Stability and wall-crossing

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1. Hearts and tilting

Definition of a Torsion pair

Let $\mathcal{A}$ be an abelian category.

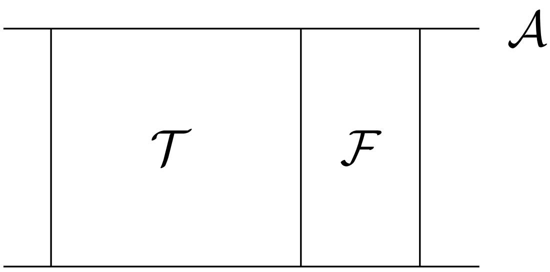
A torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a pair of full subcategories such that:

(A) $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(B) for every object $E \in \mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.



Definition of a heart

Let D be a triangulated category.

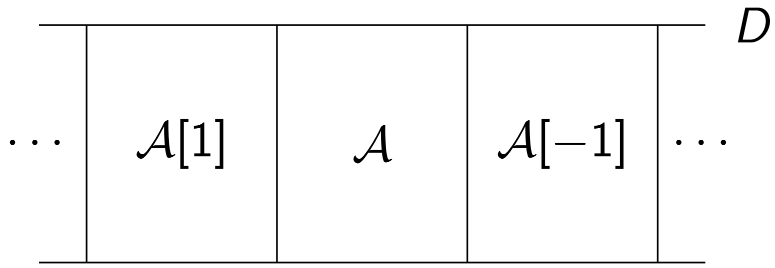
A heart $\mathcal{A} \subset D$ is a full subcategory such that:

(A) $\mathrm{Hom}(A[j], B[k]) = 0$ for all $A, B \in \mathcal{A}$ and $j > k$.

(B) for every object $E \in D$ there is a finite filtration

$$0 = E_m \to E_{m+1} \to \cdots \to E_{n-1} \to E_n = E$$

with factors $F_j = \mathrm{Cone}(E_{j-1} \to E_j) \in \mathcal{A}[-j]$.



Properties of hearts

(i) It would be more standard to say that $\mathcal{A} \subset D$ is the heart of a bounded t-structure on D. But any such t-structure is determined by its heart.

(ii) The basic example is $\mathcal{A} \subset D^b(\mathcal{A})$.

(iii) In analogy with that case we define $H^j_{\mathcal{A}}(E) := F_j[j] \in \mathcal{A}$.

(iv) $\mathcal{A}$ is automatically an abelian category.

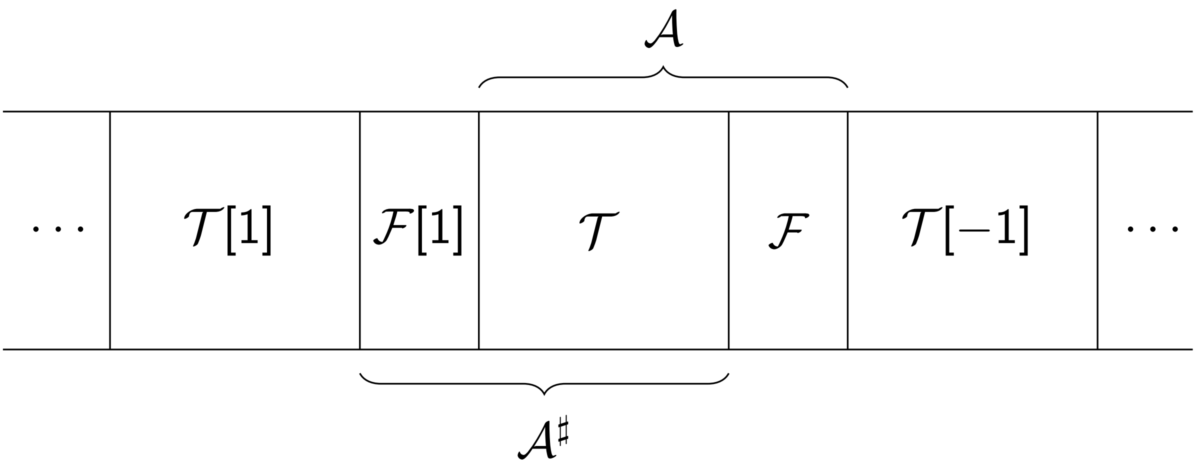
(v) The short exact sequences in $\mathcal{A}$ are precisely the triangles in D all of whose terms lie in $\mathcal{A}$.

