Contents 46

Reliability and Quality Control

Learning outcomes

You will first learn about the importance of the concept of reliability applied to systems and products. The second Section introduces you to the very important topic of quality control in production processes. In both cases you will learn how to perform the basic calculations necessary to use each topic in practice.

Reliability

46.1

Introduction

Much of the theory of reliability was developed initially for use in the electronics industry where components often fail with little if any prior warning. In such cases the hazard function or conditional failure rate function is constant and any functioning component or system is taken to be 'as new'. There are other cases where the conditional failure rate function is time dependent, often proportional to the time that the system or component has been in use. The function may be an increasing function of time (as with random vibrations for example) or decreasing with time (as with concrete whose strength, up to a point, increases with time). If we can develop a lifetime model, we can use it to plan such things as maintenance and part replacement schedules, whole system replacements and reliability testing schedules.

Prerequisites

Before starting this Section you should ...

- be familiar with the results and concepts met in the study of probability
- understand and be able to use continuous probability distributions

Learning Outcomes

On completion you should be able to ...

- appreciate the importance of lifetime distributions
- complete reliability calculations for simple systems
- state the relationship between the Weibull distribution and the exponential distribution

1. Reliability

Lifetime distributions

From an engineering point of view, the ability to predict the lifetime of a whole system or a system component is very important. Such lifetimes can be predicted using a statistical approach using appropriate distributions. Common examples of structures whose lifetimes we need to know are airframes, bridges, oil rigs and at a simpler, less catastrophic level, system components such as cycle chains, engine timing belts and components of electronic systems such as televisions and computers. If we can develop a lifetime model, we can use it to plan such things as maintenance and part replacement schedules, whole system replacements and reliability testing schedules.

We start by looking at the length of use of a system or component prior to failure (the age prior to failure) and from this develop a definition of reliability. Lifetime distributions are functions of time and may be expressed as probability density functions and so we may write

$$F(t) = \int_0^t f(t)\,dt$$

This represents the probability that the system or component fails anywhere between 0 and t.

Key Point 1

The probability that a system or component will fail only after time t may be written as

$$R(t) = 1 - F(t)$$

The function $R(t)$ is usually called the reliability function.

In practice engineers (and others!) are often interested in the so-called hazard function or conditional failure rate function which gives the probability that a system or component fails after it has been in use for a given time. This function may be defined as

$$\begin{aligned} H(t) &= \lim_{\Delta t \to o}\left(\frac{\text{P(failure in the interval } (t, t+\Delta t))/\Delta t}{\text{P(survival up to time } t)}\right) \\ &= \frac{1}{R(t)} \lim_{\Delta t \to o}\left(\frac{F(t+\Delta t) - F(t)}{\Delta t}\right) \\ &= \frac{1}{R(t)}\frac{d}{dt}F(t) \\ &= \frac{f(t)}{R(t)} \end{aligned}$$

This gives the conditional failure rate function as

$$H(t) = \lim_{\Delta t \to o} \left(\frac{\displaystyle\int_t^{t+\Delta t} f(t)\, dt/(\Delta t)}{R(t)} \right) = \frac{f(t)}{R(t)}$$

Essentially we are describing the rate of failure per unit time for (say) mechanical or electrical components which have already been in service for a given time. A graph of $H(t)$ often shows a high initial failure rate followed by a period of relative reliability followed by a period of increasingly high failure rates as a system or component ages. A typical graph (sometimes called a bathtub graph) is shown below.

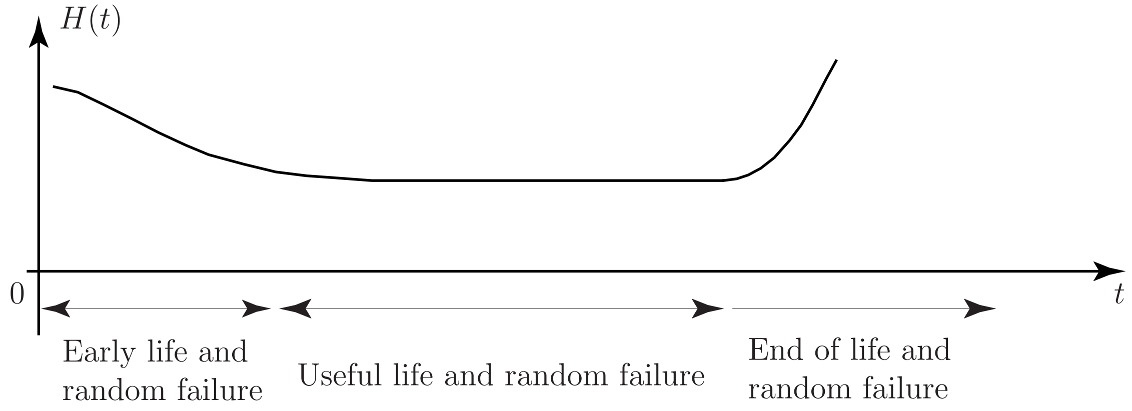


Figure 1

Note that 'early life and random failure' includes failure due to defects being present and that 'end of life and random failure' includes failure due to ageing.

