Stability and wall-crossing

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What's it all about?

(1) Calculating motivic invariants of moduli spaces of coherent sheaves on Calabi-Yau threefolds, e.g. DT invariants.

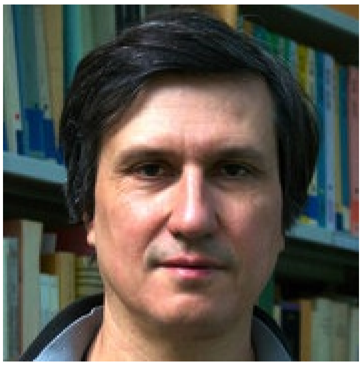
(2) Understanding the dependence of these invariants on the stability parameters.

With thanks to ...

Reineke

Joyce

Toda



Kontsevich

Soibelman

1. Introduction

Motivic invariants

The word motivic refers to invariants of varieties which satisfy

$$\chi(X) = \chi(Y) + \chi(U),$$

whenever $Y \subset X$ is closed and $U = X \setminus Y$.

Example: the Euler characteristic

$$e(X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X^{an}, \mathbb{C}) \in \mathbb{Z}.$$

Definition

The Grothendieck group $K(\mathrm{Var}/\mathbb{C})$ is the free abelian group on the set of isomorphism classes of varieties, modulo the scissor relations

$$[X] = [Y] + [U],$$

whenever $Y \subset X$ is closed and $U = X \setminus Y$.

Curve counting invariants

Let X be a Calabi-Yau threefold. Fix $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$.

$$\mathrm{Hilb}(\beta, n) = \left\{ \begin{matrix} \text{closed subschemes } C \subset X \text{ of dim } \leqslant 1 \\ \text{satisfying } [C] = \beta \text{ and } \chi(\mathcal{O}_C) = n \end{matrix} \right\},$$

$$\mathrm{DT}^{\mathrm{naive}}(\beta, n) = e(\mathrm{Hilb}(\beta, n)) \in \mathbb{Z}.$$

The genuine DT invariants are a weighted Euler characteristic

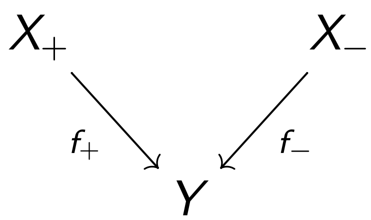
$$\mathrm{DT}(\beta, n) = e(\mathrm{Hilb}(\beta, n); \nu),$$

where $\nu \colon \mathrm{Hilb}(\beta, n) \to \mathbb{Z}$ is Behrend's constructible function, and

$$e(\mathrm{Hilb}(\beta, n); \nu) := \sum_{n \in \mathbb{Z}} n \cdot e(\nu^{-1}(n)) \in \mathbb{Z}.$$

Effect of a flop on DT invariants

Consider Calabi-Yau threefolds $X_\pm$ related by a flop:



Theorem (Toda)

The expression

$$\frac{\sum_{(\beta,n)} \mathrm{DT}^{\mathrm{naive}}(\beta,n)\, x^\beta y^n}{\sum_{(\beta,n):f_*(\beta)=0} \mathrm{DT}^{\mathrm{naive}}(\beta,n)\, x^\beta y^n}$$

is the same on both sides of the flop, under the natural identification

$$H_2(X_+,\mathbb{Z}) \cong H_2(X_-,\mathbb{Z}).$$

Definition of Stable pair invariants

Given $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, consider maps

$$f : \mathcal{O}_X \to E$$

of coherent sheaves on X such that

(A) E is pure of dimension 1 with $\mathrm{ch}(E) = (0, 0, \beta, n)$,

(B) $\dim_{\mathbb{C}} \operatorname{supp} \operatorname{coker}(f) = 0$.

There is a fine moduli scheme $\mathrm{Pairs}(\beta, n)$ for such maps, and we put

$$\mathrm{PT}^{\mathrm{naive}}(\beta, n) = e(\mathrm{Pairs}(\beta, n)) \in \mathbb{Z}.$$

Genuine stable pair invariants can be defined by weighting with the Behrend function as before.

DT versus stable pair invariants

Let X be a projective Calabi-Yau threefold.