(vi) The inclusion functor gives an identification $K_0(\mathcal{A}) \cong K_0(D)$.

The tilt of a heart at a torsion pair

Suppose $\mathcal{A} \subset D$ is a heart, and $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ a torsion pair.

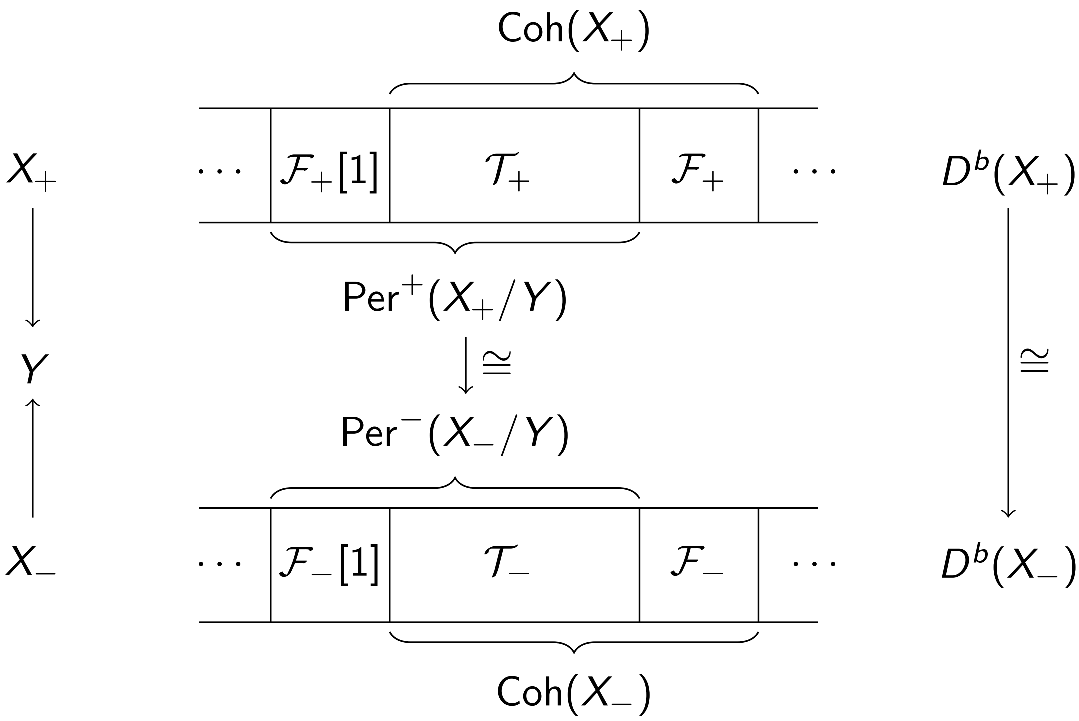
We can define a new, tilted heart $\mathcal{A}^\sharp \subset D$ as in the picture.



An object $E \in D$ lies in $\mathcal{A}^\sharp \subset D$ precisely if

$$H^{-1}_{\mathcal{A}}(E) \in \mathcal{F}, \quad H^0_{\mathcal{A}}(E) \in \mathcal{T}, \quad H^i_{\mathcal{A}}(E) = 0 \text{ otherwise.}$$

Example of tilting: threefold flop



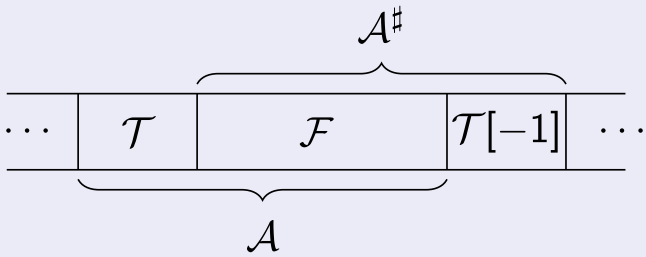
$$\mathcal{F}_+ = \langle \mathcal{O}_{C_+}(-i) \rangle_{i \geqslant 1}, \quad \mathcal{F}_- = \langle \mathcal{O}_{C_-}(-i) \rangle_{i \geqslant 2}.$$