The reliability of a system or component may be defined as the probability that the system or component functions for a given time, that is, the probability that it will fail only after the given time. Put another way, $R(t)$ is the probability that the system or component is still functioning at time t.

The exponential distribution

We have already met the exponential distribution in the form

$$f(t) = \lambda \mathrm{e}^{-\lambda t}, \qquad t \geq 0$$

However, one of the simplest distributions describing failure is the exponential distribution

$$f(t) = \frac{1}{\mu}\mathrm{e}^{-t/\mu}, \qquad t \geq 0$$

where, in this case, μ is the mean time to failure. One property of this distribution is that the hazard function is a constant independent of time - the 'good as new' syndrome mentioned above. To show that the probability of failure is independent of age consider the following.

$$F(t) = \int_0^t \frac{1}{\mu}\mathrm{e}^{-t/\mu}\, dt = \frac{1}{\mu}\Big[-\mu \mathrm{e}^{-t/\mu} \Big]_0^t = 1 - \mathrm{e}^{-t/\mu}$$

Hence the reliability (given that the total area under the curve $F(t) = 1 - \mathrm{e}^{-t/\mu}$ is unity) is

$$R(t) = 1 - F(t) = \mathrm{e}^{-t/\mu}$$

Hence, the hazard function or conditional failure rate function $H(t)$ is given by

$$H(t) = \frac{F(t)}{R(t)} = \frac{\frac{1}{\mu}\mathrm{e}^{-t/\mu}}{\mathrm{e}^{-t/\mu}} = \frac{1}{\mu}$$

which is a constant independent of time.

Another way of looking at this is to consider the probability that failure occurs in the interval $(\tau, \tau+t)$ given that the system is functioning at time τ . This probability is

$$\begin{aligned} \frac{F(\tau+t) - F(\tau)}{R(t)} &= \frac{(1-\mathrm{e}^{-(\tau+t)/\mu}) - (1-\mathrm{e}^{-\tau/\mu})}{e^{-\tau/\mu}} \\ &= 1 - \mathrm{e}^{-t/\mu} \end{aligned}$$

This is just the probability that failure occurs in the interval $(0, t)$ and implies that ageing has no effect on failure. This is sometimes referred to as the 'good as new syndrome.'

It is worth noting that in the modelling of many complex systems it is assumed that only random component failures are important. This enables us to assume the use of the exponential distribution since initial failures are removed by a 'running-in' process and the time to ultimate failure is usually long.

Example 1

The lifetime of a modern low-wattage electronic light bulb is known to be exponentially distributed with a mean of 8000 hours.

Find the proportion of bulbs that may be expected to fail before 7000 hours use.

Solution

We know that $\mu = 8000$ and so

$$\begin{aligned} F(t) &= \int_0^t \frac{1}{8000}\mathrm{e}^{-t/8000}\,dt \\ &= \frac{1}{8000}\Big[-8000\mathrm{e}^{-t/8000}\Big]_0^t \\ &= 1 - \mathrm{e}^{-t/8000} \end{aligned}$$

Hence $F(7000) = 1 - \mathrm{e}^{-7000/8000} = 1 - \mathrm{e}^{-0.675} = 0.4908$ and we expect that about 49% of the bulbs will fail before 7000 hours of use.

Task

A particular electronic device will only function correctly if two essential components both function correctly. The lifetime of the first component is known to be exponentially distributed with a mean of 6000 hours and the lifetime of the second component is known to be exponentially distributed with a mean of 7000 hours. Find the proportion of devices that may be expected to fail before 8000 hours use. State clearly any assumptions you make.

Your solution

Answer

The assumption made is that the components operate independently.

For the first component $F(t) = 1 - e^{-t/6000}$ so that $F(8000) = 1 - \mathrm{e}^{-8000/6000} = 1 - \mathrm{e}^{-4/3} = 0.7364$

For the second component $F(t) = 1 - e^{-t/7000}$ so that $F(8000) = 1 - \mathrm{e}^{-8000/7000} = 0.6811$

The probability that the device will continue to function after 8000 hours use is given by an expression of the form $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Hence the probability that the device will continue to function after 8000 hour use is

$$0.7364 + 0.6811 - 0.7364 \times 0.6811 = 0.916$$

and we expect just under 92% of the devices to fail before 8000 hours use.

An alternative answer may be obtained more directly by using the reliability function $R(t)$:

The assumption made is that the components operate independently.

