Forces in More than One Dimension

34.2

Introduction

This Section looks at forces on objects resting or moving on inclined planes and forces on objects moving along curved paths. The previous ideas are exploited in example calculations related to passenger sensations of forces on amusement rides.

Prerequisites

Before starting this Section you should ...

- be able to use vectors and to carry out scalar and vector products
- be able to use Newton's laws to describe and model the motion of particles
- be able to use coordinate geometry to study circles and parabolas
- be able to use calculus to differentiate and integrate polynomials

Learning Outcomes

On completion you should be able to ...

- compute frictional forces on static and moving objects on inclined planes and on objects moving at constant speed around bends
- calculate the forces experienced by passengers in vehicles moving along straight, curved and inclined tracks

1. Forces in two or three dimensions

Forces during circular motion

Consider a particle moving in a horizontal plane so that its position at any time t is given by

$$\underline{r} = r\cos\theta\underline{i} + r\sin\theta\underline{j}$$

where r is a constant and $\underline{i}$ and $\underline{j}$ are unit vectors at right-angles. The angle, θ, made by $\underline{r}$ with the horizontal is a function of time. We can consider four special values of θ and the associated values of $\underline{r}$. These are shown in the following table and in Figure 17.

θ	$\underline{r}$
0	$r\underline{i}$
$\pi/2$	$r\underline{j}$
π	$-r\underline{i}$
$3\pi/2$	$-r\underline{j}$

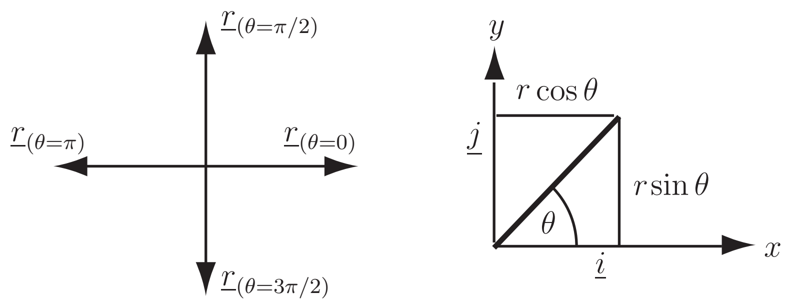


Figure 17

Note that $|\underline{r}| = r$ is a constant for all values of θ, so we must have motion in a circle of radius r. If we assume a constant angular velocity ω rad s^{-1} so that $\theta = \omega t$, then the velocity is

$$\frac{d\underline{r}}{dt} = \omega r(-\sin\omega t\underline{i} + \cos\omega t\underline{j}). \tag{2.1}$$

Hence, taking the dot product,

$$\underline{r}.\frac{d\underline{r}}{dt} = \omega r^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$$

which implies that $\dfrac{d\underline{r}}{dt}$ is always perpendicular to $\underline{r}$. Since $\dfrac{d\underline{r}}{dt}$ is the velocity vector $\underline{v}$, this means that the velocity vector is always tangential to the circle (see Figure 18). Note also that $|\underline{v}| = v = \omega r$, so $\omega = \dfrac{v}{r}$. Differentiating (2.1) again,

$$\frac{d^2\underline{r}}{dt^2} = \omega^2 r(-\cos\omega t\underline{i} - \sin\omega t\underline{j}) = -\omega^2\underline{r}. \tag{2.2}$$

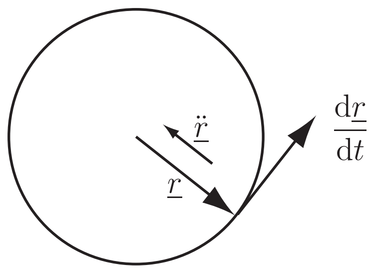


Figure 18: The velocity vector is tangential to motion in a circle

Equation (2.2) means that the second derivative, $\dfrac{d^2\underline{r}}{dt^2}$, which represents the acceleration $\underline{a}$, acts along the radius towards the centre of the circle and is perpendicular to $\dfrac{d\underline{r}}{dt}$.

The **magnitude** of the velocity (the speed) is constant and the acceleration, $\underline{a}$, is associated with the changing **direction** of the velocity. The force must act towards the centre of the circle to achieve this change in direction around the circle. Since $\underline{a}(t) = -\omega^2\underline{r}(t)$, where $\omega = \dfrac{v}{r}$, we see that the acceleration acts towards the centre of the circle and has a magnitude given by $a = \dfrac{v^2}{r}$. This is a special example of the fact that forces in the direction of motion cause changes in speed, while forces at right-angles to the direction of motion cause changes in direction.