Theorem (Toda)

(i) *For each $\beta \in H_2(X, \mathbb{Z})$ there is an identity*

$$\sum_{n\in\mathbb{Z}} \mathsf{PT}^{\mathsf{naive}}(\beta, n) y^n = \frac{\sum_{n\in\mathbb{Z}} \mathsf{DT}^{\mathsf{naive}}(\beta, n) y^n}{\sum_{n\geqslant 0} \mathsf{DT}^{\mathsf{naive}}(0, n) y^n}.$$

(ii) *This formal power series is the Laurent expansion of a rational function of y, invariant under $y \leftrightarrow y^{-1}$.*

These results also hold for genuine invariants.

Overall strategy

(A) Describe the relevant phenomenon via a change of stability condition in an abelian or triangulated category $\mathcal{C}$.

(B) Write down an appropriate identity in the Hall algebra of $\mathcal{C}$.

(C) Apply a ring homomorphism $\mathcal{I}\colon \mathrm{Hall}(\mathcal{C}) \to \mathbb{C}_q[K_0(\mathcal{C})]$ to obtain an identity of generating functions.

The first two steps are completely general, but the existence of the integration map $\mathcal{I}$ requires either

(I) $\mathcal{C}$ has global dimension $\leqslant 1$: $\mathrm{Ext}^{\geqslant 2}(M, N) = 0$,

(II) $\mathcal{C}$ satisfies the CY_3 condition: $\mathrm{Ext}^i(M, N) \cong \mathrm{Ext}^{3-i}(N, M)^*$.

2. Hall algebras

Hall algebras: the basic idea

Let $\mathcal{C}$ be an abelian category. For definiteness take $\mathcal{C} = \mathrm{Coh}(X)$.
Introduce

(I) The stack $\mathcal{M}$ of objects of $\mathcal{C}$.

(II) The stack $\mathcal{M}^{(2)}$ of short exact sequences in $\mathcal{C}$.

$$\begin{array}{ccc} & 0 \to A \to B \to C \to 0 & \\ \swarrow & & \searrow \\ (A, C) & & B \end{array}$$

$$\mathcal{M} \times \mathcal{M} \xleftarrow{(a,c)} \mathcal{M}^{(2)} \xrightarrow{b} \mathcal{M}$$

Applying a suitable 'cohomology theory' to our stacks gives

$$m \colon H^*(\mathcal{M}) \otimes H^*(\mathcal{M}) \to H^*(\mathcal{M}).$$

Grothendieck groups of stacks

As 'cohomology theory' take a relative Grothendieck group of stacks

$$H^*(\mathcal{M}) := K(\mathrm{St}/\mathcal{M}) := \Big(\bigoplus \mathbb{C} \cdot [\mathcal{S} \xrightarrow{f} \mathcal{M}]\Big)/\sim$$

where $\sim$ denotes the scissor relations

$$[\mathcal{S} \xrightarrow{f} \mathcal{M}] \sim [\mathcal{T} \xrightarrow{f|_{\mathcal{T}}} \mathcal{M}] + [\mathcal{U} \xrightarrow{f|_{\mathcal{U}}} \mathcal{M}],$$

for $\mathcal{T} \subset \mathcal{S}$ a closed substack with complement $\mathcal{U} = \mathcal{S} \setminus \mathcal{T}$.

(I) All our stacks are Artin stacks, locally of finite type over $\mathbb{C}$, with affine stabilizer groups.

(II) In the definition of $K(\mathrm{St}/\mathcal{M})$, we consider only stacks $\mathcal{S}$ of finite type over $\mathbb{C}$.

The Motivic Hall algebra

Unwrapping this definition, the motivic Hall algebra is

$$\mathrm{Hall}_{\mathrm{mot}}(\mathcal{C}) := K(\mathrm{St}\,/\mathcal{M}),$$

with product given explicitly by

$$[\mathcal{S}_1 \xrightarrow{f_1} \mathcal{M}] * [\mathcal{S}_2 \xrightarrow{f_2} \mathcal{M}] = [\mathcal{T} \xrightarrow{b \circ h} \mathcal{M}],$$

where h is defined by the Cartesian square

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{h} & \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle (a,c)} & & \\ \mathcal{S}_1 \times \mathcal{S}_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

Fibres of the correspondence

Consider again the crucial correspondence

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \Big\downarrow{\scriptstyle (a,c)} & & \\ \mathcal{M} \times \mathcal{M} & & \end{array}$$

(II) The fibre of b over $B \in \mathcal{M}$ is the Quot scheme $\mathrm{Quot}_X(B)$.

