A Multi-operator Differentiated Transport Network Model

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Abstract

We develop a network model of differentiated transport services explicitly incorporating interchangeable and rival aspects, characteristic of many transport systems, allowing exploration of the implications of strategic interaction on pricing amongst multiple rival operators within and across modes. The model offers a framework for studying the impacts of alternative policy scenarios with a wide variety of applications across the transport sector in a way that is tractable and allows meaningful analysis. We illustrate some of the uses of the framework through a series of applications which demonstrate the importance of explicitly recognising the dual rival and interchangeable aspects across *multiple* operators. Amongst other things, we show that the base model, which we characterise as $n = 2$, and which has been widely employed in the transport literature, in some respects represents a special case and that the relative size of equilibrium profit, consumer surplus and welfare across regimes as well as the rankings of different regimes across these performance indicators are non-monotonic in n, hence justifying a framework which explicitly allows n to vary. One application examines the performance of the multi-operator ticketing card scheme under guidelines operating in the UK local bus sector. This features as a key part in the UK government's local bus transport strategy but is also due to expire in 2026 and is currently under statutory review. A calibration exercise shows this regime may offer higher profit, consumer surplus and welfare as well as a more extensive service provision than the 'free-market' case. However, under non-trivial fixed costs, it may not sustain as large a network as under the 'free-market', reversing the consumer surplus and welfare rankings.

Keywords: Multi-operator; Transport Networks; Pricing; Welfare
JEL #s: D43, L13, L92, R48

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1 Introduction

A notable aspect of many transport systems is the availability of origin-destination (OD) travel via different service operators and/or modes where these services have both rival and interchangeable components. Rival components have relationships along substitute lines, with well-known strategic interaction implications in the sense that the more rival services then, typically, the more intense the competition and lower the prices. However, the interchangeable aspects of the network bring complementarity which has associated strategic interaction implications that can run counter to the rival substitute effect. Since Cournot (1838), it has been known that where goods are related as complements, an increase in the number of independent providers raises equilibrium prices, at variance with normal intuition. Indeed, splitting a monopoly along complementary lines can result in prices above even the monopoly level. Hence, the dual existence of these components on transport networks complicates policy narratives (e.g., increasing the number of service operators might have unintended consequences) but, also, modelling transport networks without these potentially countervailing channels of strategic interaction, risks important misspecification and associated erroneous analysis and conclusions. Whilst many transport studies are based on frameworks which capture one or the other of these channels, combining them is much less common. Examples of studies, based on a variety of different approaches explicitly modelling both substitute and complementary components in transport, albeit without the number of operators generally exceeding 2 within both components, include Economides and Salop (1992), Brueckner (2004), Lin (2004), McHardy and Trotter (2006), McHardy et al. (2013), Socorro and Viecens (2013), van der Weijde et al. (2013) and van den Berg et al. (2022).

In this paper we develop an n-operator transport network model which we believe will have many applications in furthering understanding of competition and regulatory issues in transport economics where services have dual complementary and substitute aspects. To date, to the best of our knowledge, existing studies have not accommodated a more extensive network setting with agents competing across complementary and substitute lines in the way we propose, including with ticketing following common 'conventions' in transport, either, (i) beyond what we characterise as the base model of $n = 2$, allowing generalising to $n > 2$, or, (ii) in a form that offers an off-the-shelf basis for extending extant $n = 2$ transport models. Many real-world transport networks are more extensive than what is implied under $n = 2$. Indeed, service under-provision is a common theme in many transport debates, hence understanding how certain market structures and policies perform in this context requires the ability to vary n. A priori it is not clear how extending a given network will play out in terms of the two countervailing strategic forces as the number of substitute and complementary relationships rise with increases in n: which will dominate - will prices go up or down? We employ a model of the Economides and Salop (1992)-type, which has a well-established presence in the literature on transport networks, with studies variously employing the framework directly or as as a model that can be nested within it's structure (e.g., see Shy, 1996; Lin, 2004; Bataille and Steinmetz, 2013; Silva and Verhoef, 2013; Socorro and Viecens, 2013; van den Berg, 2013; Clark et al., 2014; D'Alfonso et al., 2016; van den Berg et al., 2022) and which can therefore readily be extended to include all aspects of the framework, capturing substitute *and* complement channels of strategic interaction, and based on this paper, with many and variable rival operators with ticketing according to 'conventions' in transport.

The central methodological contribution of this paper is capturing this dual strategic interaction in a tractable and adaptable n-operator setting, based on an extension of the Economides and Salop (1992)-type network model and 'conventional' transport ticketing. Whilst Economides and Salop (1991) presents an n-firm extension of Economides and Salop (1992), neither model accommodates the ability, for instance, for firms to discount return-trip tickets evident in practice and the literature

(e.g., see Flores-Fillol and Moner-Colonques, 2011; van den Berg et al., 2022). Whilst Economides (1993) studies the Economides and Salop (1992)-framework under mixed bundling, which allows for such discounts on the bundled good, it does so in the case of $n = 2$. Zhou (2021), establishes itself as the first to study competitive mixed bundling with an arbitrary number of firms, based on the random utility framework of Perloff and Salop (1985). The framework is presented at a very general level with analysis of $n > 2$ being conducted with 'many' firms using an approximation. By having its foundations in Economides and Salop (1992) and Economides (1993), the framework presented here provides results directly comparable to the transport literature also based on these models under $n = 2$, but with $n > 2$ under 'conventional' transport ticketing conditions. We show that the n-operator model brings meaningful additional insights including revealing $n = 2$ to be a limiting or special case in various respects.[1] In particular, we examine the properties of two benchmark regimes for this framework illustrating the merits of the framework extending beyond $n = 2$, including revealing key variables are non-monotonic in n and, that varying n can alter the social welfare ranking of the different regimes. Given the wide range of transport network applications using, or nested within, the Economides and Salop (1992)-type model (including networks with bus, rail, car and aviation), the demonstration here of genuine value added from exploring $n > 2$, suggests this paper offers a rich platform to extend these models to see the extent to which their results generalise as well as address new questions available through the modelling capacity to vary n.

We study monopoly and make comparisons against a competitive 'free-market' regime with the same services but provided by n operators who set prices independently. We also consider different incentives between the two regimes to provide different sized networks and associated welfare implications, and demonstrate doing so can change the performance rankings of these regimes relative to a situation where each regime has the same services. Finally, we examine relative profit, consumer surplus and welfare outcomes and incentives to offer a larger or smaller network in the context of a stylised multi-operator ticketing scheme, such as that permitted in the UK under the Public Transport Ticketing Scheme Block Exemption (Competition Commission, 2001), henceforth Block Exemption, revealing some of the potential benefits as well as drawbacks of this system. This is a timely investigation given the prominence of multi-operator ticketing in the UK government's transport strategy (see Department for Transport, 2021), but also because the Block Exemption is due to expire in 2026 and is currently under statutory (e.g., see Department for Business, Energy and Industrial Strategy, 2021).

In the interests of maximising transparency and clarity, for the most part we conduct the analysis under relatively simple cost and demand conditions, typical amongst many of the extant transport network studies, which allows the clear algebraic and/or graphical presentation of the n-operator equilibria whilst capturing the impact of the key dual strategic channels of interaction. However, we do indicate how other cost and demand aspects, which feature as characteristics of the wider transport literature, can readily be accommodated into the framework but which, if taken all together, would broadly result in solutions that are not presentable either algebraically or diagrammatically limiting potential for intuitive insights.

The following Section introduces the theoretical framework. This is analysed in Section 3 under two benchmark regimes, Monopoly and Independence. Section 4 then considers the possibility that one regime might be able to sustain a larger network and analyses the impact on associated consumer surplus and welfare levels and the policy implications which follow. Section 5 introduces a stylised version of a UK multi-operator ticketing system as permitted under the Block Exemption,

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[1]Zhou (2021) shows that, relative to a regime (not studied here) where each component is supplied by an independent firm, whilst duopoly mixed bundling has ambiguous effects on prices, profits and consumer surplus, with 'many' firms, it lowers prices and profits and raises consumer surplus.

and analyses this against the Independence regime offering insights into potential benefits and drawbacks of this pricing system. Section 6 concludes.

2 Network Model

Consider an OD journey, $J = \{x, y\}$, comprising two component parts x and y, where perhaps x is an outward journey, and y is the return journey, or alternatively they represent two consecutive stages of a journey from one distinct place to another. As such, within the OD journey x and y are perfect complements. Now suppose that there are n service operators (within and/or across modes), each providing its own x_i and y_i $(i = 1, ..., n)$ component of the journey: $J_{ii} = \{x_i, y_i\}$. Let, the x_i and y_i components be horizontally differentiated across the n operators, temporally, spatially or some combination of the two. Hence, operator i might provide a service from the suburbs to a city centre and back at one pair of times, $J_{ii} = \{x_i, y_i\}$, whilst operator j provides the same, or a geographically different, service at a different set of times, $J_{jj} = \{x_j, y_j\}$ $(i \neq j = 1, ..., n)$. In addition to being able to undertake the journey via a single operator, J_{ii}, let the x and y components of the n operators' services be interchangeable. Hence, OD travel can be achieved via journey $J_{ij} = \{x_i, y_j\}$. Here, J_{ii}, J_{jj} and J_{ij} are imperfect substitutes. Demand for the composite journey J_{ij} is given by Q_{ij}. With each operator providing a complete OD service and contributing each of its x and y components to a further $n - 1$ OD services, there are $N \equiv n^2$ OD rival services across the network. At some points it is helpful to index the N services: e.g., the quantity of OD service t is Q_t with $t = 1, ..., N$. Figure 1 illustrates an example of the network in the case of $n = 2$, where the two operators' origins, O_i, and destinations, D_i $(i = 1, 2)$ might be interpreted as representing distinct geographical locations and/or times of travel and there are $N \equiv n^2 = 4$ possible OD journey configurations. I might represent an interim destination e.g., the city centre in a round-trip journey, or the point where interchange occurs with respect to an ongoing journey.

Figure 1: Simple Transport Network
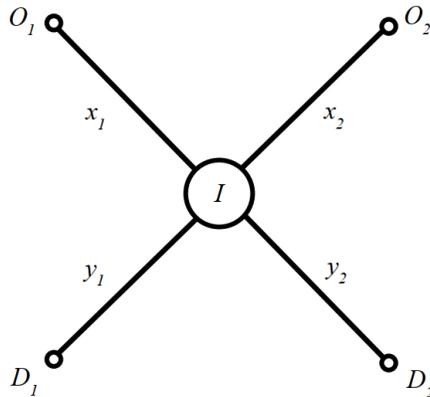
Let the total ticket cost of journey, J_{ij}, be P_{ij} $(i, j = 1, ..., n)$, where the single-operator (SO) price, P_{ii} is set by firm i, but the multi-operator (MO) price has two components, $P_{ij} = p_{ij}^i + p_{ij}^j$ $(i \neq j = 1, ..., n)$, with p_{ij}^i corresponding to the x component, set by firm i, and p_{ij}^j corresponding

to the y component, set by firm j.[2],[3] For simplicity, and where helpful, we denote SO and MO prices (quantities) as P and P_x (Q and Q_x), respectively.

In order to keep analysis accessible and transparent, and to facilitate ease of analysis and intuition, we employ a simple symmetric linear demand system which is used very widely across industrial, and specifically transport, modelling to introduce product differentiation. SO and MO service demands are given, respectively, as follows, where a, b, and d are parameters characterising the service demand conditions and substitutability:

$$Q_{ii} = a - bP_{ii} + d \left[\sum_{k \neq i} P_{kk} + (p_{ij}^i + p_{ij}^j) + \sum_{k \neq i,j} \left(p_{ik}^i + p_{ik}^k \right) + \sum_{k \neq i} \left(p_{ki}^i + p_{ki}^k \right) + \sum_{k \neq m,i} \sum_{m \neq k,i} \left(p_{mk}^m + p_{mk}^k \right) \right]$$
(1)

$$Q_{ij} = a - b \left(p_{ij}^i + p_{ji}^j \right) + d \left[P_{ii} + \sum_{k \neq i} P_{kk} + \sum_{k \neq i,j} \left(p_{ik}^i + p_{ik}^k \right) + \sum_{k \neq i} \left(p_{ki}^i + p_{ki}^k \right) + \sum_{k \neq m,i} \sum_{m \neq k,i} \left(p_{mk}^m + p_{mk}^k \right) \right]$$

Parameter a represents the base level of demand for OD travel under zero prices everywhere and can be interpreted as a measure of market size and, under asymmetry, reflect vertical differentiation across OD services: a being higher for services with superior quality. Parameter b indicates how demand for an OD service falls with increases in its own price and reflects own-price elasticity of demand. Parameter d indicates how demand for one OD service increases with increases in the price of a rival OD service and reflects the cross-price elasticity of demand.