When a particle is moving at constant speed around a circle on the end of a rope, then the force directed towards the centre is supplied by the tension in the rope. When a vehicle moves at constant speed around a circular bend in a road, then the force directed towards the centre of the bend is supplied by sideways friction of the tyres with the road. If the vehicle of mass m were to be pushed or dragged sideways by a steady force then it would be necessary to overcome the frictional force. This force depends on the normal reaction $\underline{R}$, which is equal and opposite to the weight of the vehicle (mg). The friction force is given by μmg where μ is the coefficient of friction and it must at least equal the required force towards the centre of the bend to avoid skidding. So, we must have

$$\mu mg \geq \frac{mv^2}{r}. \qquad (2.3)$$

Example 9

A car of mass 900 kg drives around a roundabout of radius 15 m at a constant speed of 10 m s^{-1}.

(a) Draw a vector diagram showing the forces on the car in the vertical and sideways directions.

(b) What is the magnitude of the force directed towards the centre of the bend?

(c) What is the friction force between the car and the road?

(d) What does this imply about the minimum value of the coefficient of friction?

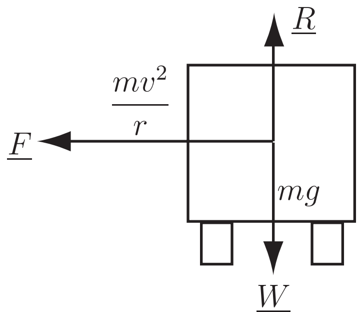


Figure 19: Forces on a vehicle negotiating a circular bend at constant speed

Solution

(a) See Figure 19. (In addition to the sideways friction involved in cornering, there will be a net force causing forward motion which is generated by the vehicle engine and exerted through friction between the tyres and the road.) The forces are shown as if they act at the centre of the vehicle, since the vehicle is being treated as a particle. Strictly speaking, the frictional forces on a road vehicle should be considered to act at the tyre/road contact and there will be differences between the forces at each wheel.

(b) The magnitude of the force is obtained by using

$$\frac{mv^2}{r} = \frac{900 \times 100}{15} = 6000.$$

So the magnitude of the force acting towards the centre of the roundabout is 6000 N.

(c) The sideways force provided by friction is $|\underline{F}| = \mu|\underline{R}|$. In this case $|\underline{R}| = mg$.

(d) Consequently we must have $\mu mg \geq 6000$, that is

$$\mu \geq \frac{6000}{900 \times 9.81} = 0.68.$$

Task

Suppose that the coefficient of friction between the car and the ground in dry conditions is 0.96.

(a) At what speed could the car drive around the roundabout without skidding?

Your solution

Answer

Equation (2.3) on page 36 states that for the vehicle to go round the bend just without skidding

$$\frac{mv^2}{r} = \mu mg, \text{ or } \frac{v^2}{r} = \mu g.$$

In this case, the maximum speed is required, so the relationship is best rearranged into the form $v = \sqrt{\mu g r}$. The values to be substituted are $\mu = 0.96$, $g = 9.81$ and $r = 15$, so

$$v = \sqrt{0.96 \times 9.81 \times 15} = 11.9.$$

So the car will skid if it drives round the roundabout at more than 11.9 m s^{-1} (nearly 27 mph).

(b) What would be the radius of roundabout that would enable a car to drive around it safely in dry conditions at 30 m s^{-1} (nearly 70 mph)?

Your solution

Answer

For this part of the question, the speed around the roundabout is known and the safe radius is to be found. Further rearrangement of the expression used in part (a) gives

$$r = \frac{v^2}{\mu g}.$$

Hence

$$r = \frac{30^2}{0.96 \times 9.81} = 95.6.$$

In reality drivers should not be exactly at the limits of the friction force while going round the roundabout. There should be some safety margin. So the roundabout should have a radius of at least 100 m to allow cars to drive round it at 30 m s^{-1}.

(c) What would be the safe radius of this roundabout when conditions are wet so that the coefficient of friction between the car and the road is reduced by a factor of 2?

Your solution

Answer

The form of the equation used in part (b) indicates that if the coefficient of friction is halved (to 0.48) by wet conditions, then the safe radius should be doubled.

When a cyclist or motorcyclist negotiates a circular bend at constant speed, the forces experienced at the points of contact between the tyres and the road are a frictional force towards the centre of the bend and the upward reaction to the combined weight of the cyclist and the cycle (see Figure 20). These forces can be combined into a resultant that acts along an angle to the vertical. Suppose that the combined mass is m and that the coefficient of friction between the tyres and road surface is μ.

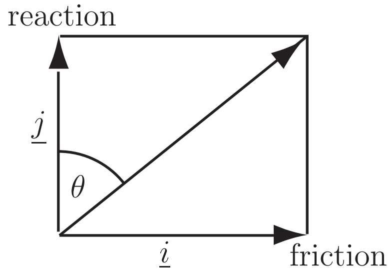


Figure 20

Forces on cycle tyres and the angle for cyclist comfort on a circular bend.
The net driving force is ignored.

