Stability and wall-crossing 3

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1. Stability in triangulated categories

Stability in triangulated categories

Definition

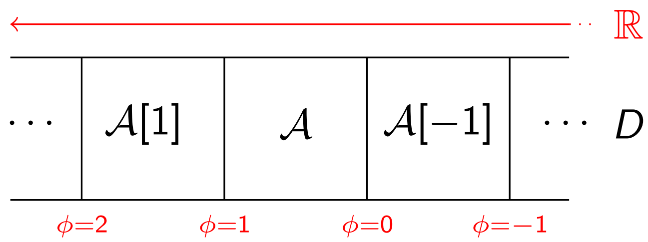
A stability condition on a tri. cat. D is a pair $(Z, \mathcal{A})$ where

(i) $\mathcal{A} \subset D$ is a heart,

(ii) $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism,

such that Z defines a stability condition on $\mathcal{A}$ with the HN property.

An object $E \in D$ is defined to be semistable if $E = A[n]$ for some Z-semistable $A \in \mathcal{A}$. The phase of E is then $\phi(E) := \phi(A) + n$.



Space of stability conditions

We consider only stability conditions satisfying the extra conditions

(A) The central charge $Z\colon K_0(D) \to \mathbb{C}$ factors via our fixed map
$$\mathrm{ch}\colon K_0(D) \longrightarrow N \cong \mathbb{Z}^{\oplus n}.$$

(B) There is a $K > 0$ such that for any semistable object $E \in D$
$$Z(E) \geqslant K \cdot \|\mathrm{ch}(E)\|.$$

The set $\mathrm{Stab}(D)$ of such stability conditions has a natural topology.

Theorem

Sending a stability condition to its central charge defines a local homeomorphism
$$\mathrm{Stab}(D) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}) \cong \mathbb{C}^n.$$
In particular, $\mathrm{Stab}(D)$ *is a complex manifold.*

Charge lattice and associated torus

(A) Consider a smooth projective Calabi-Yau threefold X and set

$$D = D^b \operatorname{Coh}(X).$$

(B) Define the charge lattice

$$N = \operatorname{im}\left(\mathrm{ch}\colon K_0(D) \to H^*(X, \mathbb{Q})\right) \cong \mathbb{Z}^{\oplus n}.$$

The Euler form $\langle -, - \rangle$ gives a skew-symmetric form on N.

(C) Introduce the algebraic torus

$$\mathbb{T} = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n,$$

with its Poisson structure

$$\{x^\alpha, x^\beta\} = \langle \alpha, \beta \rangle \cdot x^{\alpha+\beta}.$$

What we expect

(A) The space of stability conditions $\mathrm{Stab}(D)$ is non-empty.

(B) For each stability condition $\sigma \in \mathrm{Stab}(D)$ there are stacks

$$\mathcal{M}^{ss}(\alpha) = \{E \in D : E \text{ is } \sigma\text{-semistable with } \mathrm{ch}(E) = \alpha\}$$

of finite type, and corresponding DT invariants $\mathrm{DT}_\sigma(\alpha) \in \mathbb{Q}$.

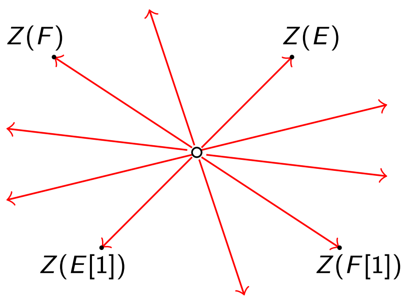
(C) As we vary $\sigma \in \mathrm{Stab}(D)$ the invariants $\mathrm{DT}_\sigma(\alpha) \in \mathbb{Q}$ undergo discontinuous changes governed by the Kontsevich-Soibelman wall-crossing formula.

The active rays

For each stability condition $\sigma \in \mathrm{Stab}(D)$ there is a countable collection of active rays

$$\ell = \mathbb{R}_{>0} \exp(i\pi\phi) \subset \mathbb{C}$$

for which there exist semistable objects of phase ϕ.



As σ varies, the active rays move and may collide and separate.