In practice, asymmetries and other factors like generalised costs, required to fit with particular modelling needs, can easily be introduced, though equilibria will not necessarily then be so algebraically transparent. For instance, the total ticket cost, P_{ij} can be re-defined as the generalised cost including a ticket cost and incorporating interchange, travel time, egress and frequency considerations (e.g., see Flores-Fillol, 2009). Similarly, it is straightforward to introduce classes of substitutability, for instance, by allocating a different coefficient d for cross effects between different sets of prices. For instance, coefficient d_1 might be used for a firm's SO service demand relative to the MO services using one of it's components (e.g., $P_{ii} \mathrm{v} P_{ik}$), d_2 for those MO services comprising only other operators' components (e.g., $P_{ii} \mathrm{v} P_{mk}$), d_3 between two SO prices (e.g., $P_{ii} \mathrm{v} P_{kk}$), d_4 between MO prices sharing a common component (e.g., $P_{ij} \mathrm{v} P_{ik}$), and d_5 between MO prices with no shared component (e.g., $P_{ij} \mathrm{v} P_{mk}$). SO demands could also have a different own-price coefficient,

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[2]Note, that since the composite price, P_{ii} here is set independently of the component prices, p_{ij}^i and p_{ji}^i (e.g., see Flores-Fillol and Moner-Colonques, 2011; van den Berg et al., 2022), as would be the case, for instance, if P_{ii} represented a return-trip ticket, the Independence regime is not structurally the same as the equivalent, Parallel Vertical Integration, introduced in Economides and Salop (1992) (with $n = 2$) and Economides and Salop (1991) (in the n-firm case), where, effectively, the following is imposed: $P_{ii} = p_{ij}^i + p_{ji}^i$. Hence, in the latter, P_{ii} is not set independently of the component prices, whilst the former is consistent with 'conventional' practices in transport ticketing.

[3]In general, the MO price should not be confused with the stylised multi-operator ticket price, which we introduce in Section 5.

b_1, from the MO own-price coefficient, b_2.[4] Hence:

$$Q_{ii} = a - b_1 P_{ii} + d_1 \left[\sum_{k \neq i} \left(p_{ik}^i + p_{ik}^k \right) + \sum_{k \neq i} \left(p_{ki}^i + p_{ki}^k \right) \right] + d_2 \sum_{k \neq m,i} \sum_{m \neq k,i} \left(p_{mk}^m + p_{mk}^k \right) + d_3 \sum_{k \neq i} P_{kk} \tag{2}$$

$$Q_{ij} = a - b_2 \left(p_{ij}^i + p_{ji}^j \right) + d_1 P_{ii} + d_2 \sum_{k \neq i} P_{kk} + d_4 \left[\sum_{k \neq i,j} \left(p_{ik}^i + p_{ik}^k \right) + \sum_{k \neq i} \left(p_{ki}^i + p_{ki}^k \right) \right]$$
$$+ d_5 \sum_{k \neq m,i} \sum_{m \neq k,i} \left(p_{mk}^m + p_{mk}^k \right)$$

Eq. (2) can readily be used to solve for profit maximising outcomes using maths software, with tractable, though not necessarily algebraically transparent, outputs. Of course, most modelling applications will have very focused asymmetries targeting the specific asymmetric aspect of interest for analysis and the model need not become unwieldy.

Although our applications later on focus on the symmetric demand system, Eq. (1), other demand characterisations, such as selected aspects of the above asymmetries or additional demand components, can easily be accommodated in the modelling. One such case of relevance in transport modelling would be to include an additional consumer type with demands for travel on a single x or y component (not in combination, as taken as given in Eq. (1)):

$$x_i = A_i - B_i p_{ij}^i + D_i \left(\sum_{j \neq i} p_{ji}^j \right), \quad y_i = A_i - B_i p_{ji}^i + D_i \left(\sum_{j \neq i} p_{ij}^j \right) \tag{3}$$

Hence, incorporating Eq. (3) with Eq. (1) would allow modelling of a network with, say, demand for round-trip, via the latter, with single outward or inward travel, via the former, with all consumer types facing the same component prices, p_{ij} and p_{ij}.[5]

In order to undertake welfare analysis across different policy scenarios it is useful to explicitly underpin the demand system with a utility function.[6] The following quasi-linear utility function, which has seen regular use in transport and industrial economics (for instance, in the two-firm, regular duopoly, case: Silva and Verhoef, 2013; D'Alfonso et al., 2016, and in the general N-firm, regular oligopoly, case: Hackner, 2000), offers one such example which produces the symmetric differentiated demand system in Eq. (1):[7]

$$U(\boldsymbol{Q}, M_0) = \alpha \sum_t^N Q_t - \frac{1}{2} \left[\sum_t^N Q_t^2 + 2\gamma \sum_t^N \sum_{r > t} Q_t Q_r \right] + M_0 \quad (r \neq t = 1, ..., N) \tag{4}$$

The degree of substitutability between pairs of the n^2 services is given by $\gamma \in [0,1]$ which takes the values of 0 under zero substitutability (independent services), and 1 under perfect substitutability.

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[4]See Choné and Linnemer (2020) for suitable quasi-linear quadratic utility functions supporting asymmetric differentiated linear demands.

[5]One simple formulation used in the $n = 2$ scenario, imposes on Eqs. (2) and (3) common parameters $a = A$, $b = B = 1$ and a common $d = D$ across dual component and individual component demands, e.g., Lin (2004).

[6]It is well known that using a specific utility function will produce quantitative results that will be different from those under a different utility function. Given most of our analysis is based on ratios of variables across regimes and in particular whether the ratios are greater or less than unity, we expect there to be a good degree of robustness to the qualitative analysis when comparing regimes under the same industry structure (the same number of services).

[7]Choné and Linnemer (2020) classify this as a Spence (1976)-type utility function, and recommend citing Shubik and Levitan (1980) given we use it in the context of N services rather than two.

$\boldsymbol{Q}$ is an N-vector of quantities, and M_0 is a composite of expenditure on all other goods. One property of the utility function that will be of use in our applications later on, is that consumer surplus, at constant prices, increases with the introduction of a new operator (n increases by 1), reflecting consumer preference for variety. Hence, the addition of an extra operator offering differentiated services has the potential to increase consumer surplus through two channels: (i) lowering prices through additional competition in the market, and (ii) expanding the variety on offer. One way to rationalise the latter in the context of a transport network would be to imagine that a new operator provides OD services between geographical locations previously served less directly, benefiting existing passengers for whom existing services were inconveniently far away from their desired origin or destination, and new ones, for whom the original services were prohibitively inconvenient. Alternatively, the new services might offer travel at times previously not available, similarly better suiting some existing passengers whilst also attracting new passengers.

One clear advantage of the use of such a utility function is that it has a closed set of feasible parameterisations of substitutability, γ, aiding analysis: solutions and results relying on γ outside this interval are irrelevant and simulations or plots across the closed and bounded interval tell the entire story of the aspect of the model being studied, for instance, facilitating graphical proofs, which we make explicit use of. In addition, maximising Eq. (4) with respect to a budget constraint, we see the three parameters in Eq. (1), a, b and d, are replaced with just two new ones, α and γ, (note $n \equiv \sqrt{N}$ is already a parameter in Eq. (1)) significantly simplifying analysis:

$$a \equiv \frac{\alpha}{(1+\gamma(N-1))}, \quad b \equiv \frac{1+\gamma(N-2)}{(1-\gamma)(1+\gamma(N-1))}, \quad d \equiv \frac{\gamma}{(1-\gamma)(1+\gamma(N-1))} \tag{5}$$

When calculating consumer surplus and welfare it is useful to recognise the symmetry, in a way that will also be evident in all later scenarios, when presenting the utility function. Here we are interested in combinations of size h from N services, in particular, pairs of services, $h = 2$. Hence, from the combinations rule for pairs with N services: $_N C_h(h=2) = \frac{N!}{h!(N-h)!} = \frac{N!}{2(N-2)!} = \frac{N(N-1)}{2}$. With $N = n^2$ being the number of differentiated services in the model with n individual operators, we have a total number of paired combinations: $_N C_2 = \frac{n^2(n^2-1)}{2}$. In order to calculate utility we need to decompose this into 3 components which we label X, Y and Z. Let X denote the number of combinations of the n SO services: $X \equiv_n C_2 = \frac{n(n-1)}{2}$, Y denote the number of combinations of $n(n-1)$ MO services: $Y \equiv_{n(n-1)} C_2 = \frac{n(n-1)[n(n-1)-1]}{2}$ and, Z denote the number of combinations of MO and SO service pairs: $Z \equiv_N C_2 - X - Y = n^2(n-1)$.

Hence, given symmetry of the equilibrium in each scenario, R, denoting SO quantity, Q^R, and MO quantity, Q_x^R, utility in regime R can be written:

$$U^R(\boldsymbol{Q}, M_0) = \alpha(nQ^R + n(n-1)Q_x^R) - \frac{1}{2}\left[n(Q^R)^2 + n(n-1)(Q_x^R)^2\right] - \gamma\left[X(Q^R)^2 + Y(Q_x^R)^2 + ZQ^R Q_x^R\right] + M_0 \tag{6}$$

It remains to note that, given π^R is total profit across the network under regime R, then consumer surplus, S, and welfare, W under regime R are given by:

$$S^R = U^R - M_0 - M^R, \quad W^R = S^R + \pi^R \tag{7}$$

where $M^R = \sum_t P_t^R Q_t^R$ ($t = 1, ..., N$) is the total expenditure across the transport network.

In terms of costs, in many transport studies marginal operating costs are treated as constant, or zero (e.g., see Clark et al., 2014) with some empirical backing (e.g., see Jørgensen and Preston, 2003). Whilst the framework can readily accommodate more nuanced cost conditions, including

additional aspects such as congestion (e.g., see Brueckner, 2004; van den Berg, 2013), we proceed with constant and common marginal costs, c, for each of the n^2 services. Let:

$$c = c_x + c_y$$

where c_z is the constant marginal cost of $z \in \{x, y\}$. With common and constant marginal cost, it is straightforward to show that the composition of the cost (the relative size of c_x and c_y) has no bearing on any aspect of the equilibrium OD prices, quantities, profits or welfare, and we can refer, henceforth, to just the composite (OD journey) cost, c. Let F be the fixed cost associated with the operator providing its x and y component services on the network. All operator costs not directly impacted by passenger numbers, e.g., interchange costs, given the fixed structure of service provision (x and y are consumed as perfect complements), can be incorporated into the fixed cost.

Finally, given there are no dual network characteristics under the scenario $n = 1$, for the most part, we limit the relevant parameter set for n using the following Assumption.

Assumption 1. [8] *(i) $n \geq 2$ and (ii) gross substitutes: $b > (n^2 - 1)d$.*

Given α plays no role in the relative equilibrium values of key variables across regimes, with γ defined on a closed and bounded interval, Assumption 1 completes what we will refer to as the relevant parameter set in Propositions and discussions.[9]

3 Benchmark Regimes

We now consider two benchmark cases: Monopoly and Independence. In the former, a single firm operates all n^2 services and sets profit-maximising prices across the network. In the latter, n independent operators undertake simultaneous and independent price setting on all prices for services they provide, as would be the case in a 'free-market' setting. For simplicity and clarity (keeping equilibrium price solutions in a form that are accessible), we solve the model under the symmetric demand system in Eq. (1), with common constant marginal cost and a fixed cost. We then report equilibrium solutions for this demand system and analyse the solutions under the conditions of the utility function given by Eq. (5) and zero marginal cost.[10]

3.1 Monopoly

Recognising the Monopolist is able to set the OD price for each service rather than be concerned with individual component prices, it solves the problem of maximising total profit across the network:[11]

$$\max_{\{\boldsymbol{P}\}} \pi = (P_{ii} - c)Q_{ii} + \sum_{k \neq i}(P_{kk} - c)Q_{kk} + + \sum_{k \neq i}(P_{ik} - c)\,Q_{ik} \tag{8}$$

$$+ \sum_{k \neq i}(P_{ki} - c)\,Q_{ki} - nF \quad (k \neq i = 1, ..., n) \tag{9}$$

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[8]Gross substitutes requires that an equal increase in the prices of all OD services leads to a fall in the demand for each service.

[9]Of course, presenting c as a percentage of α ensures the neutrality of α in variable ratios in the case where marginal cost is included. τ, introduced later, is a parameter defined on a closed and bounded set.

[10]It is straightforward to show that, since our analysis is focused on results via ratios of prices, quantities, profit, consumer surplus and welfare across regimes, rather than absolute levels of these variables, this assumption comes at no loss of generality - the ratios are invariant to the introduction of positive common, constant marginal costs. However, we report the equilibrium prices with marginal cost to facilitate further analysis not based on ratios, whereupon marginal costs play a role.

