Contingency Tables 42.2

Introduction

The practical application of statistics to engineering problems met in industry often concerns making decisions concerning probability distributions. For example you may be asked to decide whether a data set is approximately normal since much of the statistics you may apply makes this assumption. On occasions you may have to make such decisions given data concerning non-numeric variables in the form of a contingency table. Contingency tables are described in detail in this Workbook. This is one of the relatively rare occasions when hypothesis tests can be applied to non-numeric variables.

Prerequisites

Before starting this Section you should ...

- understand thoroughly what is meant by the term degrees of freedom
- have knowledge of the chi-squared distribution described in HELM 40

Learning Outcomes

On completion you should be able to ...

- explain the term contingency table
- perform hypothesis tests involving data given as a contingency table

1. Contingency tables

On occasions, it is possible that the members of a sample taken from a population can be classified by two different methods. Examples of this are:

(a) articles produced by three machines running during two shifts on a production line;

(b) the failure of electronic components and the position in which they are mounted in a machine;

(c) the failure under compression testing of steel-alloy components and the rate of cooling applied during their production.

We can represent the information obtained by observation in such situations in a *contingency table*. By using the observed data to estimate expected data on the assumption that the classification methods are independent, we can use the chi-squared test to investigate the statistical independence (or otherwise) of the classification methods.

Consider the following contingency table with r rows and c columns. Such a table is referred to as an $r \times c$ contingency table.

	1	2	3	...	c	Row Totals
1	O_{11}	O_{12}	O_{13}	...	O_{1c}	R_1
2	O_{21}	O_{22}	O_{23}	...	O_{2c}	R_2
3	O_{31}	O_{32}	O_{33}	...	O_{3c}	R_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮
r	O_{r1}	O_{r2}	O_{r3}	...	O_{rc}	R_r
Column Totals	C_1	C_2	C_3	...	C_c	N

Note that N is the total of the row totals and is the same as the total of the column totals, that is, N is the number of members of the sample taken from a population.

On the basis of the observed data we can estimate the expected frequency, say E_{ij} corresponding to the observed frequency O_{ij}. This is done as follows.

The probability that a randomly chosen element of the sample appears in row class i and column class j is given by p_{ij} where

$$p_{ij} = \frac{R_i}{N} \times \frac{C_j}{N}$$

Hence the required expected frequency is given by E_{ij} which is defined in Key Point 2.

Key Point 2

Expected Frequencies in Contingency Tables

$$E_{ij} = N \times p_{ij} = N \times \frac{R_i}{N} \times \frac{C_j}{N} = \frac{R_i \times C_j}{N}$$

Using this formula repeatedly, we can calculate the expected frequencies corresponding to the observed frequencies and hence calculate a test statistic W where

$$W = \sum_{i=1}^{c} \sum_{j=1}^{r} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

This formula tells you to calculate $\dfrac{(O_{ij} - E_{ij})^2}{E_{ij}}$ for every cell in the contingency table and sum them.

It can be shown that, provided N is large, and none of the expected frequencies are too small, say less than 3, then the quantity

$$W = \sum_{i=1}^{c} \sum_{j=1}^{r} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

follows approximately a chi-squared distribution with $(r-1) \times (c-1)$ degrees of freedom when the null hypothesis is true. This number of degrees of freedom arises since each row has $r-1$ independent entries and each column has $c-1$ independent entries.

Notes

The above statements are correct provided that we can calculate the expected frequencies without knowing the population parameters. If we have to estimate the population parameters, the number of degrees of freedom becomes $(r-1) \times (c-1) - m$ where m is the number of population parameters estimated. In the examples given here we shall not need to estimate the population parameters.

To complete the test procedure we note that the null hypothesis assumes class independence. For example, referring back to Example 2 given at the start of this Section, the null hypothesis would assume that the failure of electronic components and the position in which they are mounted in a machine are independent.

Should the test statistic exceed the critical value of χ^2 read from Table 1 at (say) the 5% level of significance, we would reject the null hypothesis and conclude that a relationship of some kind exists between the classes.

It is worth noting that in some cases (such as the following Example 3) one classification is chosen deliberately but the other is random while in other cases, both classifications are random. The same test applies in both cases.