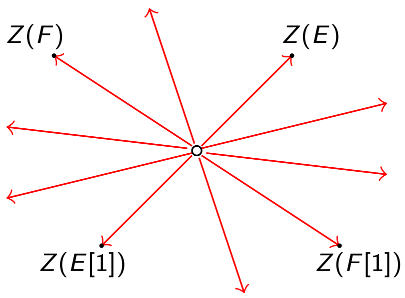
Encoding DT invariants

To each active ray is associated a formal function on $\mathbb{T}$

$$\mathsf{DT}_\ell = \sum_{Z(\alpha)\in\ell} \mathsf{DT}_\sigma(\alpha) x^\alpha.$$

Ignoring convergence issues, there is a corresponding automorphism

$$S_\ell = \exp(\{\mathsf{DT}_\ell, -\}) \in \mathrm{Aut}(\mathbb{T}).$$

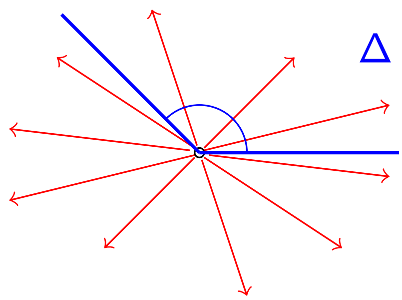


Wall-crossing formula

For any convex sector $\Delta \subset \mathbb{C}$, the clockwise product over active rays

$$\mathcal{S}_\Delta = \prod_{\ell \in \Delta} S_\ell \in \mathrm{Aut}(\mathbb{T})$$

remains constant as σ varies, providing no active ray crosses $\partial\Delta$.



This all makes good sense in a suitable completion $\mathbb{C}[[N_+]]$.

2. Irregular connections and Stokes data

Stokes matrices and isomonodromy

The wall-crossing formula resembles an isomonodromy condition for an irregular connection with values in the infinite-dimensional group

$$G = \mathsf{Aut}_{\{-,-\}}(\mathbb{T})$$

of Poisson automorphisms of the torus $\mathbb{T} \cong (\mathbb{C}^*)^n$.

We first explain such phenomena in the finite-dimensional case, so set

$$G = \mathrm{GL}(n, \mathbb{C}), \quad \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}).$$

As a warm-up we start with the case of regular singularities.

A Fuchsian connection

We will consider meromorphic connections on the trivial G-bundle over the Riemann sphere $\mathbb{CP}^1$.

Consider a connection of the form

$$\nabla = d - \sum_{i=1}^{k} \frac{A_i \, dz}{z - a_i}$$

(I) $a_i \in \mathbb{C}$ are a set of k distinct points,
(II) $A_i \in \mathfrak{g}$ are corresponding residue matrices.

Then ∇ has regular singularities at the points a_i, and also at ∞.

Isomonodromic deformations

For each based loop

$$\gamma\colon S^1 \to \mathbb{C} \setminus \{a_1, \cdots, a_k\}$$

there is a corresponding monodromy matrix $\mathrm{Mon}_\gamma(\nabla) \in G$.

If we move the pole positions $a_i \in \mathbb{C}$, we can deform the residue matrices A_i so that all monodromy matrices remain constant. Such deformations are called isomonodromic.

Isomonodromic deformations are described by a system of partial differential equations called the Schlessinger equations.

A class of irregular connections

Introduce the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}, \quad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \Phi = \{e_i^* - e_j^*\} \subset \mathfrak{h}^*.$$

Consider a connection of the form

$$\nabla = d - \left(\frac{U}{z^2} + \frac{V}{z} \right) dz,$$

(I) $U = \mathrm{diag}(u_1, \cdots, u_n) \in \mathfrak{h}$ is diagonal with distinct eigenvalues,

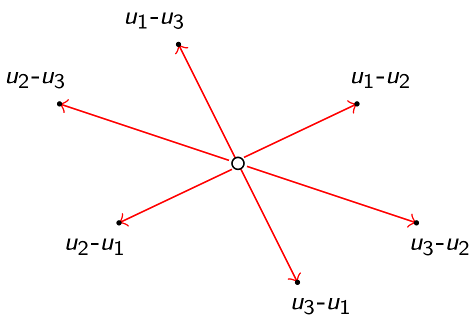
(II) $V \in \mathfrak{g}^{\mathrm{od}}$ has zeroes on the diagonal.

Then ∇ has an irregular singularity at 0 and a regular one at ∞.