For the first component $R(t) = e^{-t/6000}$ so that $R(8000) = \mathrm{e}^{-8000/6000} = \mathrm{e}^{-4/3} = 0.2636$

For the second component $R(t) = e^{-t/7000}$ so that $R(8000) = \mathrm{e}^{-8000/7000} = 0.3189$

The probability that the device will continue to function after 8000 hour use is given by

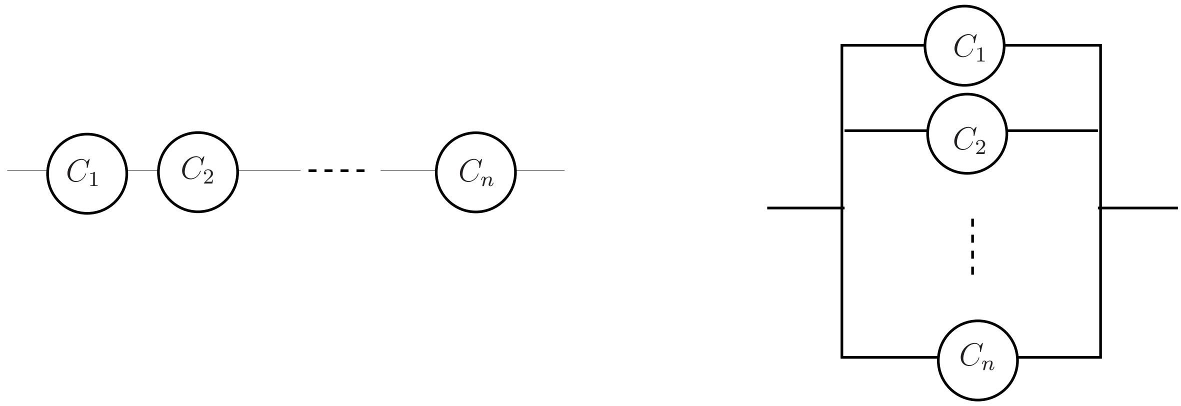
$$0.2636 \times 0.3189 = 0.0841$$

Hence the probability that the device will fail before 8000 hours use is $1 - 0.0841 = 0.916$

and we expect just under 92% of the devices to fail before 8000 hours use.

2. System reliability

It is reasonable to ask whether, in designing a system, an engineer should design a system using components in series or in parallel. The engineer may not have a choice of course! We may represent a system consisting of n components say $C_1, C_2 \ldots, C_n$ with reliabilities (these are just probability values) $R_1, R_2 \ldots, R_n$ respectively as series and parallel systems as shown below.



Series Design Parallel Design

Figure 2

With a series design, the system will fail if any component fails. With a parallel design, the system will work as long as any component works.

Assuming that the components are independent, we can express the reliability of the series design as

$$R_{\text{Series}} = R_1 \times R_2 \times \cdots \times R_n$$

simply by multiplying the probabilities.

Since each reliability value is less than one, we may conclude that a series design is less reliable than its *least reliable* component.

Similarly (although by no means as clearly!), we can express the reliability of the parallel design as

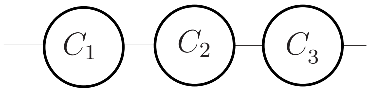
$$R_{\text{Parallel}} = 1 - (1 - R_1)(1 - R_2) \ldots (1 - R_n)$$

The derivation of this result is illustrated in Example 3 below for the case $n = 3$. In this case, the algebra involved in fairly straightforward. We can conclude that the parallel design is at least as reliable as the most reliable component.

Engineers will sometimes include 'redundant'components in parallel to improve reliability. The spare wheel of a car is a well known example.

Example 2
Series design

Consider the three components C_1, C_2 and C_3 with reliabilities R_1, R_2 and R_3 connected in series as shown below



Find the reliability of the system where $R_1 = 0.2,\ R_2 = 0.3$ and $R_3 = 0.4$.

Solution

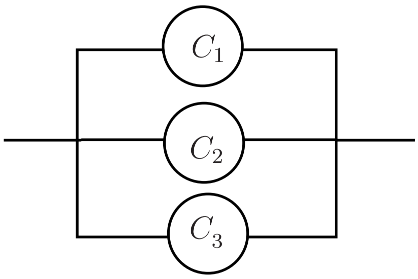
Since the components are assumed to act independently, we may clearly write

$$R_{\text{Series}} = R_1 \times R_2 \times R_3$$

Taking $R_1 = 0.2,\ R_2 = 0.3$ and $R_3 = 0.4$ we obtain the value $\quad R_{\text{Series}} = 0.2 \times 0.3 \times 0.4 = 0.024$

Example 3
Parallel design

Consider the three components C_1, C_2 and C_3 with reliabilities R_1, R_2 and R_3 connected in parallel as shown below



Find the reliability of the system where $R_1 = 0.2,\ R_2 = 0.3$ and $R_3 = 0.4$.

