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Jolian McHardy
Michael Reynolds
Stephen Trotter

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Department of Economics
University of Sheffield
9 Mappin Street
Sheffield
S1 4DT
United Kingdom
www.shef.ac.uk/economics

# The Stackelberg Model as a Partial Solution to the Problem of Pricing in a Network*

Jolian McHardy †
Department of Economics
University of Sheffield

Michael Reynolds
School of International Studies
University of Bradford

Stephen Trotter
Centre for Economic Policy
University of Hull

**Abstract**

We consider an application of the Stackelberg leader-follower model in prices in a simple two-firm network as a possible way to help resolve externalities that can be harmful to firm profit and welfare. Whilst independent pricing on the network yields lower profit and sometimes even lower welfare than monopoly pricing, we show that by allowing the firms to collude on some prices in a first-stage and set remaining prices independently (competitively) in a second stage, both profit and welfare gains can be made.

***Keywords***: Stackelberg; pricing; network
***JEL Classification***: L11; L14; L51

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†Corresponding author: Tel.: +44 (0)1142223460; Fax: +44 (0)1142223458; Email: j.mchardy@shef.ac.uk

# 1 Introduction

In a network, characterised by demand interdependencies along both substitute and complementary lines, encouraging competition in the form of independent profit maximisation can be harmful to both firm profit and welfare due to negative externalities arising through the complementary aspect of the relationship between the demand for goods. In its most simple form, this negative externality is seen in the model of complementary monopoly due to Cournot (1838), whereby welfare is higher when the complementary monopolists collude relative to where they independently maximise profit. Indeed, allowing all firms to jointly maximise profit, effectively creating a network monopoly, can improve welfare and profit (such an argument based on prices is demonstrated in Economides and Salop, 1992). With regulators rarely pursuing first-best policies due to the profound difficulties in implementing them, not least due to considerable transactions and information costs, we offer a possible solution which can yield both welfare and profit enhancements relative to the cases of joint and independent profit maximisation.[1] We illustrate our result in the context of a market in which two firms each produce an $X$ and $Y$ component of a composite good which can be interchanged in consumption, hence there are four possible consumption bundles: two cross-firm bundles and two single-firm bundles. The approach separates decision-making along complementary and substitute lines and permits firms to collude in a first-stage on cross-firm bundle prices, which captures the complementary effects, and then choose prices to independently maximise profits for the remaining single-firm prices in a second stage. This framework is solved as a Stackelberg (von. Stackelberg, 1934) equilibrium (in with prices with differentiated products) and can be shown to produce welfare outcomes which are everywhere strictly superior to network monopoly. Though the welfare outcome does not everywhere dominate that of independent profit maximisation, it does offer a resolution to the problem created where the importance of complementary ties between demands is sufficient that independent profit maximisation has a lower welfare outcome than network monopoly.

In the following Section we consider the profit and welfare outcomes under two regimes: full collusion across the network and independent pricing between the two firms. In Section 3 we introduce a new regime in which the

[1] An alternative approach to the problem of network monopoly is to separate the market along the lines once proposed (but ultimately discarded) in the Microsoft case, which will be welfare improving only if there is sufficient post-split entry into the market (see McHardy, 2006).

two firms are allowed to collude on the setting of the cross-firm prices in a first-stage and then set their single-firm prices simultaneously and independently in a second stage. Section 4 concludes the paper.

# 2 Model

We envisage a simple network with four commodity bundles $Q_{ij}$ $(i, j = 1, 2)$, where demands for the bundles are interrelated based upon the utility function:

$$U(\mathbf{Q}) = 4\alpha \sum Q_{ij} - \frac{n}{2(1+\mu)} \left[ \sum (Q_{ij})^2 + \frac{\mu}{n} (\sum Q_{ij})^2 \right] + z, \; (i, j = 1, 2). \tag{1}$$

Eq. (1) is the standard quadratic utility function where, for our purposes, $n$ is the number of commodity bundles (here $n = 4$), $\mu \in [0, \infty)$ is a measure of the degree of substitutability amongst the commodity bundles (with $\mu = 0$ for zero substitutability and $\mu \to \infty$ for perfect substitutes), $z$ is a numeraire good hence $U(\mathbf{Q})$ is quasi-linear, justifying the use of a partial equilibrium analysis, $v$ is a positive parameter and $\mathbf{P}$ is a vector of commodity bundle prices, $P_{ij}$. Consequently, demand for commodity bundle $Q_{ij}$ is linear in prices:

$$Q_{ij}(\mathbf{P}) = \alpha - \left( \frac{1}{4} + \frac{3\mu}{16} \right) P_{ij} + \frac{\mu}{16} \sum_{mn \neq ij} P_{mn}, \quad (i, j = 1, 2) \tag{2}$$

In this specification, the coefficient on each own-price, which is related to the partial own-price elasticity of demand, is common for each commodity bundle. The cross-price co-efficient is also common across all alternative commodity combinations to $ij$: all the alternative commodity bundles are equally good, but generally (for $\mu < \infty$) imperfect, substitutes. For simplicity and to aid tractability, we set costs equal to zero.