Stable pairs as quotients in a tilt

Consider tilting $\mathcal{A} = \mathrm{Coh}(X) \subset D(X)$ with respect to the torsion pair

$$\mathcal{T} = \{E \in \mathrm{Coh}(X) : \dim_{\mathbb{C}} \mathrm{supp}(E) = 0\},$$

$$\mathcal{F} = \{E \in \mathrm{Coh}(X) : \mathrm{Hom}_X(\mathcal{O}_x, E) = 0 \text{ for all } x \in X\}.$$



Note that $\mathcal{O}_X \in \mathcal{F} \subset \mathcal{A}^\sharp$. We claim that

$$\mathrm{Pairs}(\beta, n) = \left\{ \begin{array}{c} \text{quotients } \mathcal{O}_X \twoheadrightarrow E \text{ in } \mathcal{A}^\sharp \\ \text{with } \mathrm{ch}(E) = (0, 0, \beta, n) \end{array} \right\}.$$

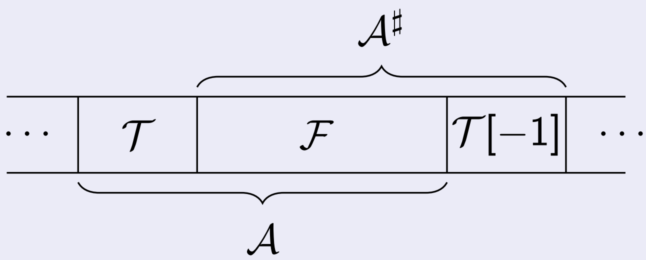
Proof of the claim about stable pairs

Given a short exact sequence in the category $\mathcal{A}^\sharp$

$$0 \longrightarrow J \longrightarrow \mathcal{O}_X \xrightarrow{f} E \longrightarrow 0,$$

we take cohomology with respect to the standard heart $\mathcal{A} \subset D$.

$$0 \to H^0_{\mathcal{A}}(J) \to \mathcal{O}_X \xrightarrow{f} H^0_{\mathcal{A}}(E) \to H^1_{\mathcal{A}}(J) \to 0 \to H^1_{\mathcal{A}}(E) \to 0.$$



It follows that $E \in \mathcal{A} \cap \mathcal{A}^\sharp = \mathcal{F}$ and $\operatorname{coker}(f) = H^1_{\mathcal{A}}(J) \in \mathcal{T}$.

Last time ...

(A) Hall algebras: $\mathrm{Hall}_{\mathrm{fty}}(\mathcal{C})$, $\mathrm{Hall}_{\mathrm{mot}}(\mathcal{C})$.

$$0 \to A \to B \to C \to 0$$

$$(A, C) \longleftarrow \qquad \longrightarrow B$$

$$\mathcal{M} \times \mathcal{M} \xleftarrow{(a,c)} \mathcal{M}^{(2)} \xrightarrow{b} \mathcal{M}$$

(B) Character map $\mathrm{ch}\colon K_0(\mathcal{C}) \to N \cong \mathbb{Z}^{\oplus n}$.

(C) Quantum torus: $\mathbb{C}_q[N] = \bigoplus_{\alpha \in N} \mathbb{C}(t) \cdot x^\alpha$ with

$$x^\alpha * x^\gamma = q^{-\frac{1}{2}(\gamma,\alpha)} \cdot x^{\alpha+\gamma}.$$

(D) Integration map: $\mathcal{I}\colon \mathrm{Hall}(\mathcal{C}) \to \mathbb{C}_q[N]$.

Positive cones and completions

Choosing a basis $(e_1, \cdots, e_n)$ for the group N gives an identification

$$\mathbb{C}[N] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

We often need to use the positive cone

$$N_+ = \Big\{ \sum_{i=1}^{n} \lambda_i e_i : \lambda_i \geqslant 0 \Big\} \subset N,$$

and the associated completion

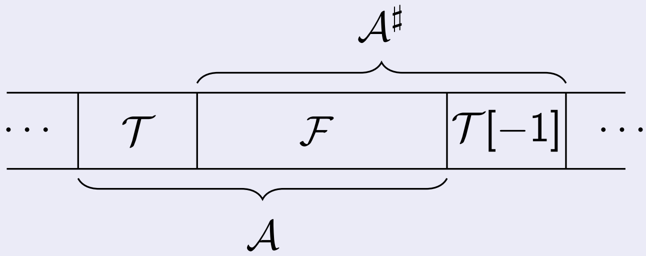
$$\mathbb{C}[[N_+]] \cong \mathbb{C}[[x_1, \cdots, x_n]].$$

We can similarly define the completed quantum torus $\mathbb{C}_q[[N_+]]$.