(III) The fibre of (a, c) over $(A, C) \in \mathcal{M} \times \mathcal{M}$ is the quotient stack

$$\left[\mathrm{Ext}^1(C, A) / \mathrm{Hom}(C, A)\right].$$

Less realistic but more fun ...

We now discuss a much less high-powered class of Hall algebras, where it is easy to make explicit calculations.

Basic assumption

Suppose that $\mathcal{C}$ is an abelian category such that

(I) Every object has only finitely many subobjects.

(II) All groups $\mathrm{Ext}^i(E, F)$ are finite.

Example

Let A be a finite dimensional algebra over $k = \mathbb{F}_q$ and take

$$\mathcal{C} = \mathrm{mod}(A)$$

to be the category of finite dimensional left A modules.

Definition of finitary Hall algebras

Definition

We define the finitary Hall algebra as follows

$$\widehat{\mathsf{Hall}}_{\mathrm{fty}}(\mathcal{C}) = \left\{ f\colon (\mathrm{Obj}(\mathcal{C})/\cong) \to \mathbb{C} \right\},$$

$$(f_1 * f_2)(B) = \sum_{A \subset B} f_1(A) \cdot f_2(B/A).$$

This is an associative, usually non-commutative, algebra.

We also define a subalgebra

$$\mathsf{Hall}_{\mathrm{fty}}(\mathcal{C}) \subset \widehat{\mathsf{Hall}}_{\mathrm{fty}}(\mathcal{C}),$$

consisting of functions with finite support.

Example: category of vector spaces

Let $\mathcal{C}$ be the category of finite dim. vector spaces over $\mathbb{F}_q$. Let

$$\delta_n \in \mathrm{Hall}_{\mathrm{fty}}(\mathcal{C})$$

be the characteristic function of vector spaces of dimension n.

$$\delta_n * \delta_m = |\,\mathrm{Gr}_{n,n+m}(\mathbb{F}_q)| \cdot \delta_{n+m},$$

$$|\,\mathrm{Gr}_{n,n+m}(\mathbb{F}_q)| = \frac{(q^{n+m}-1)\cdots(q^{m+1}-1)}{(q^n-1)\cdots(q-1)} = \binom{n+m}{n}_q.$$

It follows that there is an isomorphism of algebras

$$\mathcal{I}\colon \mathrm{Hall}_{\mathrm{fty}}(\mathcal{C}) \to \mathbb{C}[x], \qquad \mathcal{I}(\delta_n) = \frac{x^n}{(q^n-1)\cdots(q-1)}.$$

The Quantum dilogarithm

There is a distinguished element $\delta_{\mathcal{C}} \in \widehat{\text{Hall}}_{\text{fty}}(\mathcal{C})$ satisfying

$$\delta_{\mathcal{C}}(E) = 1 \quad \text{for all } E \in \mathcal{C}.$$

The isomorphism $\mathcal{I}$ maps this element $\delta_{\mathcal{C}} = \sum \delta_n$ to the series

$$\Phi_q(x) = \sum_{n \geqslant 0} \frac{x^n}{(q^n - 1) \cdots (q - 1)} \in \mathbb{C}[[x]].$$

This series is known as the quantum dilogarithm, because as $q \to 1$

$$\log \Phi_q(x) = \frac{1}{(q-1)} \cdot \sum_{n \geqslant 1} \frac{x^n}{n^2} + O(1).$$

A sample Hall algebra identity

Given a fixed object $P \in \mathcal{C}$ define elements of $\widehat{\mathrm{Hall}}_{\mathrm{fty}}(\mathcal{C})$ by

$$\delta_{\mathcal{C}}^{P}(E) = |\operatorname{Hom}_{\mathcal{C}}(P,E)|, \quad \mathrm{Quot}_{\mathcal{C}}^{P}(E) = |\operatorname{Hom}_{\mathcal{C}}^{\twoheadrightarrow}(P,E)|,$$

where $\operatorname{Hom}_{\mathcal{C}}^{\twoheadrightarrow}(P,E) \subset \operatorname{Hom}_{\mathcal{C}}(P,E)$ is the subset of surjective maps.

Lemma (Reineke)

There is an identity $\delta_{\mathcal{C}}^{P} = \mathrm{Quot}_{\mathcal{C}}^{P} * \delta_{\mathcal{C}}$.

Proof.