To begin, suppose the network is operated by a monopolist. Profit is given by $\Pi^M = \mathbf{P}'\mathbf{Q}(\mathbf{P})$. Maximising with respect to $\mathbf{P}$, and solving the four first-order conditions simultaneously, we have:

$$P_{ij}^M = 2\alpha. \tag{3}$$

Profit and welfare under monopoly are then, respectively:

$$\Pi^M = 4\alpha^2, \quad W^M = 6\alpha^2. \tag{4}$$

We now consider the profit and welfare situation in the case that the monopoly is split into two, with each rival firm producing an $X$ and $Y$ commodity and setting prices on them to independently maximise profit. Profit for firm $i$ is now:

$$\Pi_i^D = P_{ii}Q_{ii}(\mathbf{P}) + p_{xi}Q_{ij}(\mathbf{P}) + p_{yi}Q_{ji}(\mathbf{P}), \tag{5}$$

where $p_{xi}$ is the price set by firm $i$ for its $x$ commodity that is consumed in the composite $ij$ with demand $Q_{ij}$. Note, $P_{ij} \equiv p_{xi} + p_{yj}$. Maximising firm $i$'s profit with respect to its arguments, $P_{ii}$, $p_{xi}$ and $p_{yi}$, and solving the first order conditions simultaneously, yields the equilibrium composite prices, under network duopoly:

$$P_{ii}^D = \frac{16\alpha(\mu+1)}{8+9\mu+4\mu^2}, \tag{6a}$$

$$P_{ij}^D = p_{xi}^D + p_{yj}^D = \frac{16\alpha(\mu+4)}{3(8+9\mu+4\mu^2)}. \tag{6b}$$

Industry profit and welfare under independent pricing are then, respectively:

$$\Pi^D = \frac{32\alpha^2(68+179\mu+160\mu^2+40\mu^3)}{9(8+9\mu+4\mu^2)^2}, \tag{7a}$$

$$W^D = \frac{8\alpha^2(376+1084\mu+1241\mu^2+632\mu^3+144\mu^4)}{9(8+9\mu+4\mu^2)^2}. \tag{7b}$$

**Lemma 1.** [2] *Let $\Omega_{DM}(\mu) \equiv \Pi^D/\Pi^M$. (i) $\Omega_{DM}(\mu) < 1$ , (ii) $\Omega_{DM}(\mu)$ is increasing (decreasing) for $\mu \in (0,a)$ ($\mu \in (a,\infty)$) and strictly concave (convex) for $\mu \in (0,b)$ ($\mu \in (b,\infty)$), where $a \cong 0.320$ and $b \cong 1.177$ (3 d.p.), and (iii) $\lim_{\mu\to\infty} \Omega_{DM}(\mu) = 0$.*

**Lemma 2.** *Let $\Upsilon_{MD}(\mu) \equiv W^M/W^D$. (i) $\Upsilon_{MD}(\mu)$ is strictly increasing and concave, (ii) $\Upsilon_{MD}(\mu)$ is greater (less) than unity for $\mu < c$ ($\mu > c$), where*

[2] All proofs are reported in the Appendix.

$c \cong 0.282$ *(3 d.p.), and (iii)* $\lim_{\mu\to\infty} \Upsilon_{MD}(\mu) = 1.333$ *(3 d.p.).*

Hence we have shown that if $\mu$ is sufficiently small, and hence the complementary effects are sufficiently large relative to the substitute effects in the demand system, then monopoly is preferable to independent pricing in terms of welfare.[3] This is quite undesirable from a public policy point of view. In the following Section we show that this problem can be entirely eliminated by employing a leader-follower approach to price setting, exploiting the Stackelberg model.