The total force vector may be written

$$\underline{F} = mg(\mu\underline{i} + \underline{j})$$

and the angle θ is given by

$$\tan\theta = \frac{\mu mg}{mg} = \mu \qquad \text{so that} \qquad \theta = \tan^{-1}\mu.$$

Also, as argued previously (Equation (2.3)), we must have $\mu \geq \dfrac{v^2}{gr}$, which means that

$$\theta \geq \tan^{-1}\left(\frac{v^2}{gr}\right)$$

To be comfortable while riding, the cyclist likes to feel that the total force is vertical. So when negotiating the bend, the cyclist tilts towards the bend so that the resultant force acts along a 'new vertical'.

Example 10

(a) Calculate the angular velocity of the Earth in radians per second, assuming that the Earth rotates once about its axis in 24 hours.

(b) A synchronous communications satelite is launched into an orbit around the equator and appears to be stationary when viewed from the Earth. Calculate the radius of the satellite's orbit, given that $g = 9.81$ m s^{-2} and that the radius of the Earth is 6.378×10^6 metres.

Solution

(a) The angular velocity of the Earth is

$$\frac{2\pi}{24 \times 60 \times 60} = 7.272 \times 10^{-5} \text{ radians per second.}$$

(b) According to Newtonian theory of gravitation the attraction due to gravity at the Earth's surface, for a mass m, should be $\dfrac{GMm}{R^2}$, which is set equal to mg in elementary calculations. Thus we must have $g = \dfrac{GM}{R^2}$, so that the product GM equals

$$gR^2 = 9.81 \times (6.378)^2 \times 10^{12} = 3.991 \times 10^{14} \text{ m}^3/\text{s}^2.$$

For a circular satellite orbit of radius r, the gravitational force must equal mass times inward acceleration. For a mass m travelling at orbital speed v and with orbital angular velocity ω, the theory of circular orbits gives the result that the inward acceleration is

$$\frac{v^2/r}{\omega^2 r}.$$

The equation 'force equals mass times acceleration' thus gives:

$$\frac{GMm}{r^2} = m\omega^2 r.$$

We wish to ensure that the value of ω for the satellite orbit equals the value of ω for the Earth's rotation. The equation above gives the result: $\quad r^3 = \dfrac{GM}{\omega^2} = \dfrac{gR^2}{\omega^2}$

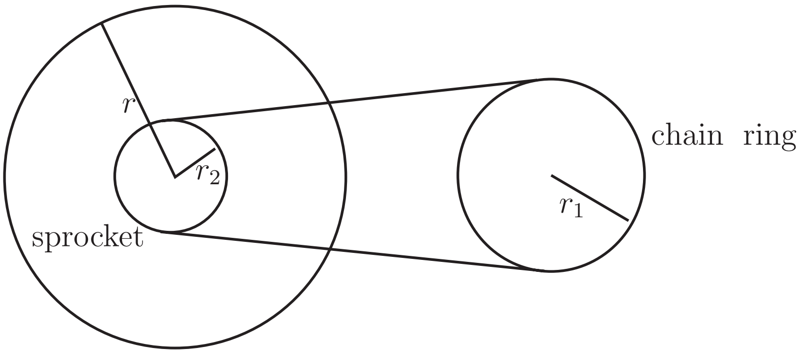
and the value of ω to be used is that which has already been calculated. This gives the result

$$r^3 = \frac{3.991 \times 10^{14}}{(7.272 \times 10^{-5})^2} = 7.547 \times 10^{22} \text{ m}^3$$

Taking the cube root gives $r = 4.23 \times 10^7$ metres. The radius of the satellite orbit is thus about 6.6 times the Earth's radius.

Task

The pedals on a bicycle drive the chain ring, which moves the chain. The chain passes around the sprocket (gear wheel) attached to the rear wheel and hence the rear wheel and the bicycle are driven (see diagram). There are cog teeth of equal width (d) cut into both the chain ring and the sprocket. If there are n_1 teeth in the chain ring and n_2 teeth in the sprocket then n_1/n_2 is the gear ratio. Suppose that the radii of the chain ring, sprocket and rear wheel are $r_1,\ r_2$ and r respectively and that the angular velocities of the chain ring and rear wheel are ω_1 and ω_2 respectively.



Bicycle drive system.