Sketch proof of the DT/PT identity

(I) Reineke's identity: $\delta^{\mathcal{O}}_{\mathcal{A}} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}} * \delta_{\mathcal{A}}$ and $\delta^{\mathcal{O}}_{\mathcal{A}^\sharp} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}^\sharp} * \delta_{\mathcal{A}^\sharp}$.

(II) Torsion pair identities: $\delta_{\mathcal{A}} = \delta_{\mathcal{T}} * \delta_{\mathcal{F}}$ and $\delta_{\mathcal{A}^\sharp} = \delta_{\mathcal{F}} * \delta_{\mathcal{T}[-1]}$.



(III) Torsion pair identities with sections:

$$\delta^{\mathcal{O}}_{\mathcal{A}} = \delta^{\mathcal{O}}_{\mathcal{T}} * \delta^{\mathcal{O}}_{\mathcal{F}} \text{ and } \delta^{\mathcal{O}}_{\mathcal{A}^\sharp} = \delta^{\mathcal{O}}_{\mathcal{F}} * \delta^{\mathcal{O}}_{\mathcal{T}[-1]}.$$

(IV) All maps $\mathcal{O}_X \to T[-1]$ are zero, so $\delta^{\mathcal{O}}_{\mathcal{T}[-1]} = \delta_{\mathcal{T}[-1]}$.

Conclusion of the sketch proof

(v) Reineke's identity again: $\delta^{\mathcal{O}}_{\mathcal{T}} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{T}} * \delta_{\mathcal{T}}$.

(vi) Putting it all together: $\mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}} * \delta_{\mathcal{T}} = \mathrm{Quot}^{\mathcal{O}}_{\mathcal{T}} * \delta_{\mathcal{T}} * \mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}^\sharp}$.

(vii) Restrict to sheaves supported in dimension $\leqslant 1$. The Euler form is then trivial so the quantum torus is commutative. Thus

$$\mathcal{I}(\mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}}) = \mathcal{I}(\mathrm{Quot}^{\mathcal{O}}_{\mathcal{T}}) * \mathcal{I}(\mathrm{Quot}^{\mathcal{O}}_{\mathcal{A}^\sharp}).$$

(viii) Setting $t = \pm 1$ then gives the required identity

$$\sum_{\beta,n} \mathrm{DT}(\beta,n) x^\beta y^n = \sum_n \mathrm{DT}(0,n) y^n \cdot \sum_{\beta,n} \mathrm{PT}(\beta,n) x^\beta y^n.$$

2. Generalized DT invariants

Moduli spaces of framed sheaves

Let X be a Calabi-Yau threefold.

So far we have been discussing moduli spaces of objects in the category $D^b \operatorname{Coh}(X)$ equipped with a kind of framing.

Example

The Hilbert scheme parameterizes sheaves $E \in \operatorname{Coh}(X)$ equipped with a surjective map $f \colon \mathcal{O}_X \twoheadrightarrow E$.

(i) This framing data eliminates all stabilizer groups, so the moduli space is a scheme, and therefore has a well-defined Euler characteristic.

(ii) In this context wall-crossing can be achieved by varying the t-structure on the derived category $D^b \operatorname{Coh}(X)$.

What about unframed DT invariants?

Fix a polarization of X and a class $\alpha \in N$, and consider the stack

$$\mathcal{M}^{ss}(\alpha) = \{E \in \mathsf{Coh}(X) : E \text{ is semistable with } \mathrm{ch}(E) = \alpha\}.$$

(A) In the case when α is primitive, and the polarization is general, this stack is a $\mathbb{C}^*$-gerbe over its coarse moduli space $M^{ss}(\alpha)$, and we set

$$\mathsf{DT}^{\text{naive}}(\alpha) = e(M^{ss}(\alpha)) \in \mathbb{Z}.$$

Genuine DT invariants are defined using virtual cycles or by a weighted Euler characteristic as before.

(B) In the general case, Joyce figured out how to define invariants

$$\mathsf{DT}^{\text{naive}}(\alpha) \in \mathbb{Q}$$

with good properties, and showed that they satisfy wall-crossing formulae as the polarization is varied.