# 3 Stackelberg Pricing

We now examine the impact upon the equilibrium prices, profits and welfare under a new regime in which the two firms are allowed to collude on the setting of their components of the cross-firm prices ($P_{ij}$ where $i \neq j = 1,2$) in stage 1 and then are required to set their single-firm prices ($P_{ii}$) simultaneously, in stage 2. Given there is full information regarding stage 1 prices when the second stage simultaneous pricing takes place, this regime is characterised by a Stackelberg model. The firms will seek to set cross-firm prices at stage 1 which ensure that they maximise their overall profit given they will be independently maximising profit on single-firm prices in stage 2. Working backwards, for the independent and simultaneous price setting in stage 2, firm $i$ will be maximising:

$$\Pi_i^S = P_{ii}Q_{ii}(\mathbf{P}) + p_{xi}Q_{ij}(\mathbf{P}) + p_{yi}Q_{ji}(\mathbf{P}), \tag{8}$$

with respect to $P_{ii}$ with the values of $p_{xi}, p_{yi}, p_{xj}$ and $p_{yj}$ given. Differentiating Eq. (8) with respect to $P_{ii}$, we get the first-order condition:

$$\frac{\partial \Pi_i^S}{\partial P_{ii}} = \alpha - \frac{1}{2}\left(1 + \frac{3\mu}{4}\right) P_{ii} + \frac{\mu}{16}(P_{jj} + 2(p_{xi} + p_{yi}) + p_{xj} + p_{yj}) = 0. \tag{9}$$

Using the implicit function theorem, it is straightforward to derive from Eq. (9) the stage 2 responses by firm $i$ (firm $j$) in terms of its optimal choice of $P_{ii}$ ($P_{jj}$) given a change in the period 1 choices of $p_{xi}$, $p_{yi}$, $p_{xj}$ and $p_{yj}$, which are as follows:

$$\frac{dP_{ii}}{dp_{xi}} = \frac{dP_{ii}}{dp_{yi}} = \frac{\mu}{4 + 3\mu}, \tag{10a}$$

[3] See Economides and Salop (1992) for an equivalent result based on prices.

$$\frac{dP_{ii}}{dp_{xj}} = \frac{dP_{ii}}{dp_{yj}} = \frac{\mu}{2(4+3\mu)}. \tag{10b}$$

Now the problem for the firms is to set the first-stage price on the cross-firm commodities given they know that their stage 2 behaviour will have the effects indicated in Eqs. (10a) and (10b). Hence, the two firms seek to set $P_{ij}$ $(i \neq j = 1, 2)$ so as to jointly maximise profit. Joint profit is given by $\Pi^S \equiv \Pi_i^S + \Pi_j^S$. Let $\mathbf{p}$ be the 4-vector with elements $p_{xi}$, $p_{xj}$, $p_{yi}$ and $p_{yj}$. Joint profit can then be expressed as:

$$\begin{aligned}
\Pi^S = {} & \frac{P_{ii}(\mathbf{p})}{16}[16\alpha - (4+3\mu)P_{ii}(\mathbf{p}) + \mu(P_{jj}(\mathbf{p}) + p_{xi} + p_{xj} + p_{yi} + p_{yj})] \\
& + \frac{P_{jj}(\mathbf{p})}{16}[16\alpha - (4+3\mu)P_{jj}(\mathbf{p}) + \mu(P_{ii}(\mathbf{p}) + p_{xi} + p_{xj} + p_{yi} + p_{yj})] \\
& + \frac{(p_{xi}+p_{yj})}{16}[16\alpha - (4+3\mu)(p_{xi}+p_{yj}) + \mu(P_{ii}(\mathbf{p}) + P_{jj}(\mathbf{p}) + p_{xj} + p_{yi})] \\
& + \frac{(p_{xj}+p_{yi})}{16}[16\alpha - (4+3\mu)(p_{xj}+p_{yi}) + \mu(P_{ii}(\mathbf{p}) + P_{jj}(\mathbf{p}) + p_{xi} + p_{yj})].
\end{aligned} \tag{11}$$

Differentiating Eq. (11) with respect to $p_{xi}$, recognising that $P_{ii}(\mathbf{p})$ and $P_{jj}(\mathbf{p})$, according to Eqs. (10a) and (10b):[4]

[4] Whilst the relevant second order conditions in all earlier cases are met trivially, the following demonstrates the conditions under which the second order condition is met for the first-period joint profit maximisation case. First, for convenience let:

$$\frac{\partial^2 \Pi^S}{\partial r \partial s} \equiv \Pi_{rs}^S, \qquad (r, s = 1, 2, 3, 4), \tag{12}$$

where $x_1, x_2, y_1$ and $y_2$ are denoted, respectively, by $1, 2, 3$ and $4$. The Hessian for the problem can therefore be summarised (given the symmetry of the problem, e.g., $\Pi_{11}^S = \Pi_{22}^S$, and given the symmetry due to Young's Theorem e.g., $\Pi_{12}^S = \Pi_{21}^S$), as:

$$\begin{vmatrix} \Pi_{11}^S & \Pi_{12}^S & \Pi_{13}^S & \Pi_{14}^S \\ \Pi_{12}^S & \Pi_{11}^S & \Pi_{14}^S & \Pi_{13}^S \\ \Pi_{13}^S & \Pi_{14}^S & \Pi_{11}^S & \Pi_{12}^S \\ \Pi_{14}^S & \Pi_{13}^S & \Pi_{12}^S & \Pi_{11}^S \end{vmatrix} \tag{13}$$

