(p_2\left(1-t_2\right)-c\right)$. The maximizer of this function is $p_2 = \frac{1}{2}\left(1 + s + \frac{c}{(1-t_2)}\right)$, and, given this price, we get $Q_2 = \frac{1}{2r}\left(1 + s - \frac{c}{(1-t_2)}\right)$. For consistency, the condition $Q_2 < Q_1^e$, i.e. $\frac{1}{2r}\left(1 + s - \frac{c}{(1-t_2)}\right) < Q_1^e$ must hold. We now check that this condition cannot hold on the equilibrium path. If $Q_2 < Q_1^e$ holds, $\pi_2 = \left(\frac{1+s-p_2}{r}\right)\left(p_2\left(1-t_2\right)-c\right)$ is independent of p_1. Hence, when choosing p_1, M maximizes the profit function $\pi_1 = \left(p_1\left(1-t_1\right)-c\right)Q_1$ with $Q_1 = \bar{x}_1 = \frac{1-p_1}{r}$. The maximizer of this function is $p_1 = \frac{1}{2}\left(1 + \frac{c}{1-t_1}\right)$, which would imply that $Q_1^e = \frac{1}{2r}\left(1 - \frac{c}{1-t_1}\right)$. However, The condition $Q_2 = \frac{1}{2r}\left(1+s-\frac{c}{(1-t_2)}\right) < \frac{1}{2r}\left(1-\frac{c}{1-t_1}\right)$ can hold only if $t_2 > t_1$, which we ruled out by assumption.

In addition, the subgame perfect equilibrium cannot be such that $p_2^e < 1+s-Q_1^e r$, i.e. $Q_2 > Q_1^e$. To see this, suppose there is no switching cost, i.e. $s=0$. Then M's profit in period 2 is independent of p_1. Furthermore, consumer demands are identical in the two periods, which implies that $p_1 = p_2$ and $Q_1 = Q_2$ in equilibrium and that p_2 must be such that the supplier's marginal revenue in period 2 equals c. Suppose now that $s > 0$. As Figure A.1 suggests, at $Q_2 = Q_1^e$ the marginal revenue drops sharply, because the marginal consumer did not buy from M in period 1. Hence, to attract this consumer the supplier must reduce p_2 sharply. Therefore, the marginal revenue at $Q_2 > Q_1^e$ must be smaller than c, which implies that M would be better off increasing p_2 and thus reducing Q_2.

Based on the above arguments, we can restrict attention to the case where $p_2^e \in [1-s-Q_1^e; 1+s-Q_1^e]$. Any value of p_2 within this interval results in the same quantity Q_2, and this quantity equals Q_1^e. Therefore, it must be that the equilibrium price is at the upper bound of the interval, i.e. $p_2^e = 1+s-Q_1^e$.

A.4 Proof of the results in Section 5.1.1

A.4.1 Equilibrium prices set by the platform

Given (5.7) and (5.8), we can express the prices set by the platform as a function of the utility levels provided to each group:

$$p_i(u_i, u_j) = \alpha_i Q_j - u_i = \alpha_i \phi_j(u_j) - u_i, \; i,j = 1,2; j \neq i, \tag{A.10}$$

We can write the expression for the profit made by the platform as

$$\pi(u_i, u_j) = \phi_1(u_1)(p_1(u_1,u_2)(1-t_1) - c_1) + \phi_2(u_2)(p_2(u_1,u_2)(1-t_2) - c_2), \tag{A.11}$$

where the price is as in (A.10). Since the platform's objective only depends on the utility levels (u_1, u_2), there is no loss in proceeding as if these utility levels were the platform's decision variables. The first-order conditions of the problem are such that

$$\frac{\partial \pi}{\partial u_i} = \phi_i'((\alpha_i \phi_j - u_i)(1-t_i) - c_i) - \phi_i + \phi_i' \phi_j \alpha_j = 0 \quad i,j = 1,2 \quad i \neq j.$$

Denote the profit-maximizing utility levels as u_i^e, that satisfy the above system of equations. We find:

$$u_i^e = -\frac{c_i}{1-t_i} - \frac{\phi_i}{\phi_i'} + \left(\alpha_i + \alpha_j \frac{1-t_j}{1-t_i}\right)\phi_j.$$

Replacing them in (A.10), we get the equilibrium prices provided in (5.10).