[11]The same result obtains if we assume a single monopoly OD price from the outset.

where $\boldsymbol{P}$ is the N-vector of the n^2 OD fares across SO and MO services.[12] A representative first-order condition, with respect to price P_{ii}, is given by:[13]

$$(P_{ii} - c)\frac{\partial Q_{ii}}{\partial P_{ii}} + Q_{ii} + \sum_{k \neq i}(P_{kk} - c)\frac{\partial Q_{kk}}{\partial P_{ii}} + \sum_{k \neq i}(P_{ik} - c)\frac{\partial Q_{ik}}{\partial P_{ii}} + \sum_{k \neq i}(P_{ki} - c)\frac{\partial Q_{ki}}{\partial P_{ii}} = 0 \qquad (10)$$

where $\frac{\partial Q_{ik}}{\partial P_{ii}} \equiv \frac{\partial Q_{mk}}{\partial P_{ii}}|_{m,k \neq i} = d$ and $\frac{\partial Q_{ii}}{\partial P_{ii}} = -b$. Recognising symmetry across all SO and MO services in the monopoly scenario, equilibrium OD price, $P^M = P_{ij} = P_{ii}$, in the general linear form demand system of Eq. (1), denoted $\bar{P}^M$, and under utility function parameterisation in Eq. (5), denoted P^M, are respectively:[14]

$$\bar{P}^M = \frac{a + c[b - d(n-1)(n+1)]}{2[b - d(n-1)(n+1)]}, \quad P^M = \frac{\alpha + c}{2} \qquad (11)$$

3.2 Independence

We now introduce the case where each operator sets, independently and simultaneously, their SO ticket prices and their component of each MO service which they are involved with providing.[15] Each firm i solves the following problem:[16]

$$\max_{\{P_{ii}, \boldsymbol{p}^i_{ij}, \boldsymbol{p}^i_{ji}\}} \pi_i = (P_{ii} - c)Q_{ii} + \left(p^i_{ij} - \frac{c}{2}\right)Q_{ij} + \sum_{k \neq i,j}\left(p^i_{ik} - \frac{c}{2}\right)Q_{ik} + \sum_{k \neq i}\left(p^i_{ki} - \frac{c}{2}\right)Q_{ki} - F \quad (k \neq j \neq i = 1,...,n)$$

$$(12)$$

where $\boldsymbol{p}^i_{ij}$ and $\boldsymbol{p}^i_{ji}$ are $(n-1)$-vectors of firm i's component of MO prices. The SO and a representative MO-component first-order condition for firm i are then, respectively:

$$\frac{\partial \pi_i}{\partial P_{ii}} = Q_{ii} + (P_{ii} - c)\frac{\partial Q_{ii}}{\partial P_{ii}} + \left(p^i_{ij} - \frac{c}{2}\right)\frac{\partial Q_{ij}}{\partial P_{ii}} + \sum_{k \neq i,j}\left(p^i_{ik} - \frac{c}{2}\right)\frac{\partial Q_{ik}}{\partial P_{ii}} + \sum_{k \neq i}\left(p^i_{ki} - \frac{c}{2}\right)\frac{\partial Q_{ki}}{\partial P_{ii}} = 0$$

$$(13)$$

$$\frac{\partial \pi_i}{\partial p^i_{ij}} = (P_{ii} - c)\frac{\partial Q_{ii}}{\partial p^i_{ij}} + Q_{ij} + \left(p^i_{ij} - \frac{c}{2}\right)\frac{\partial Q_{ij}}{\partial p^i_{ij}} + \sum_{k \neq i,j}\left(p^i_{ik} - \frac{c}{2}\right)\frac{\partial Q_{ik}}{\partial p^i_{ij}} + \sum_{k \neq i}\left(p^i_{ki} - \frac{c}{2}\right)\frac{\partial Q_{ki}}{\partial p^i_{ij}} = 0$$

where $\frac{\partial Q_{ij}}{\partial P_{ii}} = \frac{\partial Q_{mk}}{\partial p^i_{ij}}|_{(mk \neq ij)} = d$ and $\frac{\partial Q_{ii}}{\partial P_{ii}} = \frac{\partial Q_{ij}}{\partial p^i_{ij}} = -b$. Recognising symmetry across equilibrium SO prices, $P_{ii} = P_{jj}$, and MO prices, $P_{ij} = p^i_{ij} + p^j_{ij} = P_{ji}$ ($j \neq i$), and solving simultaneously, we have the equilibrium prices under Independence, for SO and MO services, under the general linear form demand system, denoted $\bar{P}$, and with parameterisations in Eq. (5) of the utility function,

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[12]In the case of Monopoly, there is only a single service operator and so here, for meaningful comparison with other regimes with multiple operators, MO tickets refer to what would be MO tickets in the multi-operator setting.

[13]All second order conditions are addressed in Appendix A.

[14]Thus, equilibrium price under Monopoly is equivalent to Joint Ownership in Economides and Salop (1991, 1992).

[15]Independence here reflects the fact individual operators are setting their prices independently. It is our stylised 'free-market' regime under 'conventional' transport ticketing. It is therefore not to be confused with the Independence regime, where all x and y components are provided by independent firms, or Parallel Vertical Integration, where each firm supplies an $\{x, y\}$ pair, setting prices for them individually but not offering a different combined price separately, as in Economides and Salop (1991, 1992).

[16]Note, that the same result obtains if we envisage the firm setting all component prices on the MO services individually, or it sets a single MO component ticket price which applies to all $2(n-1)$ MO services it jointly provides, i.e., imposing $p^i_{ij} = p^i_{ik} = p^i_{ji}$ ($\forall i \neq j, k = 1,...,n$).

denoted P, respectively:[17]

$$\bar{P}^I = \frac{3a + c[3b - d(n-1)(n+4)]}{6b - d(n-1)(4n+7)}, \quad \bar{P}^I_x = \frac{4a + c[2b - 3d(n-1)]}{6b - d(n-1)(4n+7)} \tag{14}$$

$$P^I = \frac{3\alpha(1-\gamma) + c[3 + \gamma(2n+1)(n-2)]}{6 + \gamma(n+1)(2n-5)}, \quad P^I_x = \frac{4\alpha(1-\gamma) + c[2 + \gamma(n(2n-3)-1)]}{6 + \gamma(n+1)(2n-5)}$$

3.3 Analysis and Findings

We now analyse price structure characteristics and interesting behaviours of other key variables and their relationships across these regimes. As noted before, for simplicity we do the analysis under the assumption of zero constant marginal cost, given the neutrality of positive constant marginal cost in the ratios of key variables across regimes. In order to consider consumer surplus and welfare we also employ the example utility function, Eq. (4), with parameterisations in Eq. (5), which also reduces the number of parameters involved, further facilitating ease of analysis. Imposing these simplifications on equilibrium prices, Eqs. (11) and (14), gives rise to the following Proposition.

Proposition 1. [18] *Under Independence: (i) aggregate profit is strictly lower than under Monopoly: $\pi^I < \pi^M$, (ii) consumer surplus is strictly lower than under Monopoly below a critical threshold of substitutability which is strictly decreasing in n: $S^I < S^M$ for $\gamma < \gamma_1$, where $\partial\gamma_1/\partial n < 0$, (iii) welfare is strictly lower than under Monopoly below a critical threshold of substitutability which is strictly decreasing in n: $W^I < W^M$ for $\gamma < \gamma_2$, where $\partial\gamma_2/\partial n < 0$.*

Not surprisingly, network profit is lower under Independence than Monopoly. However, consumer surplus and welfare are both lower than under Monopoly if substitutability is below a critical threshold. These thresholds are reducing in n, restricting the values of γ for which Monopoly dominates on consumer surplus and welfare. Hence, if the degree of substitutability is sufficiently low then the 'free-market' equilibrium is worse in consumer surplus, profit and welfare terms than Monopoly. That this welfare deficit is not driven entirely by superior profits under Monopoly, i.e. consumer surplus is also in deficit, reflects the potentially damaging impacts of the complementary strategic interactions on the network under Independence where they create externalities that are internalised under Monopoly.

Figure 2 reports critical thresholds (Monopoly = Independence) for consumer surplus γ_1 (grey dots) and welfare γ_2 (black dots) introduced as in Proposition 1. (n, γ) combinations below (above) these contours have Monopoly consumer surplus and welfare, respectively, dominating (dominated by) their Independence equivalents. The Figure also reports (n, γ) combinations which are required to equate an approximation of the market price elasticity of demand at the Independence equilibrium in line with upper and lower estimates of market elasticities in transport networks under zero marginal cost (solid lines) and $c = \frac{\alpha}{10}$ (dashed lines) with black lines for the $\eta = -0.4$ estimate and grey lines for the $\eta = -1.2$ estimate.[19] If the Figure represents the reality, i.e., the actual 'free-market' behaves in accordance with Independence, and the estimates are in the right range, then one estimate is of particular note. In the case of zero marginal cost and elasticity of $\eta = -1.2$

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[17]Note, these equilibrium prices do not simplify down under $n = 2$ to those reported in Economides and Salop (1992), as, stated earlier, the pricing mechanisms are different in the two papers. However, they do equate with their equivalents reported, for instance, in Economides (1993) and van den Berg et al. (2022).

[18]Proofs to Propositions are reported in Appendix C.

[19]See Appendix B for indicative elasticity workings and citations supporting the estimate levels.

Figure 2: Calibration with (n, γ) pairs supporting Equilibrium under Independence with $\eta = -0.4$ (black), $\eta = -1$ (grey), $c = 0$ (solid) and $c = \frac{\alpha}{10}$ (dash), and γ_1 (grey dots) and γ_2 (black dots) contours

(a) Wide shot

(b) Close-up
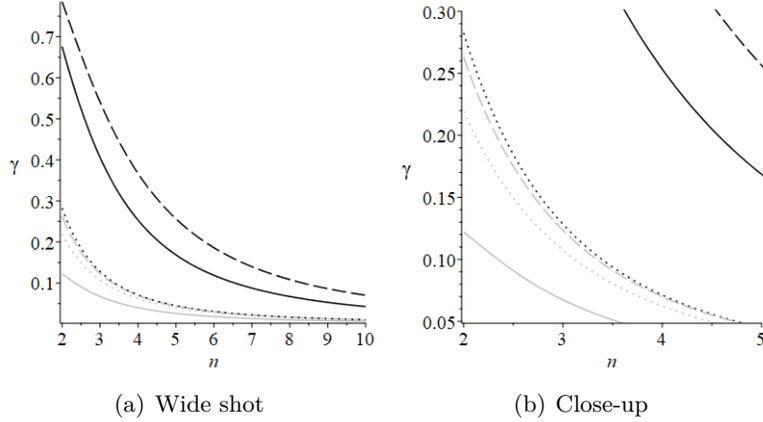
(solid grey contour), both the consumer surplus and welfare critical thresholds, γ_1 and γ_2, lie below the relevant elasticity contour: under these conditions Independence is inferior to Monopoly in profit, consumer surplus and welfare terms.[20] Hence, if a transport network operates like the Independence regime with $\eta = -1.2$, then the model suggests the performance of the market may be worse than Monopoly in all respects.

A priori, it was not obvious whether adding more rival operators would have a pro- or anti-competitive effect as additional independent services add new substitute and complementary linkages with associated countervailing strategic forces. Extending the framework beyond $n = 2$ to the n-operator case has revealed, that under these market conditions, the balance is strictly in favour of a pro-competitive response to increases in n in terms of the associated expansion in the set of γ for which Independence dominates Monopoly, providing meaningful insights available only in the n-operator framework.

Clearly from Figure 3(a), larger n has a tendency to further reduce profit performance of the industry under Independence relative to Monopoly, which might have implications for the ability to sustain a larger network or even the viability of the entire market under Independence, an issue to which we return in the following section. Figure 3 shows profit, welfare and consumer surplus are broadly improving under Independence with higher levels of n although note, at the low end of γ, there are instances in the Figures where increasing n from 2 to 3 or beyond results in a worsening in all three ratios. Hence, unlike the finding above around the pro-competitive behaviour of the critical thresholds, here we can see that relative performance for consumer surplus and welfare may or may not be pro-competitive with changes in n. Indeed, we can formalise this result in the following Proposition.

Proposition 2. *Ratios of profit, consumer surplus and welfare for Independence relative to Monopoly are non-monotonic in n.*

Adding further rival operators under Independence might guarantee a larger range of γ over

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[20]Note, under $\eta = -1.01$, consistent with estimates for long-run bus demand in the UK, the equivalent consumer surplus threshold is again below the elasticity contour but the welfare threshold is above it, meaning welfare, driven by higher profits, is higher under Monopoly, whilst consumer surplus is higher under Independence.