(a) Write down an expression for the velocity (u) of the bicycle in terms of the angular velocity of the rear wheel:

Your solution

Answer

$u = r\omega_2$

(b) Write down the relationship between the velocity of the sprocket and the velocity of the chain ring:

Your solution

Answer

$r_1\omega_1 = r_2\omega_2$

(c) Write down expressions for the circumference of the chain ring and that of the sprocket in terms of the teeth width and number of teeth:

Your solution

Answer

$2\pi r_1 = n_1 d,\ 2\pi r_2 = n_2 d$

(d) Hence derive a relationship between the angular velocities and the gear ratio:

Your solution

Answer

From (b) and (c), $\dfrac{n_1 d}{2\pi}\,\omega_1 = \dfrac{n_2 d}{2\pi}\,\omega_2$ or $n_1\omega_1 = n_2\omega_2$, so gear ratio $= \dfrac{\omega_2}{\omega_1}$

(e) Calculate the speed of the bicycle if the cyclist is pedalling at one revolution per second, the radius of the rear wheel is 0.34 m and the gear ratio is 4:

Your solution

Answer

From (a) and (d), $u = r(n_1/n_2)\omega_1 = 0.34(4)(2\pi) = 8.55$ m s^{-1} (about 19 mph).

2. Amusement rides

The design of amusement rides is intended to make the forces experienced by passengers as exciting as possible. High speeds are not enough. The production of accelerations of up to four times that due to gravity and occasional feelings of near-weightlessness are deliberate design goals. The accelerations may not only be in the forward or backward directions from the passengers' perspective but also sideways. Sideways accelerations are more limited than backwards or forwards ones, since they are less welcome to passengers and also pose particular problems for the associated structures. Upward accelerations of greater than g are avoided because of the safety risk and the need for reliable constraints to prevent passengers 'floating' out of the carriages. The rates at which the forces, or accelerations, change are important in producing the overall sensation also. The rate of change of acceleration is called **jerk**. Even the rate of change of jerk, called **jounce**, may be of interest. On roller-coasters the height, the tightness and the twistiness of turns in the track are the main parameters that influence the thrill of riding on them. Another factor that relates to thrill or discomfort is the degree of mismatch between what is observed and what is felt. For example it is odd to feel as though one's weight is acting in some other direction than the perceived vertical. In this regard the *magnitude* of the force that is experienced may not be as important as its *direction*. Although we consider only two dimensions in this Workbook, the design of roller-coasters requires calculations and considerations in 3D. We shall consider some particular examples of forces experienced by passengers. We start by exploring the forces that contribute to what we feel when riding in a vehicle. Then we shall look at the forces experienced by passengers on amusement rides ranging from rotors to roller-coasters.

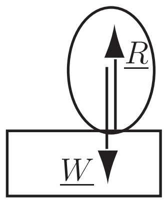


Figure 21: Forces on a seated passenger

Fearsome forces

The experiences we have on amusement rides include those of being subjected to linear and sideways accelerations that are sensed by the balancing system close to our ears and can make us feel giddy. Potentially more 'enjoyable' are sensations of unfamiliar compressive forces that act on our bodies through the vehicle in which we are travelling.

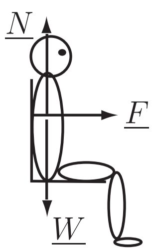


Figure 22: Forces when seated and being accelerated horizontally

Imagine that you are sitting on a stool, which is sufficiently tall so that your feet are not touching the ground. What do you feel? Your weight is acting vertically downward. However you feel an upward force, which is the normal reaction of the stool to your weight. This reaction is pushing you upward (see Figure 21). Of course the total force on you is zero. Consequently there is a difference in this case between the total force on you, which includes gravity, and the force that you experience which excludes gravity.

On the other hand if you are sitting facing forwards in a vehicle that is accelerating forwards on a flat track then you will experience the same acceleration as the vehicle, through the seat which pushes you forwards (see Figure 22). The forward force from the seat combines with the normal reaction to give a resultant that is not vertical. This simple example suggests that the force experienced by a passenger during an amusement ride can be calculated by adding up all the component forces except for the passenger's weight.

Example 11

(a) Calculate the horizontal and vertical components of the force $\underline{F}$ experienced by a passenger of mass 100 kg seated in a rollercoaster carriage that starts from rest and moves in a straight line on a flat horizontal track with a constant acceleration such that it is moving at 40 m s^{-1} after 5 s.

(b) Deduce the magnitude and direction of the force experienced by the passenger.

Solution

(a) Consider first the acceleration of the passenger's seat. The coordinate origin is chosen at the start of motion. The x-axis is chosen along the direction of travel, with unit vector $\underline{i}$, and the y-axis is vertical with y positive in the upward direction, with unit vector $\underline{j}$. The acceleration $\underline{a}$ may be calculated from

$$\underline{a} = \frac{d\underline{v}}{dt} = a\underline{i}$$

or $\underline{v} = at\underline{i} + \underline{c}$. When $t = 0, \underline{v} = \underline{0}$, so $\underline{c} = \underline{0}$. When $t = 5, \underline{v} = 40\underline{i}$. Hence $40 = 5a$ or $a = 8$. The acceleration in the direction of travel is 8 m s^{-2}, so the component of the force F in the direction of travel is given by $ma = 100 \times 8 \text{ N} = 800$ N. The seat exerts a force on the passenger that balances the force due to gravity i.e. the passenger's weight. So the vertical component of the force on the passenger is $mg = 100 \times 9.81 \text{ N} = 981$ N. Hence the total force experienced by the passenger may be expressed as

$$\underline{F} = 800\underline{i} + 981\underline{j}.$$

This is the total force exerted by the vehicle on the passenger. The force exerted by the passenger on the vehicle is the direct opposite of this i.e. $-800\underline{i} - 981\underline{j}$.