A.4.2 Effects of taxation

Assume now the monopolist's problem is solved maximizing with respect to prices. Let F_i be the first-order derivative $\frac{\partial \pi}{\partial p_i}, i = 1, 2$. The equilibrium prices, p_i^e, must satisfy the system of equations $\frac{\partial \pi}{\partial p_i} = 0, i = 1, 2$. Hence, (5.2) and (5.3) hold. In this setting, we have

$$\frac{\partial \pi^2}{\partial p_i \partial t_i} = -Q_i^e \left(\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} + 1\right) = \frac{-\frac{\partial Q_i}{\partial p_i}\left(c_i - Q_j^e \alpha_j (1-t_j)\right)}{1-t_i},\ i, j = 1, 2; j \neq i.$$
$$\frac{\partial \pi^2}{\partial p_i \partial t_j} = -p_j^e \frac{\partial Q_j}{\partial p_i} = \frac{\phi_1' \phi_2' \alpha_j}{1-\phi_1'\phi_2'\alpha_1\alpha_2} p_j^e,\ i, j = 1, 2; j \neq i.$$

Consider the effect of t_1 on p_1. Given the above expressions, the direct effect characterized in expression (5.2) is negative if and only if $c_1 < Q_2^e \alpha_2 (1 - t_2)$, whereas the indirect effect is nonpositive if and only if $\alpha_1 \frac{\partial^2 \pi}{\partial p_1 \partial p_2} \leq 0$. As for the effect of t_1 on p_2, the direct effect characterized in equation (5.3) is nonpositive if and only if $\alpha_1 \leq 0$, whereas the indirect effect is nonpositive if and only if $(c_1 - Q_2^e \alpha_2 (1 - t_2)) \frac{\partial^2 \pi}{\partial p_1 \partial p_2} \leq 0$.

A.5 Analysis of the effects of unit taxes

A.5.1 Effect of unit taxes on prices

To focus on unit taxes, we set ad valorem taxes to zero, i.e. $t_i = 0, \forall i$. Differentiating (3.2) with respect to τ_i, we find

$$\frac{\partial p_i^e}{\partial \tau_i} = -\frac{\frac{\partial^2 \pi}{\partial p_i \partial \tau_i}\frac{\partial^2 \pi}{\partial p_j^2} - \frac{\partial^2 \pi}{\partial p_j \partial \tau_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}{H},\quad \frac{\partial p_j^e}{\partial \tau_i} = -\frac{\frac{\partial^2 \pi}{\partial p_j \partial \tau_i}\frac{\partial^2 \pi}{\partial p_i^2} - \frac{\partial^2 \pi}{\partial p_i \partial \tau_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}{H},\ i = 1, 2, \quad j \neq i, \tag{A.12}$$

where

$$\frac{\partial^2 \pi}{\partial p_i \partial \tau_i} = -\frac{\partial Q_i}{\partial p_i} > 0, \quad \frac{\partial^2 \pi}{\partial p_j \partial \tau_i} = -\frac{\partial Q_i}{\partial p_j},$$
$$\frac{\partial^2 \pi}{\partial p_i^2} < 0, \quad \text{and } H \equiv \frac{\partial^2 \pi}{\partial p_1^2}\frac{\partial^2 \pi}{\partial p_2^2} - \left(\frac{\partial \pi}{\partial p_1 \partial p_2}\right)^2 > 0.$$

If the demands for the two goods are independent, it is easily shown that $\frac{\partial p_i^e}{\partial \tau_i} > 0$ and $\frac{\partial p_j^e}{\partial \tau_i} = 0$ hold for both goods. When demands are interdependent, the denominator of the expressions in (A.12) is positive by the second-order conditions of firm M's problem. So we have