Solution

Observing that $F_i = 1 - R_i$, where F_i represents the failure of the ith component and R_i represents the reliability of the ith component we may write

$$F_{\text{System}} = F_1F_2F_3 \rightarrow \quad R_{\text{System}} = 1 - F_1F_2F_3 = 1 - (1 - R_1)(1 - R_2)(1 - R_3)$$

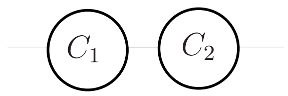
Again taking $R_1 = 0.2,\ R_2 = 0.3$ and $R_3 = 0.4$ we obtain

$$F_{\text{System}} = 1 - (1 - 0.2)(1 - 0.3)(1 - 0.4) = 1 - 0.336 = 0.664$$

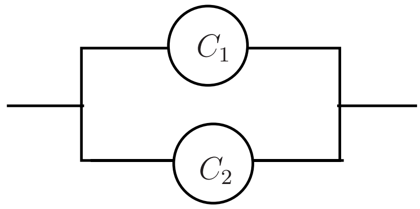
Hence series system reliability is *less* than any of the component reliabilities and parallel system reliability is *greater* than any of the component reliabilities.

Task

Consider the two components C_1 and C_2 with reliabilities R_1 and R_2 connected in series and in parallel as shown below. Assume that $R_1 = 0.3$ and $R_2 = 0.4$.



Series configuration



Parallel configuration

Let R_{Series} be the reliability of the series configuration and R_{Parallel} be the reliability of the parallel configuration

(a) Why would you expect that $R_{\text{Series}} < 0.3$ and $R_{\text{Parallel}} > 0.4$?

(b) Calculate R_{Series}

(c) Calculate R_{Parallel}

Your solution

Answer

(a) You would expect $R_{\text{Series}} < 0.3$ and $R_{\text{Parallel}} > 0.4$ because R_{Series} is less than any of the component reliabilities and R_{Parallel} is greater than any of the component reliabilities.

(b) $R_{\text{Series}} = R_1 \times R_2 = 0.3 \times 0.4 = 0.12$

(c) $R_{\text{Parallel}} = R_1 \times R_2 - R_1R_2 = 0.3 + 0.4 - 0.3 \times 0.4 = 0.58$

3. The Weibull distribution

The Weibull distribution was first used to describe the behaviour of light bulbs as they undergo the ageing process. Other applications are widespread and include the description of structural failure, ball-bearing failure and the failure of a variety of electronic components.

Key Point 2

If the random variable X is defined by a probability density function of the form

$$f(x) = \alpha\beta(\alpha x)^{\beta-1}\mathrm{e}^{-(\alpha x)^{\beta}}$$

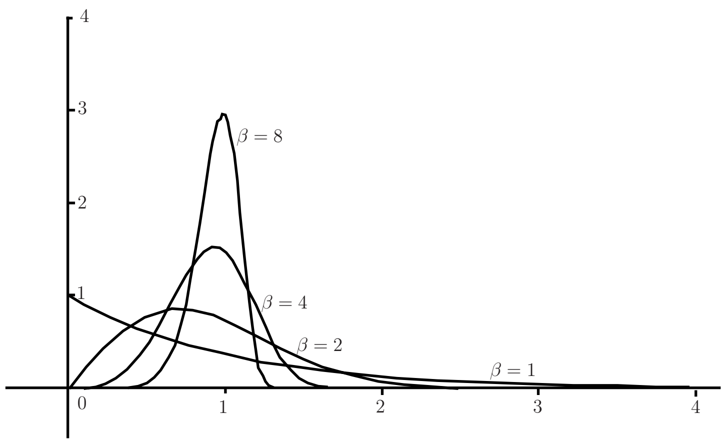
then X is said to follow a Weibull distribution.

The hazard function or conditional failure rate function $H(t)$, which gives the probability that a system or component fails after it has been in use for a given time, is constant for the exponential distribution but for a Weibull distribution is proportional to $x^{\beta-1}$. This implies that $\beta = 1$ gives a constant hazard, $\beta < 1$ gives a reducing hazard and $\beta > 1$ gives an increasing hazard. Note that α is simply a scale factor while the case $\beta = 1$ reduces the Weibull distribution to

$$f(x) = \alpha\mathrm{e}^{-\alpha x}$$

which you may recognize as one form of the exponential distribution.

Figure 3 below shows the Weibull distribution for various values of β. For simplicity, the graphs assume that $\alpha = 1$. Essentially the plots are of the function $\quad f(x) = \beta x^{\beta-1}\mathrm{e}^{-x^{\beta}}$



The Weibull distribution with $\beta = 1, 2, 4$ and 8

Figure 3

In applications of the Weibull distribution, it is often useful to know the cumulative distribution function (cdf). Although it is not derived directly here, the cdf $F(x)$ of the Weibull distribution, whose probability density function is $f(x) = \alpha\beta(\alpha x)^{\beta-1}\mathrm{e}^{-(\alpha x)^{\beta}}$, is given by the function

$$F(x) = 1 - \mathrm{e}^{-(\alpha x)^\beta}$$

The relationship between $f(x)$ and $F(x)$ may be seen by remembering that

$$f(x) = \frac{d}{dx}F(x)$$

Differentiating the cdf $F(x) = 1 - \mathrm{e}^{-(\alpha x)^\beta}$ gives the result

$$f(x) = \frac{d}{dx}F(x) = \frac{(\alpha x)^\beta \beta \mathrm{e}^{-(\alpha x)^\beta}}{x} = \alpha^\beta \beta x^{\beta-1}\mathrm{e}^{-(\alpha x)^\beta} = \alpha\beta(\alpha x)^{\beta-1}\mathrm{e}^{-(\alpha x)^\beta}$$

Mean and variance of the Weibull distribution

The mean and variance of the Weibull distribution are quite complicated expressions and involve the use of the Gamma function, $\Gamma(x)$. The outline explanations given below defines the Gamma function and shows that basically in terms of the application here, the function can be expressed as a simple factorial.