Evaluating on an object $E \in \mathcal{C}$ gives

$$|\operatorname{Hom}_{\mathcal{C}}(P,E)| = \sum_{A \subset E} |\operatorname{Hom}_{\mathcal{C}}^{\twoheadrightarrow}(P,A)| \cdot 1,$$

which holds because every map factors uniquely via its image. □

Geometric version of the identity

Let us consider the case $\mathcal{C} = \mathsf{Coh}(X)$ and $P = \mathcal{O}_X$. Define

(A) The stack $\mathcal{M}^{\mathcal{O}}$ parameterizing sheaves $E \in \mathsf{Coh}(X)$ equipped with a section $s\colon \mathcal{O}_X \to E$.

(B) The scheme Hilb parameterizing sheaves $E \in \mathsf{Coh}(X)$ equipped with a surjective section $s\colon \mathcal{O}_X \twoheadrightarrow E$.

Theorem

There is an identity in $\widehat{\mathsf{Hall}}_{\mathrm{mot}}(\mathcal{C})$

$$[\mathcal{M}^{\mathcal{O}} \xrightarrow{f} \mathcal{M}] = [\mathsf{Hilb} \xrightarrow{g} \mathcal{M}] * [\mathcal{M} \xrightarrow{\mathsf{id}} \mathcal{M}],$$

where f and g are the obvious forgetful maps.

Start of proof of the geometric case

The product on the RHS is defined by the Cartesian square

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{h} & \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle (a_1,a_2)} & & \\ \mathrm{Hilb}\times\mathcal{M} & \xrightarrow{g\times \mathrm{id}} & \mathcal{M}\times\mathcal{M} & & \end{array}$$

The points of the stack $\mathcal{T}$ over a scheme S are diagrams

$$\begin{array}{ccccccccc} & & \mathcal{O}_{S\times X} & \overset{\delta}{\searrow} & & & & & \\ & & \downarrow{\scriptstyle \gamma} & & & & & & \\ 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C & \longrightarrow & 0 \end{array}$$

of S-flat sheaves on $S\times X$ with γ surjective.

3. Integration map

Definition of the Euler form

Let $\mathcal{C}$ be an abelian category. From now on we assume

(A) $\mathcal{C}$ is linear over a field k,

(B) $\mathcal{C}$ is Ext-finite.

Example

We can take $\mathcal{C} = \mathrm{Coh}(X)$ with X smooth and projective.

Definition

The Euler form is the bilinear form

$$\chi(-,-)\colon K_0(\mathcal{C}) \times K_0(\mathcal{C}) \to \mathbb{Z}$$

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{Ext}^i(E, F).$$

Definition of the charge lattice

It is often convenient to fix a group homomorphism

$$\mathrm{ch}\colon K_0(\mathcal{C}) \longrightarrow N$$

with $N \cong \mathbb{Z}^{\oplus n}$ a free abelian group of finite rank.

Example

When $\mathcal{C} = \mathsf{Coh}(X)$, with X smooth and projective, we can take

$$\mathrm{ch}\colon K_0(\mathcal{C}) \to N = \mathrm{im}(\mathrm{ch}) \subset H^*(X, \mathbb{Q}),$$

to be the Chern character.

We always assume:

(I) The Euler form descends to a bilinear form

$$(-,-)\colon N \times N \to \mathbb{Z}.$$

We also consider the skew-symmetrization of this form

$$\langle -,-\rangle\colon N \times N \to \mathbb{Z}.$$

(II) The character $\mathrm{ch}(E)$ is locally-constant in families. This gives a decomposition

$$\mathcal{M} = \bigsqcup_{\alpha \in N} \mathcal{M}_\alpha,$$

into open-closed substacks, and induces a grading

$$\mathrm{Hall}_{\mathrm{mot}}(\mathcal{C}) = \bigoplus_{\alpha \in N} K(\mathrm{St}/\mathcal{M}_\alpha).$$

Definition of the quantum torus

Define a non-commutative algebra over the field $\mathbb{C}(t)$ by

$$\mathbb{C}_t[N] = \bigoplus_{\alpha \in N} \mathbb{C}(t) \cdot x^\alpha \qquad x^\alpha * x^\gamma = t^{-(\gamma,\alpha)} \cdot x^{\alpha+\gamma}.$$

This is a non-commutative deformation of the ring

$$\mathbb{C}[N] \cong \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}],$$

which is the co-ordinate ring of the algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong (\mathbb{C}^*)^n.$$

We use the notation $q = t^2$.