Stokes data of the connection

The Stokes rays for the connection ∇ are the rays

$$\mathbb{R}_{>0} \cdot (u_i - u_j) = \mathbb{R}_{>0} \cdot U(\alpha), \quad \alpha = e_i^* - e_j^*.$$



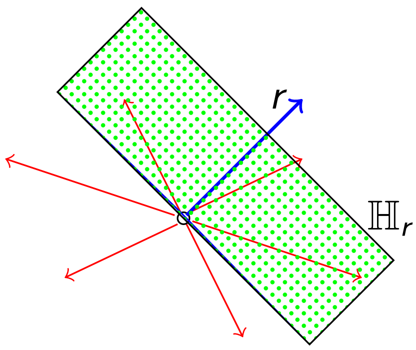
Associated to each Stokes ray ℓ is a Stokes factor

$$\mathcal{S}_\ell = \exp\Big(\sum_{U(\alpha)\in\ell} \epsilon_\alpha \Big) \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha \Big) \subset G.$$

Canonical solution on a half-plane

Given a non-Stokes ray r, there is a canonical flat section X_r of ∇ on the orthogonal half-plane $\mathbb{H}_r$, uniquely defined by the condition that

$$X_r(t) \cdot e^{U/t} \to 1 \text{ as } t \to 0 \text{ in } \mathbb{H}_r.$$



As the ray r varies, the flat section X_r remains unchanged until r crosses a Stokes ray, where it jumps by

$$X_r \mapsto X_r \cdot S_\ell.$$

Isomonodromy in the irregular case

If we now vary the diagonal matrix U, we can deform the matrix V so that the Stokes factors remain constant. Such deformations are called isomonodromic. More precisely:

For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise product

$$S_\Delta = \prod_{\ell \in \Sigma} S_\ell \in G,$$

remains constant unless a Stokes ray crosses the boundary of Σ.

Isomonodromic variations are again described by a system of partial differential equations.

Poisson vector fields on $\mathbb{T}$

Consider the group G of Poisson automorphisms of the torus

$$\mathbb{T} \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n,$$

and the corresponding Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}$, where

(A) the Cartan subalgebra

$$\mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}),$$

consists of translation-invariant vector fields on $\mathbb{T}$.

(B) the subspace $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields, and is the Poisson algebra of non-constant algebraic functions on $\mathbb{T}$

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha \in N^{\times}} \mathfrak{g}_{\alpha} = \bigoplus_{\alpha \in N^{\times}} \mathbb{C} \cdot x^{\alpha}.$$

DT invariants as Stokes data

It is tempting to interpret the elements

$$S_\ell = \exp\left(\sum_{Z(\alpha)\in\ell} \mathrm{DT}_\sigma(\alpha)\cdot x^\alpha\right) \in G$$

as defining Stokes factors for a G-valued connection of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{F}{t}\right) dt,$$

for some element $F \in \mathfrak{g}^{\mathrm{od}}$.

The wall-crossing formula is precisely the condition that this family of connections is isomonodromic as $\sigma \in \mathrm{Stab}(D)$ varies.

3. Quivers with potential

Quivers with potential

Let (Q, W) be a quiver with potential. Thus

(I) Q is an oriented graph,

(II) W is a formal sum of oriented cycles in Q.

We always assume that Q has no loops or oriented 2-cycles.

Associated to (Q, W) is a triangulated category $D^b(Q, W)$

By definition, $D^b(Q, W)$ is the subcategory of the derived category of the complete Ginzburg dg-algebra $\Pi(Q, W)$ consisting of objects with finite-dimensional total cohomology.

General properties of $D^b(Q, W)$

Let (Q, W) be a QWP as before, and set $D = D^b(Q, W)$.

(A) D has the CY_3 property:

$$\mathrm{Hom}^k(E, F) \cong \mathrm{Hom}^{3-k}(F, E)^*.$$

(B) D is generated by objects S_i indexed by the vertices of Q, and

$$\mathrm{Hom}^*(S_i, S_j) = \mathbb{C}^{\delta_{ij}} \oplus \mathbb{C}^{a_{ij}}[-1] \oplus \mathbb{C}^{a_{ji}}[-2] \oplus \mathbb{C}^{\delta_{ij}}[-3],$$

with a_{ij} the number of arrows in Q from vertex i to vertex j.