Given the matrix is symmetric, for a maximum we require that the diagonal is negative and dominant (see, for instance Theorem M.D.5, Mas-Colell et al., 1995, p. 939). The diagonal is negative if $\Pi_{11}^S < 0$, which it is since $-\left(\frac{1}{8} + \frac{7\mu}{32} + \frac{79\mu^2}{512} + \frac{89\mu^3}{2048}\right) < 0$, given $\mu \in [0, \infty)$. Similarly, the diagonal is dominant if $\Pi_{11}^S - \Pi_{12}^S - \Pi_{13}^S - \Pi_{14}^S < 0$, which it is since $-(1/256)\mu(1028\mu^2 + 3\mu^3 + 2048\mu + 1024)/(4+3\mu)^2 < 0$.

$$\frac{\partial \Pi^S}{\partial p_{xi}} = \frac{1}{16(3\mu+4)}[24\alpha\mu - 3\mu(1+\mu)P_{ii} + 2\mu^2 P_{jj} \qquad (14)$$
$$- (p_{xi} + p_{yj})(15\mu^2 + 48\mu + 32) + 2\mu(p_{xj} + p_{yi})(2+3\mu)].$$

Recognising symmetry, let $P_{ii}^S = P_{ii} = P_{jj}$ and $P_{ij}^S = P_{ij} = p_{xi} + p_{yj} = p_{xj} + p_{yi} = P_{ji}$, Eqs. (9) and (14) can be written, respectively:

$$16\alpha + 3\mu P_{ij} - (8+5\mu)P_{ii} = 0, \qquad (15a)$$

$$\mu P_{ii}(2+3\mu) - (20\mu + 3\mu^2 + 16)P_{ij} + 4\alpha(8+9\mu) = 0. \qquad (15b)$$

Solving Eqs. (15a) and (15b) simultaneously we have:

$$P_{ij}^S = \frac{2\alpha(64 + 120\mu + 57\mu^2)}{120\mu + 59\mu^2 + 64 + 3\mu^3}, \quad (i \neq j = 1, 2), \qquad (16a)$$

$$P_{ii}^S = \frac{2\alpha(64 + 104\mu + 39\mu^2)}{120\mu + 59\mu^2 + 64 + 3\mu^3}. \qquad (16b)$$

**Lemma 3.** *Under the Stackelberg equilibrium, the first-stage cross-commodity bundle price $P_{ij}^S$ is weakly greater than the second-stage own-commodity bundle price $P_{ii}^S$, with equality of prices under $\mu = 0$.*

Hence, as we would expect, the firms are able to achieve a higher price under collusion and in a first-stage "price-leader" position, than in the second-stage, independent pricing "follower" position. Industry profit and welfare under the Stackelberg equilibrium are then, respectively:

$$\Pi^S = \frac{4\alpha^2(4096 + 11264\mu + 10560\mu^2 + 3600\mu^3 + 207\mu^4)}{(64 + 120\mu + 59\mu^2 + 3\mu^3)(64 + 56\mu + 3\mu^2)}, \qquad (17a)$$

$$W^S = \frac{2\alpha^2(12288 + 34816\mu + 34240\mu^2 + 12976\mu^3 + 1299\mu^4 + 36\mu^5)}{(1+\mu)(64 + 56\mu + 3\mu^2)^2}. \qquad (17b)$$

**Lemma 4.** *The Stackelberg equilibrium only produces interior solutions for:*

$$\mu < \frac{28 + 4\sqrt{61}}{3}. \qquad (18)$$

**Proposition 1.** *Let* $\Omega_{SM}(\mu) \equiv \Pi^S/\Pi^M$. *(i)* $\Omega_{SM}(\mu)$ *is negative monotonic in* $\mu$ *and strictly concave (convex) for* $\mu \in [0, d)$ *(*$\mu \in (d, \infty)$*), where* $d \cong 2.602$ *(3 d.p.), (ii)* $\Omega_{SM}(\mu) \leq 1$*, and (iii)* $\lim_{\mu\to\infty} \Omega_{SM}(\mu) = 0$.