(b) The magnitude of the total force is $\sqrt{800^2 + 981^2}$ N ≈ 1300 N and the total force experienced by the passenger is at an angle with respect to the horizontal equal to $\tan^{-1}(981/800) \approx 51°$.

Here we considered the sudden application of a constant acceleration of 8 m s^{-2} which will cause quite a jerk for the passengers at the start. On some rides the acceleration may be applied more smoothly.

Example 12

Calculate the total force experienced by the passenger as a function of time if the horizontal component of acceleration of the vehicle is given by

$$\begin{cases} 8\sin\left(\dfrac{\pi t}{10}\right) & 0 \leq t \leq 5 \\ 8 & t > 5 \end{cases}$$

Solution

With this horizontal component of acceleration, the component of force experienced by the passenger in the direction of motion is (see Figure 23)

$$\begin{cases} 800\sin\left(\dfrac{\pi t}{10}\right) & \text{for} \quad 0 \leq t \leq 5 \\ 800 & \text{for} \quad t > 5 \end{cases}$$

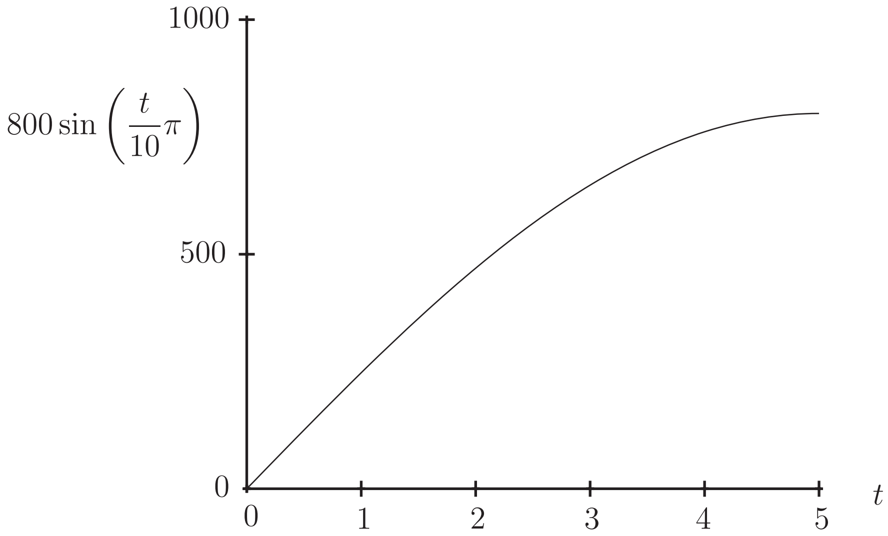


Figure 23: Horizontal component of force

The vertical component remains constant at 981 N, as in Example 11. The total force may be written

$$\underline{F} = \begin{cases} 800\sin\left(\dfrac{\pi t}{10}\right)\underline{i} + 981\underline{j} & 0 \leq t \leq 5 \\ 800\underline{i} + 981\underline{j} & t > 5 \end{cases}$$

Task

The idea of an amusement ride called the 'Rotor' is to whirl passengers around in a cylindrical container at increasing speed. When the rotation is sufficiently fast the floor is lowered but the passengers remain where they are supported by friction against the wall. Given a rotor radius of 2.2 m and a coefficient of friction of 0.4 calculate the minimum rate of revolution when the floor may be lowered.

Your solution

Answer

The reaction of the wall on the passenger will have the same magnitude as the force causing motion in a circle i.e. $\dfrac{mv^2}{r} = R$. The vertical friction force between the passenger and the wall is μR. The passenger of mass will remain against the wall when the floor is lowered if $\mu R \geq mg$. Hence it is required that $\dfrac{\mu m v^2}{r} \geq mg$ or $v \geq \sqrt{\dfrac{rg}{\mu}}$. Since $v = \omega r$, where ω is the angular velocity, the minimum required angular velocity is $\omega = \sqrt{\dfrac{g}{\mu r}}$ and the corresponding minimum rate of revolution $n = \dfrac{\omega}{2\pi}\sqrt{\dfrac{g}{\mu r}}$. Hence with $r = 2.2$ and $\mu = 0.4$, the rate of revolution must be at least 0.53 revs per sec or at least 32 revs per min.