It can be shown that if the random variable X follows a Weibull distribution defined by the probability density function

$$f(x) = \alpha\beta(\alpha x)^{\beta-1}\mathrm{e}^{-(\alpha x)^\beta}$$

then the mean and variance of the distribution are given by the expressions

$$\mathsf{E}(X) = \frac{1}{\alpha}\Gamma(1+\frac{1}{\beta}) \qquad \text{and} \qquad \mathsf{V}(X) = \left(\frac{1}{\alpha}\right)^2 \left\{\Gamma(1+\frac{2}{\beta}) - \left(\Gamma(1+\frac{1}{\beta})\right)^2\right\}$$

where $\Gamma(r)$ is the gamma function defined by

$$\Gamma(r) = \int_0^\infty x^{r-1}\mathrm{e}^{-x}dx$$

Using integration by parts gives a fairly straightforward identity satisfied by the gamma function is:

$$\Gamma(r) = (r-1)\Gamma(r-1)$$

Using this relationship tells us that

$$\Gamma(r) = (r-1)(r-2)(r-3)\ldots\Gamma(1) \qquad \text{for integral } r$$

but

$$\Gamma(1) = \int_0^\infty x^{1-1}\mathrm{e}^{-x}dx = \int_0^\infty \mathrm{e}^{-x}dx = \Big[-\mathrm{e}^{-x}\Big]_0^\infty = 1$$

so that

$$\Gamma(r) = (r-1)(r-2)(r-3)\ldots(3)(2)(1) = (r-1)! \qquad \text{for integral } r$$

Essentially, this means that when we calculate the mean and variance of the Weibull distribution using the expressions

$$\mathsf{E}(X) = \frac{1}{\alpha}\Gamma(1+\frac{1}{\beta}) \qquad \text{and} \qquad \mathsf{V}(X) = \left(\frac{1}{\alpha}\right) \left\{\Gamma(1+\frac{2}{\beta}) - \left(\Gamma(1+\frac{1}{\beta})\right)^2\right\}$$

we are evaluating factorials and the calculation is not as difficult as you might imagine at first sight.

Note that to apply the work as presented here, the expressions

$$(1+\frac{1}{\beta}) \qquad \text{and} \qquad (1+\frac{2}{\beta})$$

must both take positive integral values (i.e. $\frac{1}{\beta}$ must be an integer) in order for the factorial calculation to make sense. If the above expressions do not assume positive integral values, you will need to know much more about the values and behaviour of the Gamma function in order to do calculations. In practice, we would use a computer of course. As you might expect, the Gamma function is tabulated to assist in such calculations but in the examples and exercises set in this Workbook, the values of $\frac{1}{\beta}$ will always be integral.

Example 4

The main drive shaft of a pumping engine runs in two bearings whose lifetime follows a Weibull distribution with random variable X and parameters $\alpha = 0.0002$ and $\beta = 0.5$.

(a) Find the expected time that a single bearing runs before failure.

(b) Find the probability that any one bearing lasts at least 8000 hours.

(c) Find the probability that both bearings last at least 6000 hours.

Solution

(a) We know that $\mathsf{E}(X) = \frac{1}{\alpha}\Gamma(1+\frac{1}{\beta}) = 5000 \times \Gamma(1+2) = 5000 \times 2 = 10000$ hours.

(b) We require $\mathsf{P}(X > 8000)$, this is given by the calculation:

$$\begin{aligned} \mathsf{P}(X > 8000) = 1 - F(8000) &= 1-(1-\mathrm{e}^{-(0.0002\times 8000)^{0.5}}) \\ &= \mathrm{e}^{-(0.0002\times 8000)^{0.5}} = \mathrm{e}^{-1.265} \\ &= 0.282 \end{aligned}$$

(c) Assuming that the bearings wear independently, the probability that both bearings last at least 6000 hours is given by $\{\mathsf{P}(X > 6000)\}^2$. But $\mathsf{P}(X > 6000)$ is given by the calculation

$$\mathsf{P}(X > 6000) = 1 - F(6000) = 1 - (1 - \mathrm{e}^{-(0.0002\times 6000)^{0.5}}) = 0.335$$

so that the probability that both bearings last at least 6000 hours is given by

$$\{\mathsf{P}(X > 6000)\}^2 = 0.335^2 = 0.112$$

Task

A shaft runs in four roller bearings the lifetime of each of which follows a Weibull distribution with parameters $\alpha = 0.0001$ and $\beta = 1/3$.

(a) Find the mean life of a bearing.

(b) Find the variance of the life of a bearing.