Quantum and classical DT invariants

(A) The generating function for the quantum DT invariants is

$$\text{q-DT}_\mu = \mathcal{I}\big([\mathcal{M}^{ss}(\mu) \subset \mathcal{M}]\big) \in \mathbb{C}_q[[N_+]].$$

(B) The generating function for the classical DT invariants is

$$\text{DT}_\mu = \lim_{q \to 1}(q-1) \cdot \log \text{q-DT}_\mu \in \mathbb{C}[[N_+]].$$

A difficult result of Joyce shows that this limit exists in general.

(C) The DT invariants are also encoded by the Poisson automorphism

$$\mathcal{S}_\mu = \exp\left\{ \text{DT}_\mu, - \right\} \in \operatorname{Aut} \mathbb{C}[[N_+]].$$

This coincides with the $q = 1$ limit of conjugation by q-DT_μ.

Example: a single rigid stable bundle

Suppose there is a single rigid stable bundle E of slope μ. Then

$$\mathcal{M}^{ss}(\mu) = \{E^{\oplus n} : n \geqslant 0\} = \bigsqcup_{n \geqslant 0} \mathrm{BGL}(n, \mathbb{C}).$$

Set $\alpha = \mathrm{ch}(E) \in N$. Applying the integration map we calculate

(A) The quantum DT generating function is

$$\text{q-DT}_\mu = \sum_{n \geqslant 0} \frac{x^{n\alpha}}{(q^n - 1) \cdots (q - 1)} \in \mathbb{C}_q[[N_+]].$$

We recognise the quantum dilogarithm $\Phi_q(x^\alpha)$.

A single stable bundle continued

(B) The classical DT generating function is

$$\mathrm{DT}_\mu = \lim_{q\to 1}(q-1)\cdot \log \Phi_q(x^\alpha) = \sum_{n\geqslant 1}\frac{x^{n\alpha}}{n^2}$$

and we conclude that $\mathrm{DT}(n\alpha) = 1/n^2$.

(C) The Poisson automorphism $\mathcal{S}_\mu \in \mathrm{Aut}\,\mathbb{C}[[N_+]]$ is

$$\mathcal{S}_\mu(x^\beta) = \exp\left\{\sum_{n\geqslant 1}\frac{x^{n\alpha}}{n^2}, -\right\}(x^\beta) = x^\beta\cdot(1+x^\alpha)^{\langle\alpha,\beta\rangle}$$

where the RHS should be expanded as a power series.

3. Stability conditions

Stability conditions

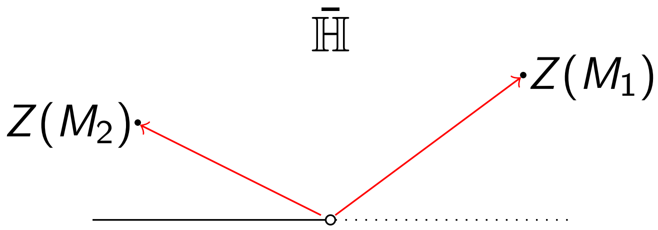
Let $\mathcal{A}$ be an abelian category.

Definition

A stability condition on $\mathcal{A}$ is a map of groups $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ such that

$$0 \neq E \in \mathcal{A} \implies Z(E) \in \bar{\mathbb{H}},$$

where $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}_{<0}$ is the semi-closed upper half-plane.



Phases and stability

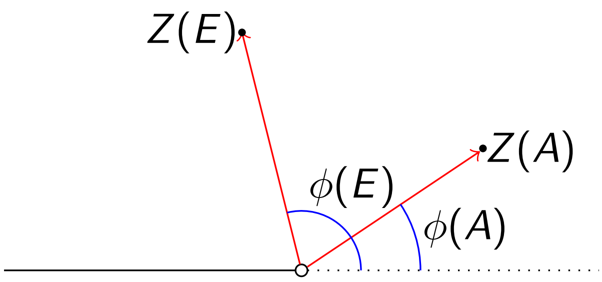
Definitions

(A) The phase of a nonzero object $E \in \mathcal{A}$ is

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1],$$

(B) An object $E \in \mathcal{A}$ is Z-semistable if

$$0 \neq A \subset E \implies \phi(A) \leqslant \phi(E).$$



Harder-Narasimhan filtrations

Definition

A stability condition Z has the Harder-Narasimhan property if every object $E \in \mathcal{A}$ has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E$$

such that each factor $F_i = E_i/E_{i-1}$ is Z-semistable and

$$\phi(F_1) > \cdots > \phi(F_n).$$

(I) If $\mathcal{A}$ has finite length this condition is automatic.