The virtual Poincaré invariant

There is an algebra homomorphism

$$\chi_t \colon K(\mathrm{St}/\mathbb{C}) \to \mathbb{Q}(t),$$

uniquely defined by the following two properties:

(I) If V is a smooth, projective variety then

$$\chi_t(V) = \sum \dim_{\mathbb{C}} H^i(V^{\mathrm{an}}, \mathbb{C}) \cdot (-t)^i \in \mathbb{Z}[t].$$

(II) If V is a variety with an action of $\mathrm{GL}(n)$ then

$$\chi_t([V/\mathrm{GL}(n)]) = \chi_t(V)/\chi_t(\mathrm{GL}(n)).$$

Note that: $\chi_t(\mathrm{GL}(n)) = q^{\binom{n}{2}} \cdot (q-1) \cdot (q^2-1) \cdots (q^n-1)$.

Integration map for curves

Theorem (Joyce)

When $\mathcal{C} = \mathrm{Coh}(X)$, with X a curve, there is an algebra map

$$\mathcal{I}\colon \mathrm{Hall}_{\mathrm{mot}}(\mathcal{C}) \to \mathbb{C}_{t^2}[N], \quad \mathcal{I}\big([\mathcal{S} \to \mathcal{M}_\alpha]\big) = \chi_t(\mathcal{S}) \cdot x^\alpha.$$

This works because

$$\dim_{\mathbb{C}} \mathrm{Ext}^1(C, A) - \dim_{\mathbb{C}} \mathrm{Hom}(C, A) = -\chi(C, A),$$

so the fibres of the crucial map

$$(a, c)\colon \mathcal{M}^{(2)} \to \mathcal{M} \times \mathcal{M}$$

over the substack $\mathcal{M}_\alpha \times \mathcal{M}_\gamma$ have Poincaré invariant $q^{-(\gamma,\alpha)}$.

Integration map: CY_3 case

(A) Kontsevich and Soibelman also construct an algebra map

$$\mathcal{I} \colon \mathrm{Hall}_{\mathrm{mot}}(\mathcal{C}) \to \mathbb{C}_t[N]$$

in the case that X is a Calabi-Yau threefold. There are still some technical problems, e.g. the existence of orientation data.

(B) It is harder to describe $\mathcal{I}$ in this case, but if S is a scheme

$$\lim_{t \to \pm 1} \mathcal{I}([S \xrightarrow{f} \mathcal{M}_\alpha]) = \begin{cases} e(S) \cdot x^\alpha & \text{if } t \to +1, \\ e(S; f^*(\nu)) \cdot x^\alpha & \text{if } t \to -1. \end{cases}$$

The integration map $\mathcal{I}$ therefore turns identities in the motivic Hall algebra into identities involving (naive or genuine) DT invariants.

Semi-classical limit: the Poisson torus

(A) The semi-classical limit of the algebra $\mathbb{C}_t[N]$ at $t = 1$ is the commutative algebra $\mathbb{C}[N]$ equipped with the Poisson bracket

$$\{x^\alpha, x^\gamma\} = \lim_{t \to 1} \frac{x^\alpha * x^\gamma - x^\gamma * x^\alpha}{t - 1} = \langle \alpha, \gamma \rangle \cdot x^{\alpha+\gamma}.$$

(B) One can use the formulae from the last slide to define semi-classical versions of the map $\mathcal{I}$ at $t = \pm 1$ that are maps of Poisson algebras. This works because

$$(\mathrm{ext}^1(C, A) - \hom(C, A)) - (\mathrm{ext}^1(A, C) - \hom(A, C)) = \chi(A, C).$$

These Poisson integration maps suffice for applications to classical (i.e. non-refined) DT invariants.

Other applications of wall-crossing

There have been several other important applications of the same technology. Some, marked $(*)$, are still work in progress:

(A) Caldero–Chapoton formula in cluster theory.

(B) Oblomkov–Shende conjecture relating DT invariants of plane curve singularities to HOMFLY polynomials.

(C) Betti numbers of moduli of sheaves on ruled surfaces.

(D) $(*)$ Crepant resolution conjecture.

(E) $(*)$ Hausel–Letellier–Rodriguez-Villegas formula on Hodge polynomials of character varieties.