Task

In an amusement ride called the 'Yankee Flyer', the passengers sit in a 'boat', which stays horizontal while executing a series of rotations on an arm about a fixed centre. Given that the period of rotation is 2.75 s, calculate the radius of rotation that will give rise to a feeling of near weightlessness at the top of each rotation.

Your solution

Answer

To achieve a feeling of 'near-weightlessness' near the top of the rotation, the force on the passenger towards the centre of rotation must be nearly equal and opposite to the reaction force of the seat on the passengers i.e. $\dfrac{mv^2}{r} = mg$. This means that $r = \dfrac{v^2}{g}$. Since $v = \omega r$, this requires that $r = \dfrac{g}{\omega^2}$. The period $T = \dfrac{2\pi}{\omega} = 2.75$ s, so $\omega = 2.285$ rad s^{-1}. Hence $r = 1.879$ m.

Example 13

An amusement ride carriage moves along a track at constant velocity in the horizontal (x-) direction. It encounters a bump of horizontal length L and maximum height h with a profile in the vertical plane given by

$$y(x) = \frac{h}{2}\left(1 - \cos\frac{2\pi x}{L}\right), \quad 0 \leq x \leq L.$$

Calculate the variation of the vertical component of force exerted on a passenger by the seat of the carriage with horizontal distance (x) as it moves over the bump.

Solution

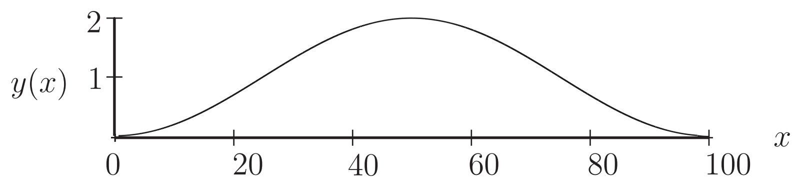


Figure 24: Profile of the bump

Figure 24 shows a graph of $y(x)$ against x for $h = 2$ and $L = 100$. Since the component of velocity of the vehicle in the x-direction is constant, then after time t, the horizontal distance moved, x, is given by $x = ut$ as long as x is measured from the location at $t = 0$. Consequently the y-coordinate may be written in terms of t rather than x, giving

$$y(t) = \frac{h}{2}\left(1 - \cos\frac{2\pi ut}{L}\right), \quad 0 \leq t \leq \frac{L}{u}.$$

The vertical component of velocity is given by differentiating this expression for $y(t)$ with respect to t.

$$\dot{y}(t) = \frac{h}{2}\frac{2\pi u}{L}\sin\left(\frac{2\pi ut}{L}\right) = \frac{h\pi u}{L}\sin\left(\frac{2\pi ut}{L}\right).$$

The vertical acceleration is given by differentiating this again.

$$\ddot{y}(t) = \frac{h\pi u}{L}\frac{2\pi u}{L}\cos\left(\frac{2\pi ut}{L}\right) = \frac{2h\pi^2 u^2}{L^2}\cos\left(\frac{2\pi ut}{L}\right).$$

Two forces contribute to the vertical force exerted by the seat on the passenger: the constant reaction to the passenger's weight and the variable vertical reaction associated with motion over the bump. The magnitude of the total vertical force R N exerted on the passenger by the seat is given by

$$R = mg + m\ddot{y}(t) = mg + m\frac{2h\pi^2 u^2}{L^2}\cos\left(\frac{2\pi ut}{L}\right) = mg\left(1 + \frac{2h\pi^2 u^2}{gL^2}\cos\left(\frac{2\pi ut}{L}\right)\right).$$

Some horizontal force may be needed to keep the vehicle moving with a constant horizontal component of velocity and ensure that the net horizontal component of acceleration is zero. Since the horizontal component of velocity is constant, there is no horizontal component of acceleration and no net horizontal component of force. Consequently $\underline{R} = R\underline{j}$ represents the total force exerted on the passenger. Figure 25 shows a graph of R/mg for $h = 2$ m, $L = 100$ m and $u = 20$ m s^{-1}.