(II) When they exist, HN filtrations are necessarily unique, because the usual argument shows that if F_1, F_2 are Z-semistable then

$$\phi(F_1) > \phi(F_2) \implies \mathrm{Hom}(F_1, F_2) = 0.$$

Another Reineke identity

Let $\mathcal{C}$ be a finitary abelian category equipped with a stability condition Z having the Harder-Narasimhan property. Let

$$\delta^{\mathrm{ss}}(\phi) \in \widehat{\mathsf{Hall}}_{\mathrm{fty}}(\mathcal{A})$$

be the characteristic function of Z-semistable objects of phase $\phi \in \mathbb{R}$.

Lemma (Reineke)

There is an identity $\delta_{\mathcal{C}} = \prod^{\rightarrow}_{\phi \in \mathbb{R}} \delta^{\mathrm{ss}}(\phi)$.

Proof.

The product is taken in descending order of phase. The result follows from existence and uniqueness of the HN filtration. □

Wall-crossing formula

(A) The LHS of the above identity is independent of Z so given two stability conditions we get a wall-crossing formula

$$\overrightarrow{\prod_{\phi\in\mathbb{R}}} \delta^{\mathrm{ss}}(\phi, Z_1) = \overrightarrow{\prod_{\phi\in\mathbb{R}}} \delta^{\mathrm{ss}}(\phi, Z_2).$$

(B) If $\mathcal{C}$ has global dimension $\leqslant 1$ we can apply the integration map $\mathcal{I}$ to get an identity in the ring $\mathbb{C}_q[[N_+]]$.

(C) We can then take the $q = 1$ limit and obtain an identity in the group of automorphisms of the Poisson algebra $\mathbb{C}[[N^+]]$.

Example: the A_2 quiver

Let $\mathcal{C}$ be the abelian category of representations of the A_2 quiver. It has 3 indecomposable representations:

$$0 \longrightarrow S_2 \longrightarrow E \longrightarrow S_1 \longrightarrow 0.$$

We have $N = K_0(\mathcal{A}) = \mathbb{Z}^{\oplus 2} = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$,

$$\langle (m_1, n_1), (m_2, n_2) \rangle = m_2 n_1 - m_1 n_2,$$

and there are isomorphisms

$$\mathbb{C}_q[[N_+]] = \mathbb{C}\langle\langle x_1, x_2 \rangle\rangle / (x_2 * x_1 - q \cdot x_1 * x_2)$$

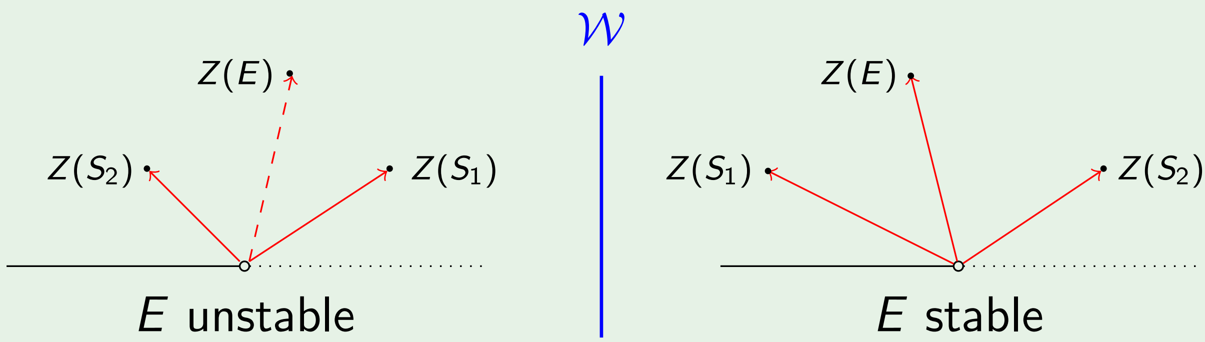
$$\mathbb{C}[[N_+]] = \mathbb{C}[[x_1, x_2]], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$

Quantum pentagon identity

The space Stab($\mathcal{A}$) is isomorphic to $\bar{\mathbb{H}}^2$ and there is a single wall

$$\mathcal{W} = \{Z \in \mathrm{Stab}(\mathcal{A}) : \mathrm{Im}\, Z(S_2)/Z(S_1) \in \mathbb{R}_{>0}\}$$