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial \tau_i}\right) = \operatorname{sgn}\left(\underbrace{-\frac{\partial^2 \pi}{\partial p_i \partial \tau_i}\frac{\partial^2 \pi}{\partial p_j^2}}_{\text{direct effect}} + \underbrace{\frac{\partial^2 \pi}{\partial p_j \partial \tau_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}_{\text{indirect effect}}\right), \; i = 1, 2, \quad j \neq i. \tag{A.13}$$

The *direct effect* is unambiguously positive, i.e. it tends to increase the price, because $\frac{\partial^2 \pi}{\partial p_j^2} < 0$ and $-\frac{\partial Q_i}{\partial p_i} > 0$ hold. Given p_j^e, M can reduce the tax burden only by reducing the quantity of good i, raising p_i^e. The *indirect effect* of τ_i on p_i^e is similar to that of an ad valorem tax: this effect is negative if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{\partial Q_i}{\partial p_j} > 0$ (e.g., when good i is a substitute to good j), or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{\partial Q_i}{\partial p_j} < 0$ (e.g., when good i is a complement to good j). Therefore, after rearranging (A.13), we obtain that. $\frac{\partial p_i^e}{\partial \tau_i} < 0$, if and only if, for $i = 1, 2, \quad j \neq i$

$$\begin{aligned} \frac{\partial Q_i}{\partial p_j} &> \frac{\frac{\partial Q_i}{\partial p_i}\frac{\partial^2 \pi}{\partial p_j^2}}{\frac{\partial^2 \pi}{\partial p_1 \partial p_2}} \qquad \text{if } \frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0, \\ \frac{\partial Q_i}{\partial p_j} &< \frac{\frac{\partial Q_i}{\partial p_i}\frac{\partial^2 \pi}{\partial p_j^2}}{\frac{\partial^2 \pi}{\partial p_1 \partial p_2}} \qquad \text{if } \frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0. \end{aligned} \tag{A.14}$$

Since the numerator on the right hand side is positive, necessary conditions for p_i^e to decrease with τ_i are that either $\frac{\partial Q_i}{\partial p_j} > 0$ when $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ holds, and that $\frac{\partial Q_i}{\partial p_j} < 0$ when $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$.

Consider now the effect of τ_i on the price of good j. We have

$$\operatorname{sgn}\left(\frac{\partial p_j^e}{\partial \tau_i}\right) = \operatorname{sgn}\left(\underbrace{-\frac{\partial^2 \pi}{\partial p_j \partial \tau_i}\frac{\partial^2 \pi}{\partial p_i^2}}_{\text{direct effect}} + \underbrace{\frac{\partial^2 \pi}{\partial p_i \partial \tau_i}\frac{\partial^2 \pi}{\partial p_1 \partial p_2}}_{\text{indirect effect}}\right), \; i = 1, 2, \quad j \neq i. \tag{A.15}$$

The *direct effect* is similar to that of an ad valorem tax: since $\frac{\partial^2 \pi}{\partial p_i^2} < 0$, the direct effect of τ_i on p_j^e is negative if and only if $\frac{\partial Q_i}{\partial p_i} > 0$ holds. Furthermore, the *indirect effect* is negative if and only if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ holds, given that $\frac{\partial F_i}{\partial \tau_i} = -\frac{\partial Q_i}{\partial p_i} > 0$. Indeed, as we have seen, the direct of τ_i on p_i is positive. Hence, if the profitability of raising p_j decreases when the price of good i goes up (that is, the prices move in opposite directions), the indirect effect tends to reduce p_j. By rearranging (A.15), we obtain that $\frac{\partial p_j^e}{\partial \tau_i} < 0$ if and only if

$$\frac{\partial Q_i}{\partial p_j} > \frac{\frac{\partial^2 \pi}{\partial p_1 \partial p_2}\frac{\partial Q_i}{\partial p_i}}{\frac{\partial^2 \pi}{\partial p_j^2}}, \; i = 1, 2, \quad j \neq i. \tag{A.16}$$

As with $\frac{\partial p_i^e}{\partial \tau_i}$, therefore, a necessary condition for p_j^e to decrease with τ_i is that the good is substitute to i when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$, or that the good is a complement to i when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$.