Example 3

In an experiment to determine the most advantageous position in a machine to mount an electronic component which may be prone to failure due to excessive heat build-up, 300 machines are tested with 100 randomly chosen examples of the component in each of 3 positions. The results obtained were as follows.

Position	1	2	3	Row Totals
Failure	40	30	50	120
Non-failure	60	70	50	180
Column Totals	100	100	100	300

Use a χ^2-test at the 5% level of significance to determine whether component failure is related to mounting position.

Solution

The hypotheses are:

H_0: component failure is independent of position,

H_1: component failure is not independent of position

The expected frequencies are calculated are follows:

$$E_{11} = \frac{120 \times 100}{300} = 40, \quad E_{12} = \frac{120 \times 100}{300} = 40, \quad E_{13} = \frac{120 \times 100}{300} = 40$$

$$E_{21} = \frac{180 \times 100}{300} = 60, \quad E_{22} = \frac{180 \times 100}{300} = 60, \quad E_{23} = \frac{180 \times 100}{300} = 60$$

The test statistic is

$$\begin{aligned} W &= \sum_{i=1}^{3}\sum_{j=1}^{2}\frac{(O_{ij}-E_{ij})^2}{E_{ij}} \\ &= \frac{(40-40)^2}{40}+\frac{(30-40)^2}{40}+\frac{(50-40)^2}{40}+\frac{(60-60)^2}{60}+\frac{(70-60)^2}{60}+\frac{(50-60)^2}{60} \\ &= 0+2.5+2.5+0+1.67+1.67 = 8.34 \end{aligned}$$

and the number of degrees of freedom is $(r-1)\times(c-1) = (2-1)\times(3-1) = 2$ so that the critical value from tables is $\chi^2_{0.05,2} = 5.99$.

Since $5.99 < 8.34$ we reject the null hypothesis and so we should conclude that there is a relationship between component failure and mounting position. Position 2 seems to be the most favourable and position 3 the least.

Washing machines are made on three production lines in a factory. A record is kept of faults reported, during the guarantee period, in machines produced by each of the three lines. The faults are classified into three types A, B and C. The results are given in the table below.

	Fault type			
Production line	A	B	C	Row Totals
1	40	28	34	102
2	27	39	32	98
3	45	26	29	100
Column Totals	112	93	95	300

Use a χ^2-test at the 5% level of significance to determine whether fault type is related to the production line on which the machine was produced.

Your solution

Answer

The hypotheses are:

H_0 : fault type is independent of production line,

H_1 : fault type is not independent of production line

The expected frequencies are calculated are follows:

$$E_{11} = \frac{102 \times 112}{300} = 38.08, \quad E_{12} = \frac{102 \times 93}{300} = 31.62, \quad E_{13} = \frac{102 \times 95}{300} = 32.30$$

$$E_{21} = \frac{98 \times 112}{300} = 36.59, \quad E_{22} = \frac{98 \times 93}{300} = 30.38, \quad E_{23} = \frac{98 \times 95}{300} = 31.03$$

$$E_{31} = \frac{100 \times 112}{300} = 37.30, \quad E_{32} = \frac{100 \times 93}{300} = 31.00, \quad E_{33} = \frac{100 \times 95}{300} = 31.70$$

The test statistic is

$$\begin{aligned} W &= \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \\ &= \frac{(40-38.08)^2}{38.08} + \frac{(28-31.62)^2}{31.62} + \frac{(34-32.30)^2}{32.30} + \frac{(27-36.59)^2}{36.59} + \frac{(39-30.38)^2}{30.38} \\ &+ \frac{(32-31.03)^2}{31.03} + \frac{(45-37.30)^2}{37.30} + \frac{(26-31.00)^2}{31} + \frac{(29-31.70)^2}{31.7} \\ &= 0.097 + 0.414 + 0.089 + 2.512 + 2.446 + 0.030 + 1.590 + 0.806 + 0.230 = 8.214 \end{aligned}$$

and the number of degrees of freedom is $(r-1) \times (c-1) = (3-1) \times (3-1) = 4$ so that the critical value from tables is $\chi^2_{0.05,4} = 9.49$.