**Proposition 2.** *Let* $\Upsilon_{MS}(\mu) \equiv W^M/W^S$. *(i)* $\Upsilon_{MS}(\mu)$ *is negative monotonic in* $\mu$ *and strictly convex, (ii)* $\Upsilon_{MS}(\mu) \leq 1$ *for* $\mu \geq 0$*, (iii)* $\lim_{\mu\to\infty} \Upsilon_{MS}(\mu) = 0.75$.

Hence, profit is weakly greater under monopoly than under the Stackelberg regime but the reverse is true in the case of equilibrium welfare. Hence we have demonstrated that the Stackelberg regime does not suffer from the same problem as independent profit maximisation, inasmuch as there exists no interval of $\mu$ for which the associated equilibrium is more damaging to welfare than monopoly. However, to understand the relationship between the Stackelberg and independent profit maximising outcomes, we complete the analysis with the following Propositions.

**Proposition 3.** *Let* $\Omega_{SD}(\mu) \equiv \Pi^S/\Pi^D$. *(i)* $\Omega_{SD}(\mu)$ *is increasing (decreasing) for* $\mu \in [0, e)$ *(*$\mu \in (e, \infty)$*)and strictly concave (convex) for* $\mu \in [0, f)$ *(*$\mu \in (f, \infty)$*), where* $e \cong 0.3495$ *(4 d.p) and* $f \cong 1.199$ *(3 d.p.), (ii)* $\lim_{\mu\to\infty} \Omega_{SD}(\mu) = 0$*, and (iii)* $\Omega_{SD}(\mu) > (\leq) 1$ *for* $g > \mu > h$ *(*$\mu \in [g, h]$*), where* $g \cong 0.349$ *and* $h \cong 0.350$ *(3 d.p.).*

**Proposition 4.** *Let* $\Upsilon_{DS}(\mu) \equiv W^D/W^S$. *(i)* $\Upsilon_{DS}(\mu)$ *is increasing (decreasing) for* $\mu \in [0, k)$ *(*$\mu \in (k, \infty)$*) and strictly concave (convex) for* $\mu \in [0, r)$ *(*$\mu \in (r, \infty)$*), where* $k \cong 2.854$ *and* $r \cong 5.538$ *(3 d.p.), (ii)* $\Upsilon_{DS}(\mu) < (\geq) 1$ *for* $\mu < (\geq) t$*, where* $t \cong 0.350$ *(3 d.p.), (iii)* $\lim_{\mu\to\infty} \Upsilon_{DS}(\mu) = 1$.

Hence, we have shown that welfare under the Stackelberg regime dominates that under independent profit maximisation for an interval of $\mu$ which extends strictly beyond that for which independent profit maximisation is inferior to monopoly. Indeed, over much the same range, we note that profit to the firms is also generally greater under the Stackelberg regime than under independent profit maximisation.

# 4 Discussion and Conclusions

In this paper we have employed a simple model based upon a network of linear demands to show that, by allowing firms to set come prices collusively and other prices independently, in a two stage process, both firm profit and welfare can be improved relative to a situation of independent pricing everywhere. In particular, if the complementary effects are sufficiently large relative to the substitute effects in the demand structure of the network, then independent pricing everywhere results in a lower welfare outcome than monopoly. In such circumstances, the 'Stackelberg' pricing structure proposed in this paper produces an outcome which is strictly superior in welfare terms compared with both monopoly and independent pricing, but is also generally more attractive than independent pricing to the firms. Hence, the Stackelberg pricing regime may not only help resolve a problem of welfare loss but will also be attractive to the firms, reducing the risk of non-compliance.

Regarding the robustness of our findings, whilst we assume a convenient form for utility which produces symmetric and linear demands we note that scope will still exist for generalisations away from our assumptions where there the demand structure includes complementary and substitute characteristics. It is the existence of externalities due to the complementarity in the demand structure which adversely affects welfare under independent profit maximisation relative to monopoly. By addressing this externality, allowing the firms to collude on some prices in a first-stage of the game, this externality is partially addressed, raising both profit and welfare. Finally, though for reasons of tractability costs were assumed away, the above argument suggests that the inclusion, at least of constant marginal cost, should not eliminate potential gains from addressing the externality due to the complementarity in the demand network.