A.5.2 Effect of unit tax on quantity of taxed good

To show the results in the most direct way, we provide the solution of M's profit maximization problem under the alternative assumption that, rather than prices, quantities are the decision variables of the monopolist. By assumption, the demand system in our model is invertible. Let $p_i(Q_i, Q_j)$ be the inverse demands for goods $i = 1, 2$. The first-order conditions of the monopolist's problem (assuming $t_1 = t_2 = \tau_j = 0$ for simplicity) write as

$$\frac{\partial \pi}{\partial Q_i} = Q_i \frac{\partial p_i}{\partial Q_i} + (p_i - c_i - \tau_i) + Q_j \frac{\partial p_j}{\partial Q_i} = 0, \quad i, j = 1, 2, \quad j \neq i. \tag{A.17}$$

To determine the effect of a change in τ_i on the equilibrium quantity Q_i^e, we totally differentiate (A.17) to obtain

$$\frac{\partial Q_i^e}{\partial \tau_i} = -\frac{\frac{\partial^2\pi}{\partial Q_i \partial \tau_i}\frac{\partial^2\pi}{\partial Q_i \partial Q_i} - \frac{\partial^2\pi}{\partial Q_i \partial Q_i}\frac{\partial^2\pi}{\partial Q_j \partial \tau_i}}{H}, \; i, j = 1, 2, \quad j \neq i, \tag{A.18}$$

where

$$\frac{\partial^2\pi}{\partial Q_i \partial \tau_i} = -1, \quad \frac{\partial^2\pi}{\partial Q_j \partial \tau_i} = 0,$$
$$\frac{\partial^2\pi}{\partial Q_j^2} < 0 \quad \text{and} \quad H \equiv \frac{\partial^2\pi}{\partial Q_1^2}\frac{\partial^2\pi}{\partial Q_2^2} - \left(\frac{\partial^2\pi}{\partial Q_1 \partial Q_2}\right)^2 > 0.$$

The denominator of (A.18) is positive by the second-order conditions of the maximization problem. The numerator is equal to $-\frac{\partial^2\pi}{\partial Q_i^2} > 0$. Hence, we obtain that $\frac{\partial Q_i^e}{\partial \tau_i} < 0$.

A.6 Effect of uniform ad valorem tax

We set unit taxes to zero and assume $t_1 = t_2 = t$. Differentiating (3.2) with respect to t, we find

$$\frac{\partial p_i^e}{\partial t} = -\frac{\frac{\partial^2\pi}{\partial p_i \partial t}\frac{\partial^2\pi}{\partial p_j^2} - \frac{\partial^2\pi}{\partial p_j \partial t}\frac{\partial^2\pi}{\partial p_1 \partial p_2}}{H}, \; \frac{\partial p_j^e}{\partial t} = -\frac{\frac{\partial^2\pi}{\partial p_j \partial t}\frac{\partial^2\pi}{\partial p_i^2} - \frac{\partial^2\pi_i}{\partial p_i \partial t}\frac{\partial^2\pi}{\partial p_1 \partial p_2}}{H}, \; i = 1, 2, \quad j \neq i, \tag{A.19}$$

where

$$\frac{\partial^2\pi_i}{\partial p_i \partial t} = -\left(p_i \frac{\partial Q_i}{\partial p_i} + p_j \frac{\partial Q_i}{\partial p_i} + Q_i\right), \quad \frac{\partial^2\pi}{\partial p_j \partial t} = -\left(p_i \frac{\partial Q_i}{\partial p_j} + p_j \frac{\partial Q_j}{\partial p_j} + Q_j\right),$$
$$\frac{\partial^2\pi}{\partial p_i^2} < 0, \quad \text{and } H \equiv \frac{\partial^2\pi}{\partial p_1^2}\frac{\partial^2\pi}{\partial p_2^2} - \left(\frac{\partial\pi}{\partial p_1 \partial p_2}\right)^2 > 0.$$