Figure 3: Profit, Consumer Surplus and Welfare under Independence relative to Monopoly

(a) Profit   (b) Consumer Surplus   (c) Welfare

$$—n = 2, \text{---} n = 3, \text{-·-·-} n = 4, \cdots\cdots n = 5$$
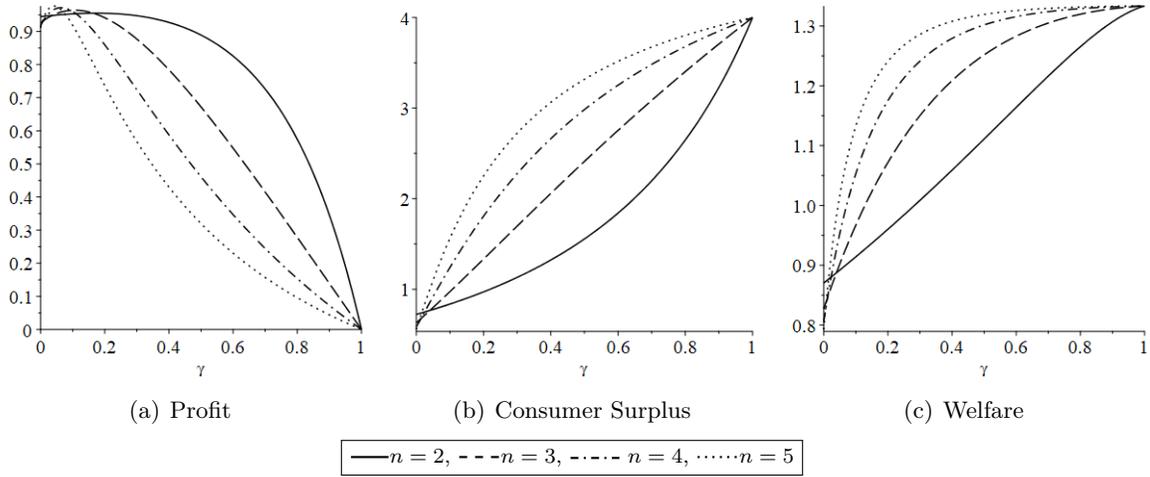
which it dominates Monopoly on welfare and consumer surplus, but depending on the level of substitutability it may have a pro- or anti-competitive effect in terms of the relative performance. This demonstration of non-monotonicities in welfare, consumer surplus and profit ratios across the regimes with changes in n, further indicates the importance of the n-operator framework for policy analysis. To understand what is driving these results, we now turn to prices.

Proposition 3. [21] *Under Independence: (i) the SO price is strictly lower than both the corresponding MO price and the Monopoly price: $P^I < P_x^I$ and $P^I < P^M$, (ii) the MO price is strictly greater than the Monopoly price below a critical threshold of substitutability which is decreasing in n: $P_x^I > P^M$ for $\gamma < \gamma_3$, where $\partial\gamma_3/\partial n < 0$, (iii) whilst the Monopoly price is invariant to change in n, the SO and MO prices under Independence are weakly reducing in n: $P_n^M = 0$, $\partial P^I/\partial n \le 0$ and $\partial P_x^I/\partial n \le 0$.*

Whilst Independence generates lower SO prices than Monopoly, the corresponding MO prices (which are strictly greater everywhere than the SO prices) are higher even than Monopoly levels for sufficiently low levels of substitutability, the threshold for which is decreasing in n: higher levels of n reduce the range of γ over which MO prices under Independence exceed Monopoly prices.[22] Hence, the inferiority of Independence in welfare and consumer surplus terms, relative to Monopoly, is driven by excessive MO prices and the associated price distortion. Firms under Independence would like to lower the MO price, but strategic interaction pushes the equilibrium price upwards. It is well known that the upward pressure on MO type prices in networks comes, under our Independence setting, from the component prices across firms having decreasing differences ($\frac{\partial^2 \pi_i}{\partial p_{ij}^j \partial p_{ij}^i} < 0$), and hence best response functions in (p_{ij}^i, p_{ij}^j)-space are downward sloping here meaning the 'cheat'

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[21]Note, although different prices have different relationships with marginal costs, the following results are still independent of marginal costs and therefore robust to the introduction of non-zero marginal costs.

[22]Note, under Parallel Vertical Integration in Economides and Salop (1992), all prices are above Monopoly levels for low substitutability and below Monopoly levels for high substitutability. Here, the overall welfare picture is not so obvious from a study of prices as, for low substitutability, some prices are higher than Monopoly under Independence, and some lower.

incentive, preventing a monopoly pricing Nash equilibrium, is to raise, not lower, p_{ij}^i prices.[23] On the other hand, there are increasing differences in other rival price relationships. The result here is consistent with the observations, for instance, on urban bus ticket prices in the commercial sector in the UK, where MO prices are reported to significantly exceed SO prices (e.g., see White, 2010). Notice that the price distortion which arises under Independence penalises those consumers for whom the match with MO services is best whilst benefiting those who are best served by SO services relative to Monopoly. This distortion is driving welfare and consumer surplus below Monopoly levels for sufficiently low γ where consumers cannot easily replace their 'expensive' preferred MO service with a 'less expensive' SO one as they are such poor substitutes.

Figure 4(a) illustrates the ratios of Independence to Monopoly prices for SO and MO services. Here we see the above story play out. Increasing the number of operators from 2 to 3 to 4 to 5 (respectively, solid, dashed, dot-dashed, dotted lines) improves prices on both SO and MO services relative to Monopoly levels, as well as achieving below Monopoly level MO prices at lower levels of γ under Independence. So, if increasing n is having a net pro-competitive effect in terms of both prices relative to Monopoly levels and lowering the threshold level of γ_3, where $P_x^I = P^M$, how do welfare and consumer surplus deteriorate relative to Monopoly for very low levels of γ as n rises above 2? To better understand what is driving this we now turn our attention to quantities.

Figures 4(b) and (c) report, respectively, SO and MO firm quantities and aggregate quantities for Independence relative to Monopoly. The following Proposition formalises some of the main quantity characteristics.

Proposition 4. *Under Independence: (i) the SO quantity is weakly greater than the Monopoly level: $Q^I \geq Q^M$, (ii) for the special case of $n = 2$, the MO quantity is everywhere strictly below the Monopoly level: $Q_x^I(n = 2) = \frac{2}{3}Q^M(n = 2)$, similarly, for all n, $Q_x^I(\gamma = 0) = \frac{2}{3}Q^M(\gamma = 0)$, (iii) otherwise for $n \geq 3$, the MO quantity is below the Monopoly level only under some critical threshold of substitutability which is strictly reducing is n: $Q_x^I < Q^M$ for $\gamma < \gamma_4$, where $\partial\gamma_4/\partial n < 0$, (iv) the aggregate quantity is below the Monopoly level only under some critical threshold of substitutability which is strictly reducing in n: $nQ^I + (n-1)nQ_x^I < n^2 Q^M$ for $\gamma < \gamma_5$, where $\partial\gamma_5/\partial n < 0$.*

Whilst SO operator-level quantities under Independence weakly exceed their monopoly equivalents (quantities under Monopoly across the same services as would be operated by a single Independent operator), this is not the case everywhere for MO and aggregate outputs levels. For $n = 2$, MO quantities are strictly below monopoly levels everywhere. Generalising to the n-operator setting exposes this to be a special-case with MO quantities in the Figure increasing beyond Monopoly levels with increases in n and γ. Aggregate output is strictly smaller under Independence for sufficiently low levels of γ with the critical γ threshold falling with n. It is now clear why increasing n above two for low levels of substitutability reduces consumer surplus and welfare under Independence relative to Monopoly. Below the critical threshold, γ_4, aggregate MO quantity under Independence is strictly below the Monopoly level whereupon increasing the number of operators increases the number of MO services which are supplying below the Monopoly level, further reducing aggregate output relative to the Monopoly level. To see this, consider γ very close to zero, then Q_x^I is close to $\frac{2}{3}Q^M$, regardless of n. Increasing n raises the number of MO services, each with almost a third below Monopoly output, a greater deficit than can be offset by higher than Monopoly output SO services under Independence, especially given there are fewer SO than MO services on the network for $n > 2$.

One take away from this analysis is that, for policy makers, if services are not poor substitutes and it is feasible to increase the number of operators beyond two this might help resolve

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[23] As Cournot (1838) first observed, such externalities arise when independent agents set different component prices in a composite good and this drives prices upwards beyond even the monopoly level.

Figure 4: Price, Quantity and Total Quantity on *SO* and *MO* Services under Independence relative to Monopoly

(a) Prices   (b) Firm Quantities   (c) Aggregate Quantities

$$\text{---}\, n = 2,\ \text{- - -}\, n = 3,\ \text{-·-·-}\, n = 4,\ \cdots\cdots n = 5,\ \text{and in (a) and (b) Black}=SO,\ \text{Grey}=MO$$

the apparent pricing anomaly arising under Independence due to the dual substitute/complement demand components. One way to do this is, obviously, through introducing new operators. But other solutions exist such as better integrating existing services that currently are not sufficiently well integrated to offer interchangeable and competing parts.[24] However, as we have seen in the Introduction, real market conditions may not sustain a sufficient number of operators and hence attracting new operators may not be feasible without subsidies. In addition, given the low profitability under Independence, it might not be straightforward to encourage service providers to better integrate - they might be deliberately disintegrating to avoid the low-profit outcome from Independence. We return to this point in the following Section.

To summarise, we have seen that under Independence, a stylised characterisation of a 'free-market' case in a deregulated public transport system with more than one operator, *MO* prices are elevated potentially above monopoly levels, damaging consumer surplus and profit. We have seen the importance of increasing the number of operators in terms of its pro-competitive impact on reducing the threshold level of γ, below which Monopoly consumer surplus and welfare dominate Independence. However, we have also seen that the relative levels of welfare and consumer surplus across the regimes are non-monotonic in n. Amongst other things, this observation supports the case for modelling $n > 2$, but also warns that increasing n under Independence may be counterproductive. Finally, if increasing n is welfare enhancing but not feasible due to being unprofitable, the inability to raise n may leave the market in a state where it is performing worse than Monopoly, an issue we turn to next.

4 Fixed Costs and Entry

We now consider the issue of incentives for network expansion, in terms of increasing the number of x and y services by increasing n, under the two benchmark regimes. For this we introduce a fixed cost, F, for each operator. Hence an additional operator brings an extra $\{x, y\}$ differentiated

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[24]Department for Transport (2021) cites situations where improved landscaping around train stations in the Home Counties of the UK have had the effect of disconnecting them with bus stops.

component pair to the network and an additional associated fixed cost. For purposes we will see later, we let f_n be the level of the fixed cost for which n operators just break even under Independence. The threshold fixed cost is given by:[25]

$$f_n \equiv \frac{\alpha^2(8\gamma n^3 - 15\gamma n^2 - 2\gamma + 8n + 1)(1 - \gamma)}{(2\gamma n^2 - 3\gamma n - 5\gamma + 6)^2(\gamma n^2 - \gamma + 1)} \tag{15}$$

which can be shown to be decreasing in n in the relevant range unless γ is sufficiently low.[26] Setting fixed cost per operator at $F_n = f_n + \delta$, where $\delta > 0$ is arbitrarily small, is sufficient to make n operators non-viable under Independence. We then examine the profitability of n operators under Monopoly with fixed cost per operator of F_n, and examine the associated consumer surplus and welfare outcomes relative to the situation with the highest affordable number of operators (maximum being $n - 1$) with F_n under Independence, where n operators cannot survive under Independence.

Figure 5(a) reports Monopoly profit (divided by α^2), after paying nF_n, across the range of γ for $n \in \{2, 3, 4, 5\}$. It is evident here that, despite incurring the fixed cost which renders Independence with n operators non-viable, Monopoly profits are everywhere positive but also broadly increasing with higher numbers of services and associated fixed costs. There are three potential drivers for this. First, as we have noted above, for a given number of operators, n, the Independence regime is inferior on profit to Monopoly with the same n, since (i) the former cannot collude to maximise profit by stifling competition across substitute lines, effectively SO prices are too low for maximum profit, and (ii) neither can they collude to eliminate the externality coming from the complementary aspect of the network resulting in MO price rising above the Monopoly level. These price distortions under Independence relative to Monopoly prices damage profit. Second, however, the Monopolist can sustain higher profits than Independence for a given n, hence where they are able to survive the fixed cost, F_n, they can exploit the extra utility that the market derives from the additional degree of service variety (higher n), with potential associated profit gains *if* it more than compensates for the additional fixed cost. Finally, higher n will, unless γ is sufficiently low (as noted above), involve a lower F_n, reducing the extent to which the total fixed costs grow with the addition of the extra services: the total fixed cost moving from F_{n-1} to F_n grows by less than F_n, being offset by reductions in the $n - 1$ intra-marginal fixed costs.
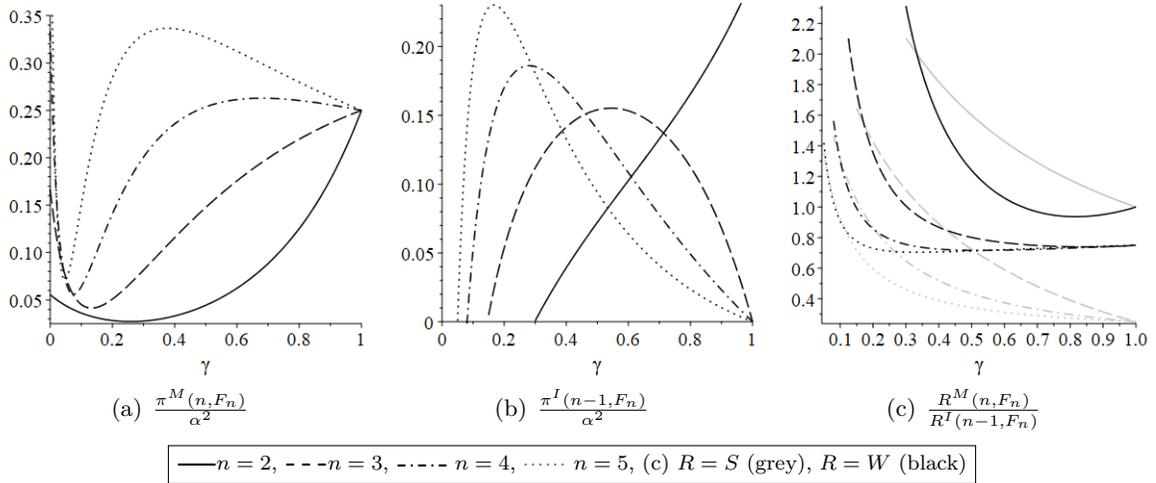
A priori, it is not obvious whether the higher provision of services under Monopoly will produce sufficient additional consumer surplus and profit such that it compensates for the additional fixed cost, relative to the consumer surplus and welfare in the smaller network under Independence. Turning to Figure 5(b), this reports profit under Independence (divided by α^2) with $n-1$ operators and fixed cost F_n. Notice that profit under Independence with $n-1$ firms reaches zero under each scenario as γ falls (thereafter becoming strictly negative). It is straightforward to show that at this point all lower levels of n under Independence are also loss-making. This means that, whereas Monopoly can sustain a network of size n under F_n, the Independence regime suffers complete market failure once γ falls below some critical level under F_n. No services are provided: profit, consumer surplus and welfare under Independence are zero, and the relative gains under Monopoly are infinite. From the Figure, these critical values become lower for higher levels of n.