(c) Find the probability that all four bearings last at least 50,000 hours. State clearly any assumptions you make when determining this probability.

Your solution

Answer

(a) We know that $\mathsf{E}(X) = \dfrac{1}{\alpha}\Gamma(1+\dfrac{1}{\beta}) = 10000 \times \Gamma(1+3) = 60000 \times 2 = 10000$ hours.

(b) We know that $\mathsf{V}(X) = \left(\dfrac{1}{\alpha}\right)\left\{\Gamma(1+\dfrac{2}{\beta}) - \left(\Gamma(1+\dfrac{1}{\beta}\right)^2\right\}$ so that

$$
\begin{aligned}
\mathsf{V}(X) &= (10000)^2\{\Gamma(1+6) - (\Gamma(1+3)]^2\} \\
&= (10000)^2\{\Gamma(7) - (\Gamma(4))^2\} \\
&= (10000)^2\{6! - (3!)^2\} = 684(10000)^2 = 6.84 \times 10^{10}
\end{aligned}
$$

(c) The probability that one bearing will last at least 50,000 hours is given by the calculation

$$
\begin{aligned}
\mathsf{P}(X > 50000) &= 1 - F(50000) \\
&= 1 - (1 - \mathsf{e}^{-(0.0001\times 50000)^{0.5}}) \\
&= \mathsf{e}^{-5^{0.5}} = 0.107
\end{aligned}
$$

Assuming that the four bearings have independent distributions, the probability that all four bearings last at least 50,000 hours is $(0.107)^4 = 0.0001$

Exercises

1. The lifetimes in hours of certain machines have Weibull distributions with probability density function

$$f(t) = \begin{cases} 0 & (t < 0) \\ \alpha\beta(\alpha t)^{\beta-1}\exp\{-(\alpha t)^\beta\} & (t \geq 0) \end{cases}$$

   (a) Verify that $\displaystyle\int_0^t \alpha\beta(\alpha u)^{\beta-1}\exp\{-(\alpha u)^\beta\}du = 1 - \exp\{-(\alpha t)^\beta\}.$

   (b) Write down the distribution function of the lifetime.

   (c) Find the probability that a particular machine is still working after 500 hours of use if $\alpha = 0.001$ and $\beta = 2$.

   (d) In a factory, n of these machines are installed and started together. Assuming that failures in the machines are independent, find the probability that all of the machines are still working after t hours and hence find the probability density function of the time till the first failure among the machines.

2. If the lifetime distribution of a machine has hazard function $h(t)$, then we can find the reliability function $R(t)$ as follows. First we find the "cumulative hazard function" $H(t)$ using

$$H(t) = \int_0^t h(t)\,dt \qquad \text{then the reliability is} \qquad R(t) = e^{-H(t)}.$$

   (a) Starting with $R(t) = \exp\{-H(t)\}$, work back to the hazard function and hence confirm the method of finding the reliability from the hazard.

   (b) A lifetime distribution has hazard function $h(t) = \theta_0 + \theta_1 t + \theta_2 t^2$. Find

      (i) the reliability function.

      (ii) the probability density function.

   (c) A lifetime distribution has hazard function $h(t) = \dfrac{1}{t+1}$. Find

      (i) the reliability function;

      (ii) the probability density function;

      (iii) the median.

      What happens if you try to find the mean?

   (d) A lifetime distribution has hazard function $h(t) = \dfrac{\rho\theta}{\rho t + 1}$, where $\rho > 0$ and $\theta > 1$.
   Find

      (i) the reliability function.

      (ii) the probability density function.

      (iii) the median.

      (iv) the mean.

Exercises continued

3. A machine has n components, the lifetimes of which are independent. However the whole machine will fail if any component fails. The hazard functions for the components are $h_1(t), \ldots, h_n(t)$. Show that the hazard function for the machine is $\sum_{i=1}^{n} h_i(t)$.

4. Suppose that the lifetime distributions of the components in Exercise 3 are Weibull distributions with scale parameters $\rho_1, \ldots, \rho_n$ and a common index (i.e. "shape parameter") γ so that the hazard function for component i is $\gamma\rho_i(\rho_i t)^{\gamma-1}$. Find the lifetime distribution for the machine.

5. *(Difficult).* Show that, if T is a Weibull random variable with hazard function $\gamma\rho(\rho t)^{\gamma-1}$,

   (a) the median is $\mathsf{M}(T) = \rho^{-1}(\ln 2)^{1/\gamma}$,

   (b) the mean is $\mathsf{E}(T) = \rho^{-1}\Gamma(1+\gamma^{-1})$

   (c) the variance is $\mathsf{V}(T) = \rho^{-2}\{\Gamma(1+2\gamma^{-1}) - [\Gamma(1+\gamma^{-1})]^2\}$.

Note that $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$. In the integrations it is helpful to use the substitution $u = (\rho t)^\gamma$.

Answers

1.