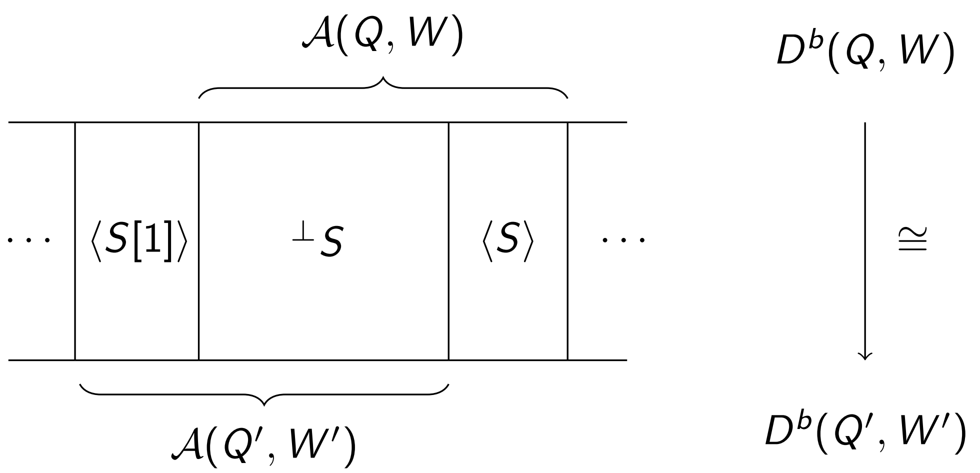
(C) There is a standard heart $\mathcal{A} \subset D$, which is finite-length, and whose simple objects are precisely the S_i.

Tilting and mutation

Let (Q, W) be a QWP and choose a vertex i of Q. Write $S = S_i$.

$$^{\perp}S = \{E \in \mathcal{A} : \mathrm{Hom}(E, S) = 0\}, \quad \langle S \rangle = \{S^{\oplus n} : n \geqslant 0\}.$$

Keller and Yang proved that there is an equivalence



where (Q', W') is a new QWP obtained by a process called mutation.

Exchange graphs

Let (Q, W) be a quiver with a generic potential.

(A) The heart exchange graph $\mathrm{EG}_{\heartsuit}(Q, W)$ has

- vertices the finite-length hearts in $D^b(Q, W)$,
- edges connecting hearts related by a simple tilt.

(B) The cluster exchange graph is the quotient

$$\mathrm{EG}(Q) = \mathrm{EG}_{\heartsuit}(Q, W)/\,\mathrm{Sph}(D)$$

where $\mathrm{Sph}(D) = \langle \mathrm{Tw}_{S_1}, \cdots, \mathrm{Tw}_{S_n} \rangle \subset \mathrm{Aut}(D)$.

Stability space versus cluster variety

(A) For each heart $\mathcal{A} \in \mathrm{EG}_{\heartsuit}(Q, W)$ there is a cell $\mathbb{H}^n \subset \mathrm{Stab}(D)$.

$$\mathrm{Stab}(D)/\,\mathrm{Sph}(D) \supset \bigcup_{\mathcal{A} \in \mathrm{EG}(Q)} \mathbb{H}^n.$$

Note that the different cells only meet in their closures.

(B) The cluster variety is a union of tori glued by birational maps

$$\mathcal{X}(Q) = \bigcup_{\mathcal{A} \in \mathrm{EG}(Q)} (\mathbb{C}^*)^n.$$

$$x^\beta \mapsto x^\beta \cdot (1 + x^\alpha)^{\langle \alpha, \beta \rangle}.$$

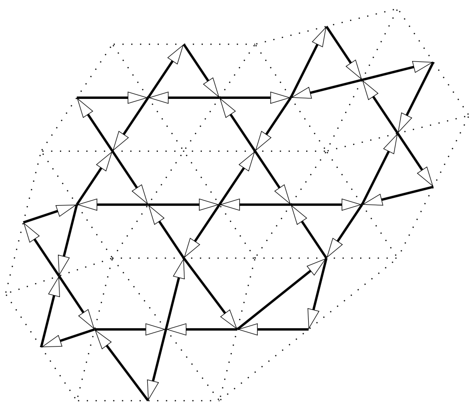
4. Examples from triangulated surfaces

From triangulations to quivers

Fix a surface S of genus g with a set $M = \{p_1, \cdots, p_d\} \subset S$.