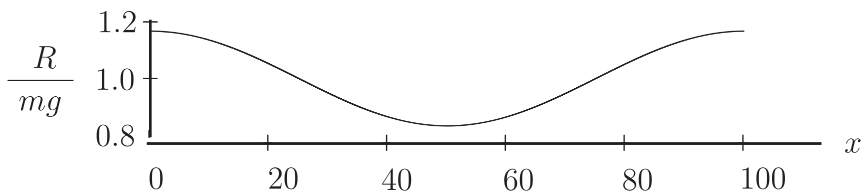
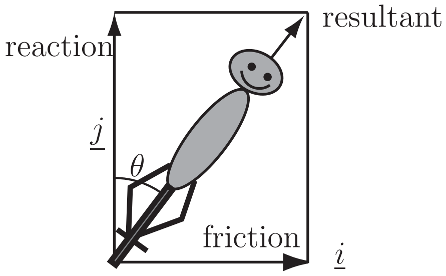


Figure 25: Vertical force acting on passenger

Banked tracks

Look back near the end of Section 34.2 subsection 1 which considered the forces on a cyclist travelling around a circular bend of radius R. We were concerned with the way in which cyclists and motorcyclists bank their vehicles to create a 'new vertical' along the direction of the resultant force. This counteracts the torque that would otherwise encourage the rider to fall over when cornering. Clearly, passengers in four wheeled vehicles, railway trains and amusement park rides are not able to bank or tilt their vehicles to any significant extent. However what happens if the road or track is banked instead? If the road or track is tilted or banked at angle $\theta = \tan^{-1}\left(\dfrac{v^2}{gr}\right)$ to the horizontal, then, at speed v around the circular bend, it is possible to obtain the same result as that achieved by tilting the cycle or motorcycle (see Figure 26).



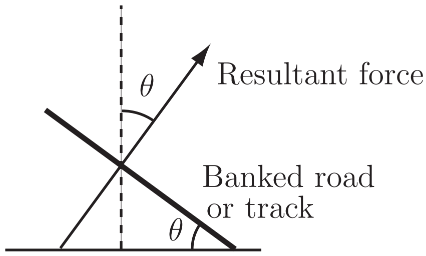


Figure 26: Equivalence of tilted cyclists and banked roads

Example 14

Calculate the angle at which a track should be tilted so that passengers in a railway carriage moving at a constant speed of 20 m s^{-1} around a bend of radius 100 m feel the resultant 'reaction' force as though it were acting vertically through their centre line.

Solution

The angle of the resultant force on passengers if the track were horizontal is given by

$$\theta = \tan^{-1}\left(\frac{v^2}{gr}\right),$$

where $v = 20, r = 100$ and $g = 9.81$.

Hence $\theta = 22.18°$ and the track should be tilted at $22.18°$ to the horizontal for the resultant force to act at right-angles to the track.

Example 15

The first three seconds of an amusement ride are described by the position coordinates

$$x(t) = 10 - 10\cos(0.5t) \qquad (0 < t < 3)$$
$$y(t) = 20\sin(0.5t)$$

in the horizontal plane where x and y are in m.

(a) Calculate the velocity and acceleration vectors.

(b) Hence deduce the initial magnitude and direction of the acceleration and the magnitude and direction of the acceleration after three seconds.

Solution

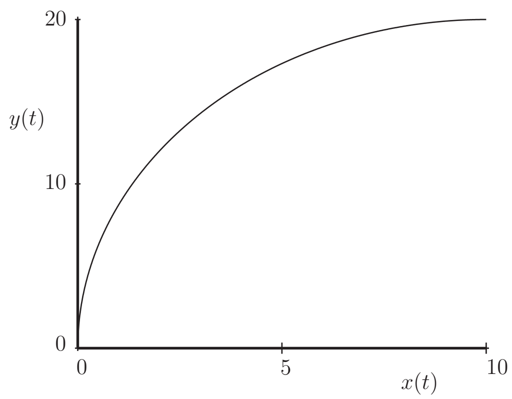


Figure 27: Path of ride

In this Example, the path of the ride is not circular (see Figure 27). In fact it is part of an ellipse. If we choose unit vectors $\underline{i}$ along the x-direction and $\underline{j}$ along the y-direction, and origin at $t = 0$, the position vector may be written

$$\underline{r}(t) = (10 - 10\cos(0.5t))\underline{i} + 20\sin(0.5t)\underline{j}.$$

The velocity vector is obtained by differentiating this with respect to time.

$$\underline{v}(t) = 5\sin(0.5t)\underline{i} + 10\cos(0.5t)\underline{j}.$$

The acceleration vector is obtained by differentiating again.

$$\underline{a}(t) = 2.5\cos(0.5t)\underline{i} - 5\sin(0.5t)\underline{j}$$

At $t = 0, \underline{a}(t) = 2.5\underline{i}$. So the initial acceleration is 2.5 m s^{-2} in the x-direction.

At $t = 3, \underline{a}(t) = 2.5\cos(1.5)\underline{i} - 5\sin(1.5)\underline{j} = 0.177\underline{i} - 0.487\underline{j}$.

So after three seconds the acceleration is 4.99 m s^{-2} at an angle of 88° in the negative y-direction. This means a sideways acceleration of about $0.5g$ towards the inside of the track and almost at right-angles to it.