If the demands for the two goods are independent, it is easily shown that $\frac{\partial p_1^e}{\partial t} > 0$ and $\frac{\partial p_2^e}{\partial t} > 0$ hold. When demands are interdependent, the denominator of the expressions in (A.19) is positive by the second-order conditions of firm M's problem. So we have

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial t}\right) = \operatorname{sgn}\left(\underbrace{-\frac{\partial^2\pi}{\partial p_i\partial t}\frac{\partial^2\pi}{\partial p_j^2}}_{\text{direct effect}} + \underbrace{\frac{\partial^2\pi}{\partial p_j\partial t}\frac{\partial^2\pi}{\partial p_1\partial p_2}}_{\text{indirect effect}}\right),\ i = 1, 2, \quad j \neq i. \tag{A.20}$$

The sign of the *direct effect* depends on the sign of $\left(p_i\frac{\partial Q_i}{\partial p_i} + p_j\frac{\partial Q_i}{\partial p_i} + Q_i\right)$, because $\frac{\partial^2\pi}{\partial p_j^2} < 0$. In equilibrium, given the FOC (3.2), we have that $p_i\frac{\partial Q_i}{\partial p_i} + p_j\frac{\partial Q_i}{\partial p_i} + Q_i = \frac{c_i}{1-t}\frac{\partial Q_i}{\partial p_i} + \frac{c_j}{1-t}\frac{\partial Q_j}{\partial p_i}, i, j = 1, 2, i \neq j$. This term is negative if $\frac{\partial Q_j}{\partial p_i} < 0$ (goods are complements), and can be positive only if $\frac{\partial Q_j}{\partial p_i} > 0$ (goods are substitutes). As for the indirect effect, this effect is negative if $\frac{\partial^2\pi}{\partial p_1\partial p_2} > 0$ and $\frac{c_i}{1-t}\frac{\partial Q_i}{\partial p_j} + \frac{c_j}{1-t}\frac{\partial Q_j}{\partial p_j} < 0$, or if $\frac{\partial^2\pi}{\partial p_1\partial p_2} < 0$ and $\frac{c_i}{1-t}\frac{\partial Q_i}{\partial p_j} + \frac{c_j}{1-t}\frac{\partial Q_j}{\partial p_j} > 0$.

To proceed, let us specify the demand functions: assume these functions are linear and symmetric, i.e. $Q_i = \alpha - \beta p_i - \gamma p_j, \quad i, j = 1, 2,\ i \neq j$,where $\alpha > 0$ and $\beta < 0$. Furthermore, $\gamma > 0$ if the goods are complements (i.e., $\frac{\partial Q_i}{\partial p_j} < 0$), whereas $\gamma < 0$ if the goods are substitutes ($\frac{\partial Q_i}{\partial p_j} > 0$). Let us also assume the goods have identical unit cost c. Under these assumptions, we have $\frac{\partial^2\pi}{\partial p_1^2} = \frac{\partial^2\pi}{\partial p_2^2} = -2\beta\,(1-t)$ and $\frac{\partial^2\pi}{\partial p_1\partial p_2} = 2\gamma(1-t)$. Furthermore, we have $\frac{\partial^2\pi}{\partial p_1\partial t} = \frac{\partial^2\pi}{\partial p_2\partial t} = \frac{c}{1-t}\,(\beta - \gamma)$. Given these assumptions, we can rewrite (A.20) as

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial t}\right) = \operatorname{sgn}\left(2c\left(\beta^2 - \gamma^2\right)\right),\ i = 1, 2. \tag{A.21}$$

Which implies that $\frac{\partial p_i^e}{\partial t} > 0$ if and only if $\beta > |\gamma|$. This condition is also necessary and sufficient to obtain that $\frac{\partial Q_i^e}{\partial t} < 0$.