(a) $$\frac{d}{du}(-\exp\{-(\alpha u)^\beta\})$$

so

$$\int_0^t \alpha\beta(\alpha u)^{\beta-1}\exp\{-(\alpha u)^\beta\}du = \left[-\exp\{-(\alpha u)^\beta\}\right]_0^t = 1-\exp\{-(\alpha t)^\beta\}.$$

(b) The distribution function is $F(t) = 1-\exp\{-(\alpha t)^\beta\}$.

(c) $R(500) = \exp\{-(0.001 \times 500)^2\} = 0.7788$

(d) The probability that all of the machines are still working after t hours is

$$R_n(t) = \{R(t)\}^n = \exp\{-(\alpha t)^{n\beta}\}.$$

Hence the time to the first failure has a Weibull distribution with β replaced by $n\beta$. The pdf is

$$f_n(t) = \begin{cases} 0 & (t<0) \\ \alpha n\beta(\alpha t)^{n\beta-1}\exp\{-(\alpha t)^{n\beta}\} & (t \geq 0) \end{cases}$$

2.

(a) The distribution function is

$$F(t) = 1-R(t) = 1-e^{-H(t)} \quad \text{so the pdf is} \quad f(t) = \frac{d}{dt}F(t) = h(t)e^{-H(t)}.$$

Hence the hazard function is

$$h(t) = \frac{f(t)}{R(t)} = \frac{h(t)e^{-H(t)}}{e^{-H(t)}} = h(t).$$

(b) A lifetime distribution has hazard function $h(t) = \theta_0 + \theta_1 t + \theta_2 t^2$.

(i) Reliability $R(t)$.

$$\begin{aligned} H(t) &= \left[\theta_0 t + \theta_1 t^2/2 + \theta_2 t^3/3\right]_0^t \\ &= \theta_0 t + \theta_1 t^2/2 + \theta_2 t^3/3 \\ R(t) &= \exp\{-(\theta_0 t + \theta_1 t^2/2 + \theta_2 t^3/3)\}. \end{aligned}$$

(ii) Probability density function $f(t)$.

$$\begin{aligned} F(t) &= 1-R(t) \\ f(t) &= (\theta_0 + \theta_1 t + \theta_2 t^2)\exp\{-(\theta_0 t + \theta_1 t^2/2 + \theta_2 t^3/3)\}. \end{aligned}$$

Answers

(c) A lifetime distribution has hazard function $h(t) = \dfrac{1}{t+1}$.

(i) Reliability function $R(t)$.

$$\begin{aligned} H(t) &= \int_0^t \frac{1}{u+1} du = \Big[\ln(u+1)\Big]_0^t = \ln(t+1) \\ R(t) &= \exp\{-H(t)\} = \frac{1}{t+1} \end{aligned}$$

(ii) Probability density function $f(t)$. $F(t) = 1 - R(t)$, $f(t) = \dfrac{1}{(t+1)^2}$

(iii) Median M. $\dfrac{1}{M+1} = \dfrac{1}{2}$ so $M = 1$.

To find the mean: $\mathsf{E}(T+1) = \displaystyle\int_0^\infty \frac{1}{t+1}\, dt$ which does not converge, so neither does $\mathsf{E}(T)$.

(d) A lifetime distribution has hazard function

$$h(t) = \frac{\rho\theta}{\rho t + 1}, \qquad \text{where } \rho > 0 \text{ and } \theta > 1.$$

(i) Reliability function $R(t)$.

$$\begin{aligned} H(t) &= \int_0^t \frac{\rho\theta}{\rho t + 1}\, dt = \Big[\theta \ln(\rho t + 1)\Big]_0^t = \theta \ln(\rho t + 1) \\ R(t) &= \exp\{-\theta \ln(\rho t + 1)\} = (\rho t + 1)^{-\theta} \end{aligned}$$

(ii) Probability density function $f(t)$.

$$\begin{aligned} F(t) &= 1 - R(t) \\ f(t) &= \theta\rho(\rho t + 1)^{-(1+\theta)} \end{aligned}$$

(iii) Median M.

$$\begin{aligned} (\rho M + 1)^{-\theta} &= 2^{-1} \\ \rho M + 1 &= 2^{1/\theta} \qquad \therefore \qquad M = \rho^{-1}(2^{1/\theta} - 1) \end{aligned}$$

(iv) Mean.

$$\begin{aligned} \mathsf{E}(\rho T + 1) &= \theta\rho \int_0^\infty (\rho t + 1)^{-\theta}\, dt \\ &= \frac{\theta}{\theta - 1}\Big[-(\rho t + 1)^{1-\theta}\Big]_0^\infty = \frac{\theta}{\theta - 1} \\ \mathsf{E}(T) &= \rho^{-1}\left(\frac{\theta}{\theta - 1} - 1\right) = \rho^{-1}(\theta - 1)^{-1} \end{aligned}$$

Answers

3. For each component

$$\begin{aligned} H_i(t) &= \int_0^t h_i(t)\,dt \\ R_i(t) &= \exp\{-H_i(t)\} \end{aligned}$$