Figure 5(c) demonstrates the potential for the larger network under Monopoly to yield consumer surplus and welfare benefits with network size n relative to the lower provision $n-1$ under

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[25]Hence, setting $n = 3$ in Eq. 15 generates f_3 which reduces profit under Independence with three operators to zero.

[26]Note, there are potentially countervailing forces in determining F_n. Increasing n increases strategic interaction (more so for higher γ) under Independence, which can adversely impact profit, reducing F_n. On the other hand, increasing n introduces new services with which each operator can combine, raising profit, less so with higher γ.

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Figure 5: Monopoly v Independence Profits, Consumer Surplus and Welfare on an Extended Network with Fixed Cost F_n

(a) $\frac{\pi^M(n,F_n)}{\alpha^2}$ (b) $\frac{\pi^I(n-1,F_n)}{\alpha^2}$ (c) $\frac{R^M(n,F_n)}{R^I(n-1,F_n)}$

$\quad$ —— $n=2$, - - - $n=3$, - · - · - $n=4$, · · · · · · $n=5$, (c) $R=S$ (grey), $R=W$ (black)

Independence. Discontinuities in the lines occur, for sufficiently low γ, where welfare and consumer surplus are still strictly positive under Monopoly, but the market under Independence ceases to function, as noted above, yielding infinite relative gains under Monopoly. From observation, in the case of $n=2$, consumer surplus gains under Monopoly stretch across the full feasible range of γ.[27] Gains under Monopoly are still clearly available, relative to the smaller network under Independence as n increases to 3 and beyond, but not for all γ. The following Proposition follows directly.

Proposition 5. *Under Independence with fixed costs F_n, equilibrium consumer surplus is (weakly) lower than under Monopoly with $n=2$. This inequality doesn't generalise everywhere for $n \geq 3$.*

However, for the cases with $n \geq 3$ reported in Figure 5, these gains only arise below a critical level of γ which is decreasing in n. Also the relative gains themselves are broadly everywhere falling with n. In other words, in the presence of fixed costs, the market under Independence has the potential to fail to deliver the number of services that would have been available under Monopoly. Indeed, despite the Monopoly exercising its market power uncontested, it still has the potential to raise consumer surplus and welfare beyond the levels available under Independence. These relative gains appear to reduce with n and higher substitutability, which both favour higher consumer surplus under Independence.

In terms of a potential real-world parallel, in the UK commercial urban bus sector, it is believed that the sector often under-provides services and has insufficient profitability to support the number of operators needed to achieve competitive benefits (e.g., see White, 2010; Department for Transport, 2021). From this analysis, it appears possible that the adverse profit situation arising due to the dual substitute and complement strategic externalities under Independence (that are internalised under Monopoly) might help explain these shortcomings. It also suggests the potential merits of investigating alternative pricing structures which can help address these externalities under Independence and improve upon the outcomes under Monopoly. This is something we turn attention to in the following section.

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[27]Note, at $\gamma = 1$ the consumer surplus and welfare ratios are exactly 1.

However, it is also apparent from Figure 5, that if γ is sufficiently large (strong substitutes) and $n-1$ operators are viable under Independence, then despite providing a lower number of services than Monopoly, the lower prices due to competition alongside the lower total fixed cost (one less F_n than under Monopoly), provide consumer surplus and welfare gains over the more extensive Monopoly equilibrium. That said, if the calibration exercise, above, is reliable and realistic then it suggests γ is low and in the region where Independence performs badly here.

This is not the end of the story though, as we have not sought to include other benefits from the larger network under Monopoly. For instance, sizeable externality benefits of urban public transport (e.g., pollution and congestion) identified in Adler and van Ommeren (2016) suggest that the enhanced quantity gains under the larger network, with fixed costs under Monopoly relative to Independence, could well add heavily to the welfare benefit calculation under Monopoly. One interesting avenue for further inquiry would be to adapt this model towards evaluating the associated potential environmental and wider externality gains.

5   Multi-Operator Ticketing

The network framework can readily be applied to study the potential benefits of *MO* pricing arrangements such as those allowed in the UK under the Block Exemption, introduced earlier. The Block Exemption was one of several measures taken to improve the functioning of the local bus market in the UK that was deregulated (outside London) in 1986 under the Local Transport Act (1985).[28] Alongside some notable benefits, such as innovation and cost efficiencies (e.g., see White, 2019), there have been a number of long-standing problems including a collapse in multi-operator ticket (*MT*) use and insufficient passenger volumes to sustain multiple operators to the detriment of competition (e.g., see White, 2010). The Block Exemption was introduced to try and encourage better integrated ticketing provision towards enhancing quality, flexibility, geographical coverage and raising patronage. In particular, it permitted a variety of ticketing options which it was hoped would help reduce the risk of infringing competition law in the process of providing integrated ticketing options across rival operators. Amongst other things, it was suggested that *MT*s can increase competition between operators (see Competition Commission, 2011). However, evidence suggests that even after its introduction, despite *MT*s being widely available, the use of them has remained low and prices have remained high. For instance, Department for Transport (2013, p. 43) reports evidence of *MT* ticket prices exceeding their *SO* ticket prices by as much as 40%, whilst problems with *MT* pricing and usage, inadequate network provision and patronage are cited as recently as Department for Transport (2021). As noted in the Introduction, the application of the framework to the study of *MTs* is particularly timely given their prominence in the UK government's local bus strategy, but also because the Block Exemption is itself currently under statutory review and due to expire in 2026.

One class of *MT*s permitted under the Block Exemption is multi-operator travel cards (*MTC*s). This ticketing arrangement allows two or more operators to offer services which can be used to make a particular journey, using whichever services they like, multiple times (e.g., daily or weekly travel cards). In this Section we introduce a stylised pricing regime which seeks to capture some of the key aspects of the *MTC*, in particular employing the *MTC* pricing framework specifically recommended by the Competition Commission (Department for Transport, 2013). We derive the equilibrium pricing structure in the *n*-operator framework and compare it with the 'free-market' Independence regime in order to better understand what might be the particular benefits of the

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[28]Under deregulation local bus networks outside London were split into 'supported' services (with local authority support fostering competition via competitive tendering) and 'commercial' (unsupported, 'free-market') services.

MTC but also shed light on some potential limitations, including why the recommended model for *MTC* pricing may not be in widespread use.

The UK local bus market represents a relatively isolated case of deregulated public transport services amongst developed economies.[29] This, alongside an ongoing unresolved debate on the relative merits of public versus private provision of public transport (e.g., see Gagnepain et al., 2011), makes the UK something of a test-case with innovations and developments therein being of global interest. The continued failure of policy initiatives to broaden the use of *MTC*s amongst other things, is of particular importance in this context, as *MTC*s are seen as a significant part of solving the UK's ongoing local public transport woes (e.g., see Department for Transport, 2021). In the previous Sections we noted that the calibration exercise placed γ in its low range, potentially where Independence under performs against even Monopoly. In this Section we will see whether the *MTC* pricing strategy offers an improvement over Independence in this parameter range where Independence performs so badly.

5.1   MTC Equilibrium

In modelling the *MTC* we adopt the pricing framework recommended by the Competition Commission (see Department for Transport, 2013, pp.22):

> The *MTC* price is set at "Average or median fare x Estimated [typical] ticket usage x Passenger discount for purchasing a multi-journey ticket."

In order to focus on the effects of the strategic interaction amongst operators in the n-operator setting associated with price incentives, we adopt a simplified, stylised representation of the *MTC*. We assume that passengers selecting the *MTC* would have usage exactly equal to the estimated [typical] ticket usage. For simplicity we set this usage to one with no further loss of generality. In addition, we assume away any benefits that the *MTC* offers in terms of flexibility and transactions costs (e.g., buying one ticket in place of multiple tickets and being able to change travel plans at no extra cost). Both these aspects of the *MTC* offer real potential benefits and are worthy of analysis in their own right. We sacrifice this for greater clarity regarding the strategic interaction implications of the pricing system and recognise any associated benefits we find in so doing to be a likely understatement.

Under the rules of the *MTC*, operators do not have to join an *MTC*, but they cannot be prevented from joining one. For simplicity, we model the situation where all operators are participating in the *MTC*. In addition, operators are allowed to agree a price for the *MTC*, although, as the recommendations suggest, this is envisaged as a discount on average fares elsewhere on the network and it is this vision we are seeking to analyse. One obvious question is, if this is the recommended option, why does the evidence not support it being widely adopted (reflecting on the earlier Department for Transport, 2013, evidence of *MT* ticket prices exceeding their *SO* ticket prices by as much as 40%)? Operators joining the *MTC* are also required to continue to provide independently-priced *SO* tickets. Implementing these conditions in our n-operator framework, the *MO* price is now the *MTC* price which, in the spirit of the above recommended pricing, we take

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[29]With private urban public transport provision being widespread in the developing world (e.g., see Gwilliam, 2001), extensive private provision in the UK outside London is something of a special case amongst developed economies.

as a percentage τ of the weighted average of the *SO* prices on the network:[30]

$$P_x = \tau \frac{P_{ii}Q_{ii}(P_{ii}, \sum_{k \neq i} P_{kk}, P_x)) + \sum_{k \neq i} P_{kk}Q_{kk}(P_{ii}, \sum_{k \neq i} P_{kk}, P_x)}{Q_i(P_i, \sum_{k \neq i} P_{kk}, P_x)) + \sum_{k \neq i} Q_{kk}(P_{ii}, \sum_{k \neq i} P_{kk}, P_x)}, \quad (i \neq k = 1, ..., n) \quad (16)$$

where $\tau \in [0, 1]$, determines the discount applied to the *MO* price for participating operators. Whilst we know, from the earlier study of the Independence regime, that operators would like to reduce *MO* prices, by having some limiting mechanism for the *MO* price through the *MTC* it may have some attraction. However, it is straightforward to show that if the operators selected an optimising τ it would not everywhere lie in the weak-discount interval that we wish to study in analysing the recommended pricing framework, in itself indicating a potential shortcoming to the expectation of operators voluntarily providing discounts on the *MTC*: $\tau \in [0, 1]$. Hence, we model a situation where operators independently set their *SO* prices knowing that the *MO* price will be set in accordance with Eq. (16) and then we will study the impact of different levels of τ on the resulting market equilibrium.

The *SO* and *MO* demand functions can be expressed as:

$$Q_{ii} = a - bP_{ii} + d \left[\sum_{k \neq i} P_{kk} + n(n-1)P_x \right] \quad (i \neq k = 1, ..., n) \quad (17)$$

$$Q_x = a - bP_x + d \left[P_{ii} + \sum_{k \neq i} P_{kk} + (n(n-1)-1)P_x \right]$$

Operator i solves the following maximisation problem:

$$\max_{\{P_{ii}\}} \pi_i = (P_{ii} - c)Q_{ii} + (n-1)(P_x - c)Q_x - F \quad (18)$$

where, given symmetry, each firm receives half the ticket price on each of its $2(n-1)$ *MO* services. The *SO* first-order condition is then:

$$\frac{\partial \pi_i}{\partial P_{ii}} = Q_{ii} + (P_{ii} - c)\frac{\partial Q_{ii}}{\partial P_{ii}} + (n-1)(P_x - c)\frac{\partial Q_x}{\partial P_{ii}} + (n-1)Q_x\frac{\partial P_x}{\partial P_{ii}} = 0 \quad (19)$$

where $\frac{\partial Q_{ii}}{\partial P_{ii}} = -b + dn(n-1)\frac{\partial P_x}{\partial P_{ii}}$, $\frac{\partial Q_x}{\partial P_{ii}} = d + d[n(n-1)-1]\frac{\partial P_x}{\partial P_{ii}}$, and $\frac{\partial P_x}{\partial P_{ii}} = \frac{\tau}{n}$. Recognising symmetry across equilibrium *SO* prices, $P_{ii} = P_{jj} = P$, and the *MO* price in equilibrium is $P_x = \tau P$, we have the equilibrium prices under the *MTC* regime, for *SO* services, under the general linear-form demand system, Eq.(1), denoted $\bar{P}$, and with specific parameterisations Eq. (5), denoted P, of the example utility function, respectively:

$$\bar{P}^{MTC} = \frac{(n + (n-1)\tau)a + c[(n + (n-1)\tau)b - d(n-1)(n - \tau + n^2\tau)]}{2b(n + (n-1)\tau^2) - d(n-1)[(2\tau + 1)\tau n^2 - n(2\tau^2 - 3\tau - 1) - 2\tau^2]}, \quad (20)$$

$$P^{MTC} = \frac{(1 - \gamma)(\tau - n(1 + \tau))\alpha + c[\tau(n-1)(1 - \gamma) + n(n(n-1)\gamma + (1 - \gamma))]}{n[\gamma(3 + n - 2n^2) - 2] + n\gamma(n-1)(n+3)\tau - 2(n-1)(1 + \gamma(n-1))\tau^2}$$

[30]It is straightforward to show under symmetry this yields the same outcome under optimisation as the simple average of independent prices: $P_x = \tau \frac{P_{ii} + \sum_{k \neq i} P_{kk}}{n}$, $(i \neq k = 1, ..., n)$.