# Appendix

*Proof of Lemma 1.* First, note $\Omega_{DM}$ is continuous. From Eqs. (4) and (7a):

$$\Omega_{DM}(\mu) = \frac{8(179\mu + 160\mu^2 + 40\mu^3 + 68)}{9(8 + 9\mu + 4\mu^2)^2}, \tag{19}$$

which is clearly increasing (decreasing) for sufficiently small (large) $\mu$, since:

$$\frac{\partial \Omega_{DM}(\mu)}{\partial \mu} = -\frac{8(-208 + 139\mu + 1188\mu^2 + 920\mu^3 + 160\mu^4)}{9(8 + 9\mu + 4\mu^2)^3}, \tag{20}$$

and is strictly concave (convex) for sufficiently small (large) $\mu$, since:

$$\frac{\partial^2 \Omega_{DM}(\mu)}{\partial \mu^2} = \frac{16(-3364 - 10749\mu - 4304\mu^2 + 6944\mu^3 + 4800\mu^4 + 640\mu^5)}{9(8 + 9\mu + 4\mu^2)^4}. \tag{21}$$

Indeed, Eq. (21) changes from concave to convex at $\mu = b \cong 1.177$ (3 d.p.). Finally, $\Omega_{DM}(\mu) < 1$ follows from noting that $\Omega_{DM}(\mu)$ obtains a minimum at $\mu = a \cong 0.320$ (3 d.p.) whereupon $\Omega_{DM}$ is strictly positive. (iii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu\to\infty} \Omega_{DM}(\mu) = 0$. $\square$

*Proof of Lemma 2.* (i) First note that $\Upsilon_{MD} \equiv W^M/W^D$ is continuous. From Eqs. (4) and (7a):

$$\Upsilon_{MD}(\mu) = \frac{4(144\mu^4 + 632\mu^3 + 1241\mu^2 + 1084\mu + 376)}{27(8 + 9\mu + 4\mu^2)^2}, \tag{22}$$

which is clearly increasing for $\mu \geq 0$, since:

$$\frac{\partial \Upsilon_{MD}(\mu)}{\partial \mu} = \frac{16(92\mu^3 + 16\mu^4 + 540\mu^2 + 1021\mu + 476)}{27(8 + 9\mu + 4\mu^2)^3} > 0, \tag{23}$$

and is strictly concave for $\mu \geq 0$, since:

$$\frac{\partial^2 \Upsilon_{MD}(\mu)}{\partial \mu^2} = -\frac{32(11536\mu^2 + 4064\mu^3 + 480\mu^4 + 64\mu^5 + 10581\mu + 2342)}{27(8 + 9\mu + 4\mu^2)^4} < 0. \tag{24}$$

(ii) Setting Eq. (22) equal to unity and solving for $\mu$ yields $\mu = c \cong 0.282(3d.p.)$. It is straightforward to see that Eq. (22) is greater (less) than unity for higher (lower) values of $\mu$. (iii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu\to\infty} \Upsilon_{MD}(\mu) = 13824/10368$. $\square$

*Proof of Lemma 3.* It follows from Eqs. (16a) and (16b) that $P^S_{ij} - P^S_{ii} = (+)(16\mu + 18\mu^2)$

which is zero for $\mu = 0$ and strictly positive for $\mu > 0$. $\square$

*Proof of Lemma 4.* Using Eqs. (16a) and (16b) in Eq. (2), we get the demand for the cross-firm good under the Stackelberg pricing regime is:

$$Q_{ij}^{S} = \frac{\alpha(64 + 56\mu - 3\mu^2)}{2(3\mu^2 + 56\mu + 64)} \tag{25}$$

which is only positive iff Eq. (18) holds. $\square$

*Proof to Proposition 1.* (i) First note that $\Omega_{SM}(\mu) \equiv \Pi^S/\Pi^M$ is continuous. From Eqs. (7a) and (17a):

$$\Omega_{SD}(\mu) = \frac{207\mu^4 + 3600\mu^3 + 10560\mu^2 + 11264\mu + 4096}{(120\mu + 59\mu^2 + 64 + 3\mu^3)(3\mu^2 + 56\mu + 64)}, \tag{26}$$

which is clearly negative monotonic in $\mu$, since:

$$\frac{\partial \Omega_{SM}(\mu)}{\partial \mu} = -\frac{\mu(621\mu^5 + 10008\mu^4 + 42912\mu^3 + 74496\mu^2 + 57344\mu + 16384)}{(3\mu^2 + 56\mu + 64)^2(1 + \mu)(120\mu + 59\mu^2 + 64 + 3\mu^3)}. \tag{27}$$