For machine

$$R(t) = \prod_{i=1}^{n} R_i(t) = \exp\{-\sum_{i=1}^{n} H_i(t)\}$$

the probability that all components are still working.

$$\begin{aligned} F(t) &= 1 - R(t) \\ f(t) &= \exp\{-\sum_{i=1}^{n} H_i(t)\}\sum_{i=1}^{n} H_i(t) \\ h(t) &= f(t)/R(t) = \sum_{i=1}^{n} h_i(t). \end{aligned}$$

4. For each component

$$\begin{aligned} h_i(t) &= \gamma\rho_i(\rho_i t)^{\gamma-1} \\ H_i(t) &= \int_0^t \gamma\rho_i(\rho_i t)^{\gamma-1}\,dt = (\rho_i t)^{\gamma} \end{aligned}$$

Hence, for the machine,

$$\begin{aligned} \sum_{i=1}^{n} H_i(t) &= t^{\gamma}\sum_{i=1}^{n}\rho_i^{\gamma} \\ R(t) &= \exp\left(-t^{\gamma}\sum_{i=1}^{n}\rho_i^{\gamma}\right) \\ F(t) &= 1 - R(t) \\ f(t) &= \gamma t^{\gamma-1}\sum_{i=1}^{n}\rho_i^{\gamma}\exp\left(-t^{\gamma}\sum_{i=1}^{n}\rho_i^{\gamma}\right) \\ h(t) &= \gamma\rho(\rho t)^{\gamma-1} \end{aligned}$$

so we have a Weibull distribution with index γ and scale parameter ρ such that

$$\rho^{\gamma} = \sum_{i=1}^{n}\rho_i^{\gamma}.$$

Answers

5. Suppose the distribution function is

$$F(t) = 1 - e^{-(\rho t)^\gamma}.$$

Then the pdf is

$$f(t) = \frac{d}{dt}F(t) = \gamma\rho(\rho t)^{\gamma-1}e^{-(\rho t)^\gamma}$$

and the hazard function is

$$H(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)} = \gamma\rho(\rho t)^{\gamma-1}$$

as required.

(a) At the median, M, $F(M) = 0.5$. So

$$\begin{aligned} 1 - e^{-(\rho M)^\gamma} &= 0.5 \\ e^{-(\rho M)^\gamma} &= 0.5 \\ (\rho M)^\gamma &= -\ln(0.5) = \ln(2) \\ \rho M &= (\ln 2)^{1/\gamma} \\ M &= \rho^{-1}(\ln 2)^{1/\gamma} \end{aligned}$$

(b) The mean is

$$\begin{aligned} \mathsf{E}(T) &= \int_0^\infty t\ f(t)\ dt = \int_0^\infty t\ \gamma\rho(\rho t)^{\gamma-1}e^{-(\rho t)^\gamma}\ dt \\ &= \int_0^\infty \gamma(\rho t)^\gamma e^{-(\rho t)^\gamma}\ dt \end{aligned}$$

Let $u = (\rho t)^\gamma$, so $du/dt = \gamma\rho(\rho t)^{\gamma-1}$ and $du = \gamma\rho(\rho t)^{\gamma-1}\ dt$. Then

$$\begin{aligned} \mathsf{E}(T) &= \int_0^\infty \rho^{-1}u^{1/\gamma}e^{-u}\ du \\ &= \rho^{-1}\int_0^\infty u^{1+1/\gamma-1}e^{-u}\ du \\ &= \rho^{-1}\Gamma(1+1/\gamma) \end{aligned}$$

Answers

5. continued

(c) To find the variance we first find

$$\begin{aligned} \mathsf{E}(T^2) &= \int_0^\infty t^2\ f(t)\ dt = \int_0^\infty t^2\ \gamma\rho(\rho t)^{\gamma-1}e^{-(\rho t)^\gamma}\ dt \\ &= \int_0^\infty \gamma\rho^{-1}(\rho t)^{\gamma+1}e^{-(\rho t)^\gamma}\ dt \end{aligned}$$

Let $u = (\rho t)^\gamma$, so $du/dt = \gamma\rho(\rho t)^{\gamma-1}$ and $du = \gamma\rho(\rho t)^{\gamma-1}\ dt$. Then

$$\begin{aligned} \mathsf{E}(T^2) &= \int_0^\infty \rho^{-2}u^{2/\gamma}e^{-u}\ du \\ &= \rho^{-2}\int_0^\infty u^{1+2/\gamma-1}e^{-u}\ du \\ &= \rho^{-2}\Gamma(1+2/\gamma) \end{aligned}$$

So

$$\begin{aligned} \mathsf{V}(T) &= \mathsf{E}(T^2) - \{\mathsf{E}(T)\}^2 \\ &= \rho^{-2}\Gamma(1+2/\gamma) - \rho^{-2}\{\Gamma(1+1/\gamma)\}^2 \\ &= \rho^{-2}\{\Gamma(1+2/\gamma) - \{\Gamma(1+1/\gamma)\}^2\} \end{aligned}$$