5.2 Analysis and Findings

For the analysis, as before, we set marginal costs to zero, which has stronger empirical support than before given the application is primarily the local bus market (e.g., see Jørgensen and Preston, 2003), and adopt the equilibrium price solutions based on the example utility function Eq. (4), for simplicity. The following Proposition sets out the comparison of the MTC regime with Independence (our approximation of the 'free-market' alternative) regarding profit, consumer surplus and welfare outcomes.

Proposition 6. *Under the MTC, (i) profit is strictly higher (lower) than under Independence for sufficiently low (high) γ and n and high (low) τ: $\pi^{MTC} > \pi^I$ below the contour (in n and γ terms) in Figure 9(a) (see Appendix C), (ii) consumer surplus is strictly higher than under Independence in the relevant range, $S^{MTC} > S^I$, (iii) welfare is strictly higher than under Independence for $\tau \in [0,1]$ and $\gamma \in [0,1)$: $W^{MTC} \geq W^I$*

Hence, whilst the MTC is successful in improving welfare and consumer surplus over the Independence regime, it can come at the cost of a profit hit, although, it also has the potential to raise profitability. Whilst the MTC helps address the complementary externality which drives up the MO price under Independence (an externality which damages profit and consumer surplus) any strategic interaction along substitute lines acting to reduce SO prices under the MTC is further channelled into MO prices at an added discount, and of course, MO services are substitutes for SO ones. The balance of these two forces can favour profit under the MTC at low levels of γ where the strategic interaction along substitute lines is weakened by lower substitutability between services. This is offset by higher numbers of operators (for any given level of substitutability, more rivals intensify competitive pressures) and higher MTC discounts (exacerbating the follow through of SO price pressure on linked MO prices).

Figure 6: Profit, Consumer Surplus and Welfare under Independence relative to MTC(τ)

| (a) Profit | (b) Consumer Surplus | (c) Welfare |

$\boxed{\text{———}n=2, \text{- - -}n=3, \text{-·-·-}n=4, \text{·······}n=5, \text{Black } \tau=1, \text{Grey } \tau=0.9}$
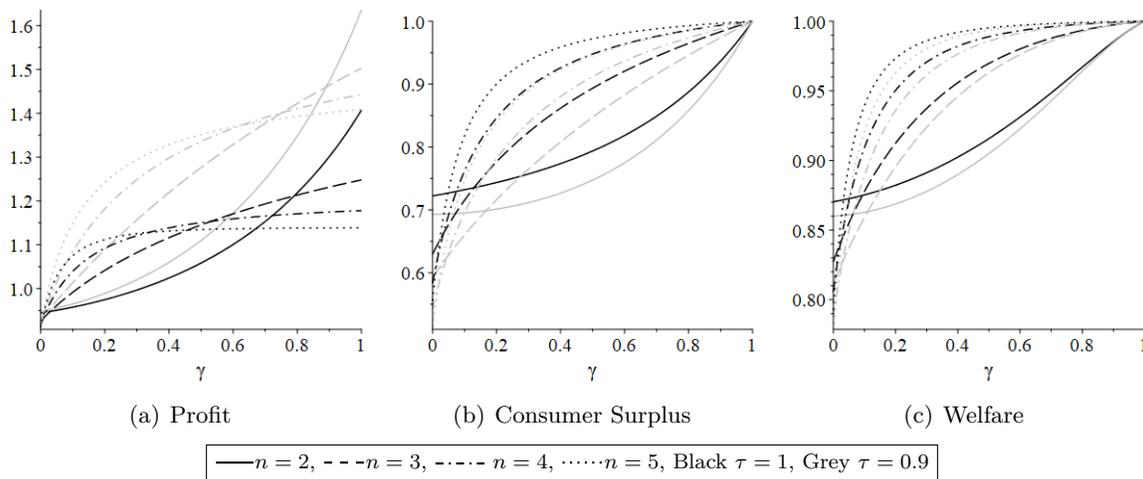
Figure 6 illustrates profit, consumer surplus and welfare ratios for Independence relative to the MTC in the cases of $n \in \{2,3,4,5\}$ and $\tau \in \{0.9,1\}$. From the selection of cases in the Figure, relative profit under the MTC is broadly decreasing with lower τ and higher n and γ, supporting the above discussion. It is clear from the Figure that there exist parameterisations where for a

given value of n, MTC profit exceeds that under Independence, whilst at higher n the situation is reversed, leading directly to the following Proposition.

Proposition 7. *Operators may be discouraged from setting up an MTC even if it yields profit improvements over Independence, if there is a fear of new entry, into the network and MTC arrangement, reversing the profit ranking of the regimes.*

In terms of incentives to engage in an MTC, across the parameter set illustrated in Figure 6, profits are mostly lower than under Independence - suggesting that if it is thought there are gains socially from the MTC, based on the recommended interpretation of the pricing framework which is analysed here, then it might be necessary to make it compulsory to join an MTC with the recommended pricing regime, rather than optional as it currently is in the UK. If our calibration exercise is to be believed, with zero marginal cost, it indicates the 'free-market' is operating at the low substitutability range where the MTC might achieve higher profits than Independence.

In the Figure, relative consumer surplus and welfare under the MTC are broadly increasing in τ, but like profit, broadly decreasing in n and γ. Hence, the above argument supporting Proposition 6 also supports relative improvements in welfare and consumer surplus reinforcing, amongst other things, the poor performance of the Independence regime with low γ and n. However, the relationship between these ratios and n is not completely monotonic, as, similar to the result of Proposition 2, we have a situation indicated in Figure 6 with profit, welfare and consumer surplus at low levels of γ increasing under the MTC relative to Independence with an increase in n, leading directly to the following Proposition.

Proposition 8. *The performance of the MTC regime relative to Independence in profit, consumer surplus and welfare terms is non-monotonic in n.*

The reasoning behind this is as discussed above, and further reinforces the importance of the $n > 2$ framework.
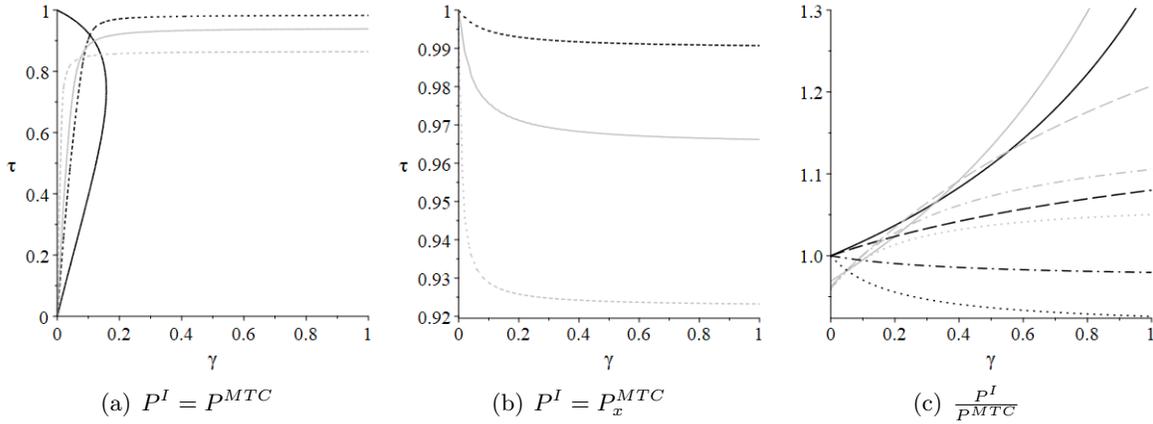
Turning now to aspects of the underlying price story, one might expect that under the recommended MTC price framework, the MO price would be lower than the Independence MO price, and for sufficiently low τ, lower also than the Independence SO price. However, the following Proposition shows that whilst the former is everywhere true, the latter would be true for all τ and γ in the case of $n \in \{2, 3\}$, but is does not hold everywhere for higher n, hence, a model limited to $n = 2$ would indicate $P_x^{MTC} < P^I$, which need not be the case.

Proposition 9. *Under the MTC, (i) the MO price is strictly lower than the MO price under Independence: $P_x^{MTC} < P_x^I$, (ii) there exist triples (γ, τ, n) in the relevant range for which the SO price exceeds the SO price under Independence, and, for $\tau = 1$ this is everywhere guaranteed for $n \geq 4$, (iii) the MO price is strictly lower than the SO price under Independence for $n \leq 3$, and similarly for a subsection of triples (γ, τ, n) in the relevant range for $n \geq 4$.*

Figure 7(a) and (b) illustrate contours in (γ, τ)-space where $P^I = P^{MTC}$ and $P^I = P_x^{MTC}$, respectively for $n \in \{2, 4, 5, 10\}$ and where P^I is lower (higher) than P^{MTC} (P_x^{MTC}) to the left of the contours. Clearly, $P^I < P^{MTC}$ for sufficiently high τ and/or sufficiently low γ, with the role of n being non-monotonic. On the one hand, under the MTC for a given SO price, a lower τ translates the SO price into a lower MO price. On the other hand, the MTC commitment to provide the $1 - \tau$ percentage discount in setting the MO price provides countervailing upward pressure to SO prices acting against the usual downward pressure on SO prices through the 'cheat' incentive, which eliminates the collusive outcome as a Nash equilibrium. Indeed, this upward pressure can potentially lead to the SO prices under the MTC being higher than under Independence, with

this countervailing pressure being stronger when the services are weak substitutes (γ is low). From Figure 7(b), $P^I < P_x^{MTC}$ holds for sufficiently high τ, with higher γ and n tending to reduce the critical level of τ supporting this inequality. Broadly, higher discounts under the MTC can eventually bring the MO price below the Independence SO price, with higher n appearing to work in opposition to the lower τ in this respect suggesting higher n has a greater impact on reducing SO prices under Independence relative to the MO price under the MTC, consistent with story of Proposition 9(iii) where $P_x^{MTC} > P^I$ for low n. We can perhaps see what is driving this in Figure 7(c), as higher n has the ability to raise P_x^{MTC} - insulating the firms from the discount applied to the MO price. However, Figure 7(c) also reveals a further non-monotonicity: for low γ a decrease in τ can cause P^{MTC} to rise (noting the P^I is invariant to changes in τ): when the services are poor substitutes, firms are able to insulate themselves from the penalties of the MTC discount by raising SO prices.