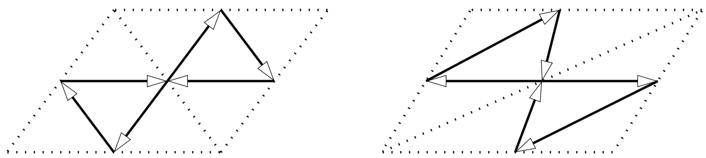
Consider triangulations of S with vertices at the points p_i.

Associated to any such triangulation is a quiver:



Flips and the exchange graph

A flip of the triangulation induces a mutation of the quiver:



(A) Fomin, Shapiro and Thurston proved that the exchange graph is the set of (tagged) triangulations, with the edges being flips.

(B) Labardini-Fragoso dealt with the potentials associated to degenerate triangulations.

Cluster variety

Let (S, M) be a marked surface as above, choose a triangulation and let Q be the corresponding quiver. Set $G = \mathrm{PGL}(2, \mathbb{C})$.

Theorem (Fock and Goncharov)

The cluster variety $\mathcal{X}(Q)$ is a dense open subset of the stack of labelled G-local systems on $S \setminus M$

$$\mathcal{X}(Q) \subset \mathrm{Loc}_G^*(S \setminus M) \xrightarrow{2^d:1} \mathrm{Loc}_G(S \setminus M).$$

Space of stability conditions

Choose a generic potential W and set $D = D^b(Q, W)$.

Theorem (-, Ivan Smith)

$$\mathrm{Stab}(D)/\,\mathrm{Aut}(D) \cong \mathrm{Quad}(g, d).$$

The space $\mathrm{Quad}(g, d)$ parameterizes pairs (S, ϕ) with

(A) S is a Riemann surface of genus g,

(B) $D = \sum_{i=1}^{d} p_i$ is a reduced divisor,

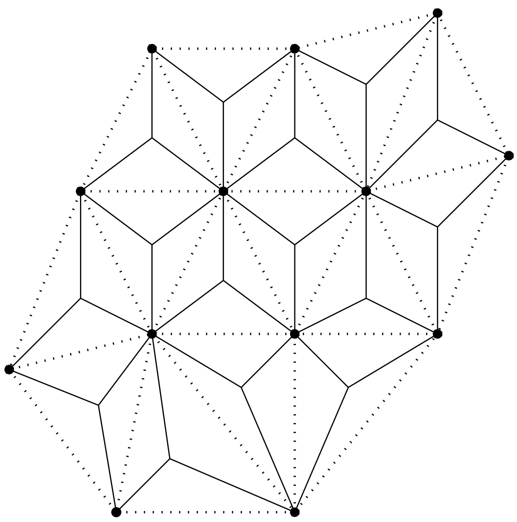
(C) $\phi \in H^0(S, \omega_S(D)^{\otimes 2})$ has simple zeroes.

Horizontal strip decomposition

A quadratic differential defines a foliation

$$\langle \sqrt{\phi(p)}, X \rangle \in \mathbb{R}, \qquad X \in T_p S.$$

For a generic point $\phi \in \mathrm{Quad}(g, d)$ the trajectories split the surface into a disjoint union of horizontal strips.



Relating $\mathrm{Stab}(D)$ to $\mathcal{X}(Q)$

Two stories (like Frobenius versus tt^* in $\mathrm{GL}(n)$ case):

(1) Non-holomorphic version (Gaiotto-Moore-Neitzke):

$$\begin{array}{ccccc} & & \mathcal{M}^0_{Higgs} & \hookrightarrow & \mathcal{M}_{Betti} \cong \mathcal{X}(Q) \\ & & \Big\downarrow {\scriptstyle (S^1)^n} & & \\ \frac{\mathrm{Stab}(D)}{\mathrm{Aut}(D)} \cong \mathrm{Quad}(g,n) & \xleftarrow[\mathbb{C}-str.]{fix} & B_0 & & B_0 \subset H^0(S, K_S(D)^2) \end{array}$$

(2) Holomorphic version ('conformal limit'):

$$\begin{array}{ccccc} \frac{\mathrm{Stab}(D)}{\mathrm{Aut}(D)} \cong \mathrm{Quad}(g,n) & \xrightarrow[non-canon.]{\cong} & \mathrm{Proj}(g,n) & \longrightarrow & \mathcal{M}_{Betti} \cong \mathcal{X}(Q) \\ & \searrow & & \swarrow & \\ & & \mathcal{M}(g,n) & & \end{array}$$