Given:

$$\begin{aligned}\frac{\partial^2 \Omega_{SM}(\mu)}{\partial \mu^2} = &\left(\frac{2}{(3\mu^2 + 56\mu + 64)^2(1 + \mu)(120\mu + 59\mu^2 + 64 + 3\mu^3)^2}\right) \\ &\times (1863\mu^8 + 27648\mu^7 + 140832\mu^6 + 248544\mu^5 - 526080\mu^4 \\ &- 2543616\mu^3 - 3588096\mu^2 - 2228224\mu - 524288),\end{aligned} \tag{28}$$

$\Omega_{SM}(\mu)$ is clearly strictly concave (convex) for sufficiently small (large) values of $\mu$. Setting Eq. (28) equal to zero yields $\mu = d \cong 2.602$ (3 d.p). (ii) First, note that $\Omega_{SM}(0) = 1$. Observing that $\Omega_{SM}(\mu)$ is negative monotonic, as established in (i), it follows that $\Omega_{SM}(\mu > 0) < 1$, which completes the proof. (iii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu \to \infty} \Omega_{SM}(\mu) = 0$. $\square$

*Proof of Proposition 2.* (i) First note that $\Upsilon_{MS} \equiv W^M/W^S$ is continuous. From Eqs. (4) and (17b):

$$\Upsilon_{MS}(\mu) = \frac{12288 + 34816\mu + 34240\mu^2 + 12976\mu^3 + 1299\mu^4 + 35\mu^5}{3(1 + \mu)(64 + 56\mu + 3\mu^2)^2}, \tag{29}$$

which is clearly decreasing for $\mu \geq 0$, since:

$$\frac{\partial \Upsilon_{MS}(\mu)}{\partial \mu} = -\left(\frac{3(3\mu^2+56\mu+64)}{(1299\mu^4+12976\mu^3+34240\mu^2+36\mu^5+34816\mu+12288)^2}\right) \\ \times (10152\mu^5+59328\mu^4+179456\mu^3+243\mu^6+285696\mu^2+221184\mu+65536) < 0, \tag{30}$$

and is strictly convex for $\mu \geq 0$, since:

$$\frac{\partial^2 \Upsilon_{MS}(\mu)}{\partial \mu^2} = \left(\frac{24}{(1299\mu^4+12976\mu^3+34240\mu^2+36\mu^5+34816\mu+12288)^3}\right) \\ \times(201403138048\mu^3+139997478912\mu^2+54559506432\mu+178569609216\mu^4 \\ +101849677824\mu^5+38617888768\mu^6+10261702656\mu^7+2009340000\mu^8 \\ +265008240\mu^9+18506880\mu^{10}+594864\mu^{11}+6561\mu^{12}+9126805504) > 0. \tag{31}$$

(ii) First, note that $\Upsilon_{MS}(0) = 1$. Second, it follows from the strict negative monotonicity of $\Upsilon_{MS}(\mu)$ as established in (i), that $\Upsilon_{MS}(\mu > 0) < 1$. (iii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu\to\infty} \Upsilon_{MS}(\mu) = 3240/4320$. □

*Proof of Proposition 3.* (i) First note that $\Omega_{SD}(\mu) \equiv \Pi^S/\Pi^D$ is continuous. From Eqs. (4) and (17a):

$$\Omega_{SM}(\mu) = \frac{8(68+179\mu+160\mu^2+40\mu^3)(120\mu+59\mu^2+64+3\mu^3)(3\mu^2+56\mu+64)}{9(8+9\mu+4\mu^2)^2(207\mu^4+3600\mu^3+10560\mu^2+11264\mu+4096)}, \tag{32}$$

which is clearly increasing (decreasing) in $\mu$ if $\mu$ is sufficiently small (large), since:

$$\frac{\partial \Omega_{SD}(\mu)}{\partial \mu} = -\left(\frac{8}{9(8+9\mu+4\mu^2)^3(207\mu^4+3600\mu^3+10560\mu^2+11264\mu+4096)^2}\right) \\ \times(-17431527424\mu-16374562816\mu^2+81595465728\mu^3+300426264576\mu^4 \\ +488007536640\mu^5+472673600512\mu^6+292014724864\mu^7+115412832016\mu^8 \\ +28059227097\mu^9+3844182348\mu^{10}+253318896\mu^{11}+6093360\mu^{12}-3489660928). \tag{33}$$