Figure 7: SO and MO Prices under Independence and MTC(τ)

(a) $P^I = P^{MTC}$ (b) $P^I = P_x^{MTC}$ (c) $\frac{P^I}{P^{MTC}}$

(a) and (b) ——$n = 2$, - - -$n = 4$, —— $n = 5$, - - - $n = 10$, (c) ——$n = 2$, - - -$n = 3$, -·-·- $n = 4$, ······$n = 5$, Black $\tau = 1$, Grey $\tau = 0.9$

To summarise, the MTC has the potential to raise profit, consumer surplus and welfare relative to Independence in the case of a low number of operators, low substitutability (such as indicated in the calibration) and low MTC discount. This partly reflects the poor performance of the 'free-market' Independence case under these conditions. However, it is also despite these also being the conditions under which the MTC SO and MO prices can exceed the Independent SO price. Recalling the discussion in Section 3, though, the poor performance of Independence relative to Monopoly was driven by the price distortion taking P_x^I above Monopoly levels, but the MTC ensures all prices are strictly below P_x. However, the MTC is only more profitable than Independence over a limited parameter subset, and as we have seen in the case of Monopoly versus Independence, profitability may have a role to play in determining the viability of a given number of operators, and indeed whether there is any viable provision at all under a given pricing regime. We turn our attention to this now.
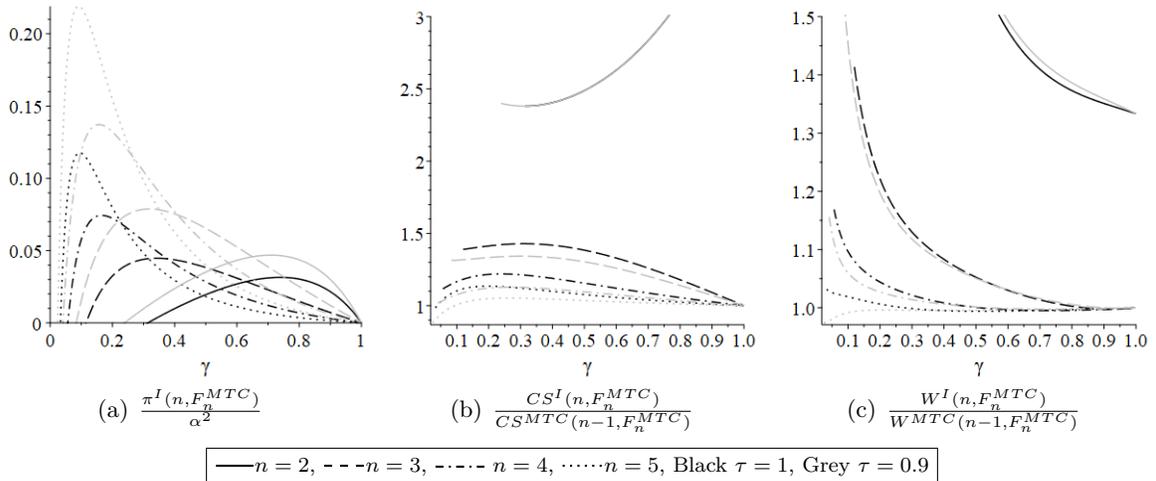
5.3   Independence, MTC and Entry

As noted above, although the MTC regime improves consumer surplus and welfare relative to Independence for a given number of operators, for much of the (γ, n, τ) parameter set it is inferior in profit terms. Given the express concern, noted above, around profitability under private provi-

sion being inadequate to maintain multiple operators to the detriment of competition and service availability in the UK local bus sector, this might present a drawback to the MTC regime in the presence of non-trivial fixed operating costs. We therefore undertake an analysis comparing the profit, consumer surplus and welfare outcomes under an MTC with the 'free-market' Independence regime where there is a fixed cost per operator, $F_n^{MTC} \equiv f_n^{MTC} + \delta$, and where, for a given level of τ, the fixed cost f_n^{MTC} brings profit equal to zero with n operators under the MTC. As before, δ is an arbitrarily small positive term ensuring that under F_n^{MTC}, n operators will not be viable under the MTC regime.[31]

Figure 8 reports profit (divided by α^2) under Independence and ratios of consumer surplus and welfare for Independence relative to MTC with fixed cost, F_n^{MTC}. Figure 8(a) shows that, whereas under F_n^{MTC} profit is zero under the MTC, there are large intervals of γ over which Independence yields positive profits despite the fixed cost impact. Therefore, in this parameter set a larger network is feasible under Independence than the MTC. In Figure 8(a), the lower zero-profit cut-off point - the lowest γ for which n operators are viable under Independence - becomes lower as n increases thereby extending the range of γ where (i) $\pi^I(n, F_n^{MTC}) > 0$ but where, (ii) n operators are not viable under the MTC. As we will see, in some circumstances in this parameter set, provision entirely fails under the MTC: there are no services provided under the MTC and the relative gains under Independence become infinite.

Figure 8: MTC v Independence Profits and Welfare on an Extended Independence Network with Fixed Cost F_n

(a) $\frac{\pi^I(n, F_n^{MTC})}{\alpha^2}$ (b) $\frac{CS^I(n, F_n^{MTC})}{CS^{MTC}(n-1, F_n^{MTC})}$ (c) $\frac{W^I(n, F_n^{MTC})}{W^{MTC}(n-1, F_n^{MTC})}$

—— $n = 2$, - - - $n = 3$, -·-·- $n = 4$, ······ $n = 5$, Black $\tau = 1$, Grey $\tau = 0.9$

Again, a priori, it is not obvious whether or not the higher profitability under Independence, allowing a larger network provision, will generate higher consumer surplus and welfare over the smaller network with associated smaller fixed cost burden under the MTC. Figures 8(b) and (c) report consumer surplus and welfare ratios for Independence relative to the MTC under the fixed cost F_n^{MTC} for $n \in \{2, 3, 4, 5\}$ and $\tau \in \{0.9, 1\}$. The lines are discontinuous at the point where γ is sufficiently low that, under Independence with n operators bearing F_n^{MTC}, profit becomes zero (although the case of $n = 2$ is prematurely constrained vertically for presentational convenience). Table 1 reports the values of γ for these zero profit thresholds under Independence for $\tau = 1$ and

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[31]We do not report F_n^{MTC} as it is unwieldy as an expression.

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$\tau = 0.9$.[32] The Table also reports the values of γ below which the MTC is unprofitable for $n-1$, or any lower level of n, and hence the market under the MTC entirely fails. Note, only in the case of $n=2$ and $\tau=1$ (shaded) does the Independence regime with n operators become unprofitable whilst the MTC regime with $n-1$ operators remains profitable. In all other cases reported, the Independence regime is still profitable, offering positive consumer surplus and welfare, where the MTC over a range of γ, entirely fails to provide any services.

Table 1: Independence versus MTC Zero Profit Critical Values of γ

n	$\tau = 1$		$\tau = 0.9$	
	$\pi^I(n, F_n^{MTC}) = 0$	$\pi^{MTC}(n-1, F_n^{MTC}) = 0$	$\pi^I(n, F_n^{MTC}) = 0$	$\pi^{MTC}(n-1, F_n^{MTC}) = 0$
2	0.314	0.300	0.239	0.300
3	0.116	0.123	0.084	0.127
4	0.057	0.065	0.041	0.071
5	0.033	0.034	0.024	0.046

It is clear from Figure 8 that consumer surplus and welfare under Independence with n operators is greater than under the MTC with $n-1$ operators over a wide range of γ. However, it is plain that Independence, having higher profitability here, does not guarantee the larger network it can sustain always feeds into welfare and consumer surplus gains over the smaller network under the MTC with the lower fixed cost burden. The welfare and consumer surplus ratios do dip below 1, favouring the MTC with $n-1$ operators, within the relevant range.

It follows that the potentially punitive pricing regime under the MTC can damage profit to the extent that it cannot support as large a network as Independence for γ above some critical lower threshold and, supplying the larger network under Independence, despite the additional fixed cost, provides utility gains. This result is a product of underlying tensions between the welfare gains of a larger network (albeit at extra fixed cost) against the welfare gains associated with the downward price pressure under the MTC but with a smaller network (and the lower associated total fixed cost), with the latter being dominant in the Figure for higher n and at lower levels of γ. Notice that Figure 8(b), which reports the ratio of consumer surplus for Independence relative to MTC, demonstrates that the welfare gains through the provision of the additional operator come through both profit and consumer gains, with consumer surplus strictly greater under Independence for $n \in \{2, 3\}$.

In the earlier analysis, it was apparent that the MTC can dominate Independence in profit terms, albeit on a limited subset of parameters and, in particular, low substitutability, in the region the calibration suggests might represent the UK bus market. We now briefly explore the potential for the MTC to offer a more extensive network than Independence. As in Section 4, the relevant fixed cost is now F_n, whereupon n operators cannot survive in the market under Independence. It is straightforward to show that there are parameterisations which result in (i) the MTC with n operators and fixed cost, F_n, not only being profitable and making positive consumer surplus and welfare contributions, but (ii) where the market entirely fails under Independence. For instance, Figure 9(b), in Appendix C, reports profit levels (divided by α^2) for the n-operator MTC with $\tau = 1$ ($\tau = 0.9$) in black (grey). It also reports, as vertical lines, the critical levels of γ below which profit under Independence with $n-1$ (or any smaller number of) operators is zero. In the case of $n=2$ and $\tau=1$, the associated $n=2$ Independence zero-profit cut-off (marked by the solid vertical line) sits marginally to the left of the horizontal intercept of the MTC profit line (solid and black). The interval of γ between these, the vertical line and MTC profit line

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[32]Note, although the Independence regime is independent of τ, the fixed cost F_n^{MTC} does, of course, depend on τ, and so these thresholds depend on τ.

intercept, represent parameterisations where both the MTC with n operators and Independence with $n-1$ operators would be functioning profitably. At all lower levels of γ, only the MTC would operate, and the market under Independence would entirely fail. Notice, that for all other (n, τ) combinations illustrated in the Figure, there is no such overlap: where the MTC with n-operators makes profit and contributes to welfare and consumer surplus, the market under Independence would entirely fail.

Hence, in the presence of sufficiently low or zero fixed costs, for a given number of operators in the network the MTC regime provides consumer surplus and welfare gains over Independence. However, unless substitutability is at the low or very low end, the MTC comes with a hit to profit which, in the presence of sufficiently large fixed costs might result in a smaller number of operators supplying the market with an associated reduction in consumer surplus and welfare relative to Independence, making the equilibrium inferior in these terms. On the other hand, if γ is sufficiently low (in the region indicated by the calibration) then the MTC protects profits relative to Independence, such that for a given n across both regimes it dominates on profit, consumer surplus and welfare terms. Indeed, in the presence of fixed costs, it results in the provision of a welfare-generating MTC where the market under Independence may entirely fail.

6 Conclusions

Our central methodological contribution was capturing dual rival and interchangeable network aspects, common in transport settings, in a tractable and adaptable differentiated n-operator modelling framework. The framework is based on the 2-operator model due to Economides and Salop (1992), which has a well-established presence in the transport literature. In common with many transport applications of this framework, we employ a different pricing structure, reflecting conventions in transport ticketing in line with Economides (1993).

We solved the model under the benchmark regimes of (i) Monopoly, and, (ii) a stylised representation of existing 'free-market' conditions: Independence. We showed that the result that Independence can serve consumers less well than Monopoly under $n = 2$ (e.g., see Economides and Salop, 1992, but under different pricing assumptions) generalises to some extent in the n-operator setting. However, importantly, we showed that the ratios of profit, consumer surplus and welfare across the two regimes are non-monotonic in n. Similarly, we showed that the ranking of the regimes in terms of consumer surplus and welfare varied with n for a given level of substitutability. Both of these results serve to justify an n-operator approach, where the real market of interest is likely to extend beyond $n = 2$, since, a priori, we cannot say whether increasing n will aid the performance of the market under Independence relative to Monopoly, or indeed, change the ranking order of the two regimes in consumer surplus or welfare terms. $n = 2$ was also demonstrated to be a special case in a number of respects. We, also consider the possibility that the equilibrium network size may vary under different pricing regimes with fixed costs. This has particular importance in a network setting where service under-provision is a common policy concern. Introducing a fixed cost per operator we showed that, by sustaining a larger network with associated added utility potential, the parameter set over which Monopoly can dominate Independence in profit, consumer surplus and welfare terms becomes quite extensive. Indeed, this can be shown to hold everywhere under the $n = 2$ setting, which we therefore establish as a special case.

We also provided analysis of a stylised representation of the multi-operator ticket card (MTC) pricing system available to operators in the UK deregulated local bus market. This is a timely analysis, as (i) the Block Exemption under which this system is permitted is due to expire in 2026 and is under statutory review, but also (ii) MTC is at the heart of the UK government's

strategy to address failings in the deregulated local bus market. The analysis, in which the MTC pricing system is compared with the 'free-market' Independence regime, aims to shed light on benefits and shortcomings of the MTC. We show that, in the absence of fixed costs, the MTC everywhere outperforms the Independence regime in consumer surplus and welfare terms, and for low substitutability, dominates in profit terms, too. However, the relative performance of the MTC across these market performance indicators is non-monotonic in n. Indeed, operators may fear joining an MTC which offers higher profit than Independence, if there is potential for future entry of a new operator into the market and the MTC, as increasing n can reverse the profit ranking of the regimes for a given level of substitutability.

In the presence of fixed costs, we showed that if substitutability is sufficiently low, the MTC can sustain a larger network than is feasible under Independence. However, conversely, if substitutability is not sufficiently low then Independence can sustain a larger network and dominate the MTC regime in terms of profit, consumer surplus and welfare.

We performed a basic calibration exercise based on demand elasticity estimates consistent with the UK local bus market and found relatively low levels of substitution are required to bring the calibrated model to the predictions of the 'free-market' Independence regime. This lends support to the idea that the UK local bus market might be operating under Independence in the region where the MTC provides higher profit, consumer surplus and welfare along with a more extensive network than Independence. Indeed, at low levels of substitutability Independence even performs less well than Monopoly. If this is an accurate representation of the UK local bus market then it suggests the current 'free-market' pricing system may be destined to under perform, and that continuation of the MTC scheme might indeed help resolve this issue. However, the MTC appears neither to have been widely implemented along the recommended guidelines nor to be in high usage amongst travellers. We have suggested one reason why operators may not wish to join an MTC under the recommended price structure and pointed at the voluntary aspect of taking part in an MTC as something that might be reviewed. However, this work represents a first shot at this issue as a short application of the n-operator model and many further avenues of enquiry remain, including other benefits of the MTC, such as environmental and other externality gains associated with better public transport provision, and a more extensive empirical exercise to produce a better calibration of the model.