Since $8.214 < 9.49$ we do not have sufficient evidence to reject the null hypothesis and so we should conclude that there is no evidence that the distribution of fault types differs between production lines.

Exercises

1. A new compound for the drive belt of domestic vacuum cleaners is tested. Twenty cleaners are fitted with belts made from the new material and twenty are fitted with standard belts. The cleaners are run for a fixed period after which the belts are examined for signs of wear. The numbers showing significant wear are counted. The data are as follows.

	Wear	No wear
Standard	12	8
New compound	6	14

Test the hypothesis that there is no difference between the standard belts and those made with the new compound in terms of the probability of showing wear. Use the 5% level of significance.

2. Electronic devices are made on three production lines. Records are kept of faults found on devices made on each line. Faults are classified as "electronics", "power supply" or "mechanical". The data are as follows.

	Production Line		
	1	2	3
Electronic	13	33	15
Power supply	7	4	11
Mechanical	18	10	14

Test the hypothesis that there is no association between production line and type of fault. Use the 5% level of significance.

Answers

1. Observed frequencies:

	Wear	No wear	Total
Standard	12	8	20
New compound	6	14	20
Total	18	22	40

Expected frequencies: $20 \times 18/40 = 9$, $20 \times 22/40 = 11$.

	Wear	No wear	Total
Standard	9	11	20
New compound	9	11	20
Total	18	22	40

Test statistic

$$W = \sum \frac{(O-E)^2}{E} = \frac{(12-9)^2}{9} + \frac{(8-11)^2}{11} + \frac{(6-9)^2}{9} + \frac{(14-11)^2}{11}$$
$$= 1 + 0.82 + 1 + 0.82 = 3.636$$

Degrees of freedom: $(2-1) \times (2-1) = 1$.

Critical value: $\chi^2_1(5\%) = 3.841$.

The result is not significant at the 5% level. There is insufficient evidence to conclude that there is a difference between the wear rates.

Answers

2. Observed frequencies:

	Production Line 1	2	3	Total
Electronic	13	33	15	61
Power supply	7	4	11	22
Mechanical	18	10	14	42
Total	38	47	40	125

Expected frequencies, e.g. $61 \times 38/125 = 18.544$.

	Production Line 1	2	3	Total
Electronic	18.544	22.936	19.520	61
Power supply	6.688	8.272	7.040	22
Mechanical	12.768	15.792	13.440	42
Total	38.000	47.000	40.000	125

Test statistics

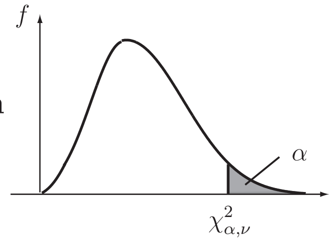
$$W = \sum \frac{(O-E)^2}{E} = \frac{(13-18.544)^2}{18.544} + \cdots + \frac{(14-13.440)^2}{13.440} = 15.860.$$

Degrees of freedom: $(3-1) \times (3-1) = 4$.

Critical value: $\chi^2_4(5\%) = 9.488$.

The test statistic is significant at the 5% level. We reject the null hypothesis and conclude that there is an association between fault type and production line. In particular there seems to be an excess of electronic faults on Line 2.

Table 1: Percentage Points $\chi^2_{\alpha,\nu}$ of the χ^2 distribution



α	0.995	0.990	0.975	0.950	0.900	0.500	0.100	0.050	0.025	0.010	0.005
v											
1	0.00	0.00	0.00	0.00	0.02	0.45	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.01	0.21	1.39	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	2.37	6.25	7.81	9.35	11.34	12.28
4	0.21	0.30	0.48	0.71	1.06	3.36	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	4.35	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	5.35	10.65	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	6.35	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	7.34	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	8.34	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	9.34	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	10.34	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	11.34	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	12.34	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	13.34	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	8.55	14.34	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	15.34	23.54	26.30	28.85	31.00	34.27
17	5.70	6.41	7.56	8.67	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.87	17.34	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.28	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.65
28	12.46	13.57	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	89.33	107.57	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	99.33	118.50	124.34	129.56	135.81	140.17