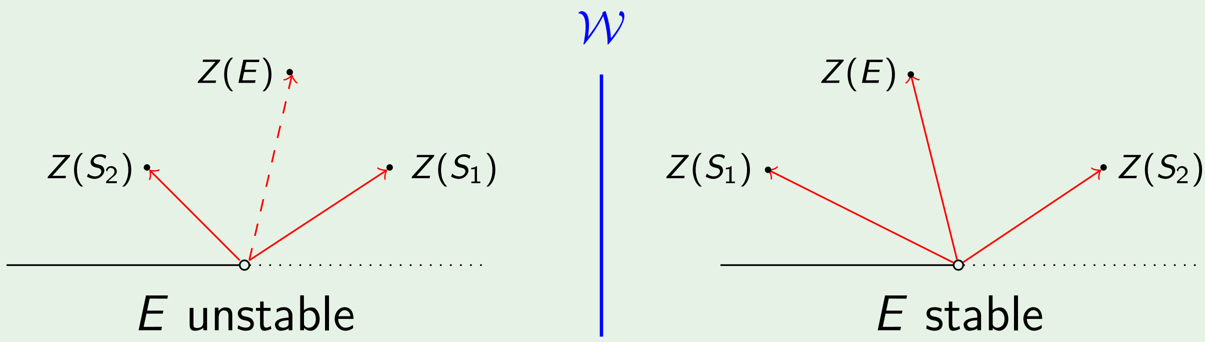
where the object E is strictly semistable.



The wall-crossing formula in $\mathbb{C}_q[[N_+]]$ becomes the pentagon identity

$$\Phi_q(x_2) * \Phi_q(x_1) = \Phi_q(x_1) * \Phi_q(\sqrt{q} \cdot x_1 * x_2) * \Phi_q(x_2).$$

Semi-classical version



The semi-classical version of the wall-crossing formula is the cluster identity

$$C_{(0,1)} \circ C_{(1,0)} = C_{(1,0)} \circ C_{(1,1)} \circ C_{(0,1)}.$$

$$C_\alpha \colon x^\beta \mapsto x^\beta \cdot (1 + x^\alpha)^{\langle \alpha, \beta \rangle} \in \operatorname{Aut} \mathbb{C}[[x_1, x_2]].$$

It can be viewed in the group of birational automorphisms of $(\mathbb{C}^*)^2$.

4. Stability in triangulated categories

Stability in triangulated categories

Let D be a triangulated category.

Definition

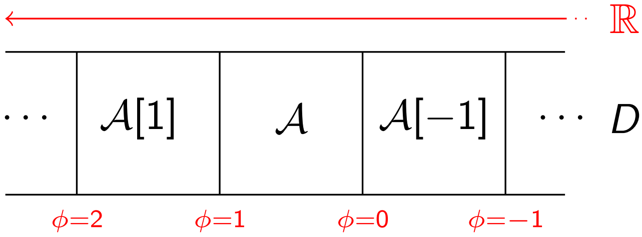
A stability condition on D is a pair $(Z, \mathcal{A})$ where

(i) $\mathcal{A} \subset D$ is a heart,

(ii) $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism,

such that Z defines a stability condition on $\mathcal{A}$ with the HN property.

An object $E \in D$ is defined to be semistable if $E = A[n]$ for some Z-semistable $A \in \mathcal{A}$. The phase of E is then $\phi(E) := \phi(A) + n$.



Space of stability conditions

We consider only stability conditions satisfying the extra conditions

(A) The central charge $Z\colon K_0(D) \to \mathbb{C}$ factors via our fixed map

$$\mathrm{ch}\colon K_0(D) \longrightarrow N \cong \mathbb{Z}^{\oplus n}.$$

(B) There is a $K > 0$ such that for any semistable object $E \in D$

$$Z(E) \geqslant K \cdot \|\mathrm{ch}(E)\|.$$

The set $\mathrm{Stab}(D)$ of such stability conditions has a natural topology.

Theorem

Sending a stability condition to its central charge defines a local homeomorphism

$$\mathrm{Stab}(D) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}) \cong \mathbb{C}^n.$$

In particular, $\mathrm{Stab}(D)$ *is a complex manifold.*