Given the wide range of transport network studies employing, or nested within, the Economides and Salop (1992) model (and often, explicitly employing the pricing structure in Economides, 1993), with $n = 2$, this paper offers a rich platform to extend these models under 'conventional' transport ticketing practices to see the extent to which their results generalise as well as address new questions available through the modelling capacity to vary n.

Appendix

A   Second Order Conditions

For Monopoly and Independence, the Hessian matrix for the firm profit function (where each element is a second derivative with respect to one of it's prices) is symmetric. Given the matrix is symmetric, for a maximum we require that the diagonal is negative and dominant (see, for instance Theorem M.D.5, Mas-Colell et al., 1995, p. 939). A dominant diagonal requires that, for a square matrix with elements a_{ij}, $|a_{ii}| > \sum_{j \neq i} |a_{ij}|\ \forall i$. In the case of Monopoly, who sets n^2 prices, the Hessian is an $N \times N$ matrix. Under Independence, each firm sets $2n-1$ prices (its SO price and $2(n-1)$ prices for each of its $n-1$ x-component MO services and $n-1$ y-component MO services). The Hessian is therefore a $(2n-1) \times (2n-1)$ matrix. In both cases, the principal diagonal comprises $-2b$ with $2d$ elsewhere. The diagonal is negative since $-b < 0$ and the dominant diagonal requires, $b > (n^2 - 1)d$ under Monopoly and $b > 2(n-1)d$ under Independence, which are guaranteed under Assumption 1. In the case of the stylised MTC, each firm only sets one price: it's SO price. Hence, the second order condition is satisfied since $-b < 0$. Note, the restrictions on the relationship between b and d in Eq. (5) satisfy gross substitutes strictly for $\gamma < 1$.

B   Calibration

A rough calibration of the model to fit the Independence regime, which we treat as a stylised 'free-market' case, is conducted using market elasticities of $\eta \in \{-0.4, -1.2\}$ that are at the upper and lower ends of estimates for transport networks. For instance, Goodwin (1992), who takes averages across various studies, reports long-run price elasticities for bus and rail of -0.6 and -1.1 to 1 decimal place. Small and Winston (1999) produce U.S. urban (intercity) price elasticities of demand to one decimal place: car -0.5, rail -0.6 (-0.7), bus -0.9 (-1.2) and air -0.4. Paulley et al. (2006) cite mean UK long-run bus elasticity at -1.01, which is in line with more recent estimates (e.g. Dunkerley et al., 2018). We solve for the (n, γ) combinations which produce the equilibrium total quantity and weighted average price under independence with the lower and the upper elasticity. To derive an expression approximating the elasticity across the market under Independence we differentiate total quantity with respect to price, allowing all prices to change, giving $n^2(-b + d(n-1)(1+n))$. The approximate elasticity calculation is then found by multiplying this derivative by weighted average price ($[n^2 P^I + n(n-1)P_x^I]/n^2$) and dividing by total quantity ($n[a - bP^I + d(n-1)(P^I + nP_x^I)] + n(n-1)[a - bP_x^I + d(nP^I + (n(n-1) - 1)P_X^I)]$, as produced algebraically by the model at the equilibrium. Setting this alternatively equal to -0.4 and -1.2 produces the solid line relationships between γ and n as given in Figure 2. We follow the same approach with non-zero marginal cost, specifically $c = \alpha/10$, with associated dashed line relationships in the Figure.

C   Proof to Propositions

C.A   Proof to Proposition 1

(i) Let $H \equiv \frac{\pi^I}{\pi^M} = \frac{4(8\gamma n^3 - 15\gamma n^2 - 2\gamma + 8n + 1)(1 - \gamma)}{(2\gamma n^2 - 3\gamma n - 5\gamma + 6)^2 n)}$, which, as a ratio of two polynomials, is a continuous function in the relevant range. Since there are no solutions in the relevant range for $H = 1$ and there are feasible (n, γ) combinations yielding $H < 1$ (e.g., see Figure 3(a)), H lies everywhere below unity in the relevant range. (ii) Let $H \equiv \frac{S^I}{S^M} = \frac{4[(4n^5 - 12n^4 + 5n^3 + n^2 + 8n + 3)\gamma^2 + (8n^3 - 7n^2 - 11n - 8)\gamma + 4n + 5]}{n(2\gamma n^2 - 3\gamma n - 5g + 6)^2}$, which is a continuous function. There is only one valid root for the threshold value $H = 1$, given by $\gamma_1 = \frac{2(-2n^2 + (64n^4 - 104n^3 + 83n^2 - 31n + 4)^{0.5} - 4n - 8)}{12n^4 - 24n^3 + 7n^2 - 19n - 12}$. Given the element in the square root is strictly positive in the relevant range and the expression is otherwise a quotient of two polynomials, it is also a continuous function. A simple plot reveals it is strictly decreasing in n in the relevant range. Given the existence of a feasible (n, γ) combination below γ_1 for a given n for which $H < 1$ and vice versa (e.g., see Figure 3(b)), completes the proof. Figure 2 illustrates the threshold contour for $H = 1$ (grey dotted line). (iii) Let $H \equiv \frac{W^I}{W^M} = \frac{(16n^5 - 48n^4 - 44n^3 + 124n^2 + 32n + 28)\gamma^2 + (96n^3 - 148n^2 - 108n - 56)\gamma + 80n + 28}{3(2\gamma n^2 - 3\gamma n - 5\gamma + 6)^2 n}$, which, as a ratio of polynomials, is continuous in the relevant range. The threshold solution γ_2 is unique

and strictly decreasing in n in the relevant range. Given the existence of a feasible (n, γ) combination below γ_2 for a given n for which $H < 1$ and vice versa (e.g., see Figure 3(c)), completes the proof.

C.B Proof to Proposition 3

(i) Let $H \equiv \frac{P^M}{P^I} = \frac{6+(2n^2-3n-5)\gamma}{6(1-\gamma)}$ and $H_x \equiv \frac{P_x^I}{P^I} = \frac{4}{3}$. First, trivially $H_x > 1$. Second, the maximal parameter solutions for $H = 1$ are $\gamma = 0$ and $n = 1$, which lie outside the relevant range. However, $H_n = \frac{(4n-3)\gamma}{6(1-\gamma)}$ and $H_\gamma = \frac{2n^2-3n+1}{(1-\gamma)^2}$ are both strictly positive in the relevant range, ensuring $H > 1$ in the relevant range, completing the proof.

(ii) Let $H \equiv \frac{P_x^I}{P^M} = \frac{8(1-\gamma)}{2n^2-3n+3}$. Note, $H_\gamma = -\frac{8}{2n^2-3n+3}$ and $H_n = \frac{-2(4n-3)}{(2n^2-3n+3)^2}$ which are both strictly positive in the relevant range. $H = 1$ yields $\gamma_3 = \frac{2}{2n^2-3n+3}$ which is a continuous function of n where $\gamma_{3n} = -\frac{2(4n-3)}{2n^2-3n+3} < 0$ in the relevant range. Given the strict negative monotonicity of H in its arguments, whilst (n, γ) satisfying γ_3 yield $H = 1$, increasing (decreasing) n and/or γ in the feasible range will result in H falling below 1 (rising above 1), completing the proof. (iii) $P_n^M = 0$ follows directly from inspection of Eq. (11). Partially differentiating Eqs. (14), $P_n^I = -\frac{12\gamma(\alpha-c)\left(n-\frac{3}{4}\right)(1-\gamma)}{\nabla^2} \leq 0$, $P_{xn}^I = -\frac{16\gamma(\alpha-c)\left(n-\frac{3}{4}\right)(1-\gamma)}{\nabla^2} \leq 0$, where $\nabla \equiv 2\gamma n^2 - 3n\gamma - 5\gamma + 6 > 0$ in the relevant range.

C.C Proof to Proposition 4

(i) Let $H \equiv \frac{Q^I}{Q^M} = \frac{6+(6n^2-8n-4)\gamma}{6+(2n^2-3n-5)\gamma}$, which has a single solution $H = 1$ at $\gamma = 0$, and is everywhere else, in the relevant range, strictly greater than one, completing the proof. (ii) Let $H_x \equiv \frac{Q_x^I}{Q^M} = \frac{4+(4n^2-8n-2)\gamma}{6+(2n^2-3n-5)\gamma}$. For $\gamma = 0$ and for $n = 2$ $H_x = \frac{2}{3}$, completing the proof. (iii) From (ii) solving $H_x = 1$ for γ yields, $\gamma_4 = \frac{2}{2n^2-5n+3}$, which is strictly decreasing in n in the relevant range. Noting H_x is continuous on n and g and $H_x(\gamma = 0) < 1$, completes the proof. (iv) Let $H \equiv \frac{nQ^I+n(n-1)Q_x^I}{n^2Q^M} = \frac{4n^3-6n^2-2n-2)\gamma+4n+2}{(2n^3-3n^2-5n)\gamma+6n}$, which is continuous in n and γ. Solving $H = 1$ for γ we have $\gamma_5 = \frac{2}{2n^2-n+2}$, which is strictly decreasing in n in the relevant range. Noting, at $\gamma = 0$ we have $H = \frac{1+2n}{3n}$, which is strictly less than one in the relevant range, completes the proof.

C.D Proof to Proposition 6

(i) Solving $\pi^I = \pi^{MTC}$ for n, the plot in Figure 9(a) shows this contour in (γ, τ, n)-space with clear negative (positive) relationship between n and γ (τ), for in the relevant range. (ii) Solving $CS^I = CS^{MTC}$, for n, a 3D plot over $\tau \in [0, 1]$ and $\gamma \in [0, 1]$ reveals there are no solutions for $n > 1$. (iii) The same applies as in (ii) for $W^I = W^{MTC}$ over $\tau \in [0, 1]$ and $\gamma \in [0, 1)$.

C.E Proof to Proposition 9

(i) Let $H \equiv \frac{P_x^I}{P_x^{MTC}}$. Note, H is continuous in the relevant range. Solving $H = 1$, for n and performing a 3D plot to see there are no solutions for $n \geq 2$. Continuity alongside solutions for $H > 1$ but no solutions for $H = 1$ in the relevant range completes the proof. (ii) Let $H \equiv \frac{P^I}{P^{MTC}}$. Note, H is continuous in the relevant range. Solving $H = 1$ for n yields the contour plot in Figure 7(a) with contours at $n = 2, 4, 5, 10$ and $H < 1$ ($H > 1$) for values to the left (right) of the contours, completing the proof. In the case of $\tau = 1$, let $H(1) \equiv \frac{P^I}{P^{MTC(1)}} = \frac{(3n^3-3n^2-12n+6)\gamma+12n-6}{(4n^3-8n^2-7n+5)\gamma+12n-6}$, which is continuous in the relevant range. Solving for $H(1) = 1$ gives $\bar{n} = 2 + 3^{0.5}$ with the integer set in the relevant range, $n \in \{2, 3\}$, below $\bar{n}$ and the set $n \geq 4$ above $\bar{n}$, Figure 7(c) illustrates the former set, where, $H(1) > 1$, and a subset of the latter where, $H(1) < 1$, completing the proof. (iii) Let $H \equiv \frac{P^I}{P^{MTC}}$. Note, H is continuous in the relevant range. Solving $H = 1$, for n yields the contour plot in Figure 7(b) with contours at $n = 4, 5, 10$ and $H < 1$ ($H > 1$) for values to the right (left) of the contours. Note, there are no solutions for $H = 1$ in the relevant range for $n \in \{2, 3\}$, hence continuity and the existence of (γ, τ) combinations for $n \in \{2, 3\}$ where $H > 1$ (e.g., see Figure 8(c)), completes the proof.

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Figure 9: Independence Versus MTC

(a) $\pi^I = \pi^{MTC}$

(b) $\frac{\pi^{MTC}(F_n)}{2}$ and $\pi^I(F_n) = 0$ thresholds
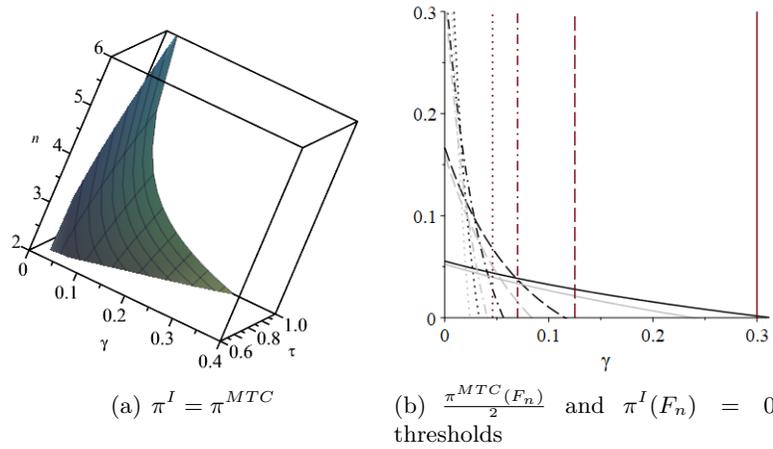
(b) —— $n = 2$, - - - $n = 3$, -·-·- $n = 4$, ······ $n = 5$, Black $\tau = 1$, Grey $\tau = 0.9$, vertical thresholds where $\pi^I(F_n, n-1) = 0$

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