Setting Eq. (33) equal to zero and solving for $\mu$ yields, $\mu = e \cong 0.3495$ (4 d.p.). Given:

$$\frac{\partial^2 \Omega_{SD}(\mu)}{\partial \mu^2} = \left(\frac{16}{3(8+9\mu+4\mu^2)^4(207\mu^4+3600\mu^3+10560\mu^2+11264\mu+4096)^3}\right)$$
$$\times(-845661880713216\mu - 4286940002123776\mu^2 - 12672640309264384\mu^3$$
$$- 23973745492230144\mu^4 - 29436990785060864\mu^5 - 21318466171568128\mu^6$$
$$- 3576595470090240\mu^7 + 10929409200062464\mu^8 + 14367546390040576\mu^9$$
$$+ 9868024438793472\mu^{10} + 4394664490046336\mu^{11} + 1317772443151044\mu^{12}$$
$$+ 262788896301381\mu^{13} - 73942156967936 + 33436492270416\mu^{14}$$
$$+ 2538719996256\mu^{15} + 102982034664\mu^{16} + 1681767360\mu^{17}), \tag{34}$$

$\Omega_{SM}(\mu)$ is clearly strictly concave (convex) for sufficiently small (large) values of $\mu$. Setting Eq. (34) equal to zero yields $\mu = f \cong 1.199$ (3 d.p). (ii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu\to\infty} \Omega_{SM}(\mu) = 0$. (iii) First, note that $\Omega_{SM}(0.3495) > 1$, whilst $\Omega_{SM}(0) < 1$ and from (ii), $\lim_{\mu\to\infty} \Omega_{SM}(\mu) = 0$. Given the properties of $\Omega_{SM}(\mu)$ from (i), we know that $\Omega_{SM}(\mu)$ must equal unity at two points in the relevant range of $\mu$. Setting Eq. (32) equal to unity and solving for $\mu$, we find the roots $\mu = g \cong 0.349$ and $\mu = h \cong 0.350$ (3 d.p.). □

*Proof of Proposition 4.* (i) First note that $\Upsilon_{DS} \equiv W^D/W^S$ is continuous. From Eqs. (7a) and (17b):

$$\Upsilon_{DS}(\mu) = \frac{4(144\mu^4+632\mu^3+1241\mu^2+1084\mu+376)(1+\mu)(3\mu^2+56\mu+64)^2}{9(8+9\mu+4\mu^2)^2(1299\mu^4+12976\mu^3+34240\mu^2+36\mu^5+34816\mu+12288)}, \tag{35}$$

which is clearly increasing (decreasing) for sufficiently small (large) values of $\mu$, since:

$$\frac{\partial \Upsilon_{DS}(\mu)}{\partial \mu} = -\left(\frac{4}{9(8+9\mu+4\mu^2)^3(1299\mu^4+12976\mu^3+34240\mu^2+36\mu^5+34816\mu+12288)^2}\right)$$
$$\times(3\mu^2+56\mu+64)(-8806465536\mu - 25413812224\mu^2 - 40627978240\mu^3$$
$$- 38899857408\mu^4 - 22079425152\mu^5 - 6267853520\mu^6 + 222363204\mu^7$$
$$+ 862537792\mu^8 + 331806643\mu^9 + 65740644\mu^{10} + 6351696\mu^{11}$$
$$+ 133056\mu^{12} - 1300234240), \tag{36}$$

and is clearly strictly concave (convex) for sufficiently small (large) values of $\mu$, since:

$$\begin{aligned}\frac{\partial^2 \Upsilon_{DS}(\mu)}{\partial \mu^2} = &\left(\frac{32}{9(8+9\mu+4\mu^2)^4(1299\mu^4+12976\mu^3+34240\mu^2+36\mu^5+34816\mu+12288)^3}\right)\\
&\times(-18327072128630784\mu - 105205505235877888\mu^2 - 360170887770537984\mu^3\\
&\quad - 825429319634386944\mu^4 - 1343488454855491584\mu^5 - 1604454731011850240\mu^6\\
&\quad - 1429980672502530048\mu^7 - 956663665954275328\mu^8 - 478415605887645696\mu^9\\
&\quad - 175977789100670592\mu^{10} - 45842990698845024\mu^{11} - 7621631524112918\mu^{12}\\
&\quad - 448042813761015\mu^{13} - 1425104508551168 + 148732848593492\mu^{14}\\
&\quad + 52788607837920\mu^{15} + 8924481301365\mu^{16} + 897657699312\mu^{17}\\
&\quad + 50651845344\mu^{18} + 1415169792\mu^{19} + 14370048\mu^{20}).\end{aligned} \tag{37}$$

(ii) First, note that $\Upsilon_{MS}(0) = 1$. Second, it follows from the strict negative monotonicity of $\Upsilon_{MS}(\mu)$ as established in (i), that $\Upsilon_{MS}(\mu > 0) < 1$. (iii) Using *Bernoulli-L'Hôspital's Rule*, $\lim_{\mu\to\infty} \Upsilon_{MS}(\mu) = 3240/4320$. $\square$

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