Engineering Example 1

Car velocity on a bend

Problem in words

A road has a bend with radius of curvature 100 m. The road is banked at an angle of 10°. At what speed should a car take the bend in order not to experience any (net) side thrust on the tyres?

Mathematical statement of the problem

Figure 28 below shows the forces on the car.

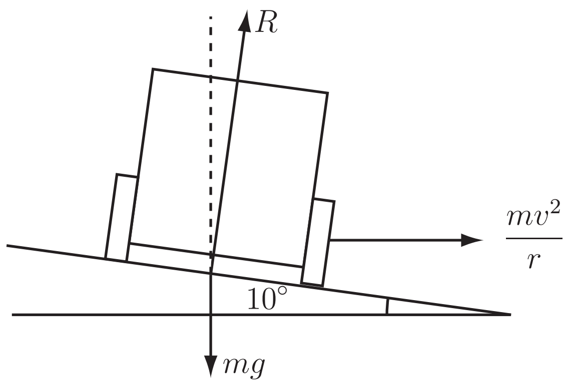


Figure 28: A vehicle rounding a banked bend in the road.

In the figure R is the reactive force of the ground acting on the vehicle. The vehicle provides a force of mg, the weight of the vehicle, operating vertically downwards. The vehicle needs a sideways force of $\dfrac{mv^2}{r}$ in order to maintain the locally circular motion.

We have used the following assumptions:

(a) The sideways force needed on the vehicle in order to maintain it in circular motion (called the centripetal force) is $\dfrac{mv^2}{r}$ where r is the radius of curvature of the bend, v is the velocity and m the mass of the vehicle.

(b) The only force with component acting sideways on the vehicle is the reactive force of the ground. This acts in a direction normal to the ground. (That is, we assume no frictional force in a sideways direction.)

(c) The force due to gravity of the vehicle is mg, where m is the mass of the vehicle and g is the acceleration due to gravity ($\approx$ 9.8 m s^{-2}). This acts vertically downwards.

The problem we need to solve is 'What value of v would be such that the component of the reactive force of the ground exactly balances the sideways force of $\dfrac{mv^2}{r}$?' This will give us the maximum velocity at which the vehicle can take the bend.

Mathematical analysis

We can split the reactive force of the ground into two components. One component is in the horizontal direction and the other in a vertical direction as in the following figure:

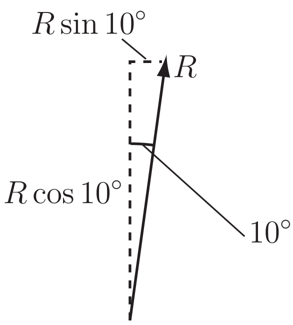


Figure 29: Reaction forces on the car

The force of $\dfrac{mv^2}{r}$ must be provided by a component of the reactive force in the horizontal direction i.e.

$$R\sin(10°) = \frac{mv^2}{r} \qquad (1)$$

However the reactive force must balance the force due to gravity in the vertical direction therefore

$$R\cos(10°) = mg \qquad (2)$$

We need to find v from the above equations. Dividing Equation (1) by Equation (2) gives

$$\tan(10°) = \frac{v^2}{rg} \Rightarrow v^2 = rg\tan(10°)$$

We are given that the radius of curvature is 100 m and that $g \approx 9.8$ m s^{-2}. This gives

$v^2 \approx 100 \times 9.8 \times 0.17633$

$\Rightarrow \quad v^2 \approx 172.8$

$\Rightarrow \quad v \approx 13.15$ m s^{-1} (assuming v is positive)

Interpretation

We have found that the maximum speed that the car can take the bend in order not to experience any side thrust on the tyres is 13.15 m s^{-1}. This is $13.15 \times 60 \times 60/1000$ kph $= 47.34$ kph. In practice, the need for a margin of safety would suggest that the maximum speed round the bend should be 13 m s^{-1}.

Exercises

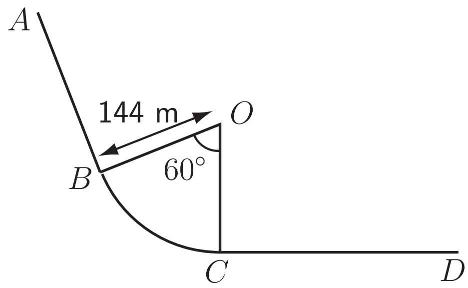
1. A bend on a stretch of railway track has a radius of 200 m. The maximum sideways force on the train on this bend must not exceed 0.1 of its weight.

   (a) What is the maximum possible speed of the train on this bend?

   (b) How far before this bend should a train travelling at 30 m s^{-1} begin to decelarate given that the maximum braking force of the train is 0.2 of its weight?

   (c) What modelling assumptions have you made? Comment on their validity.

2. The diagram shows a portion of track of a one-way fairground ride on which several trains are to run. AB and CD are straight. BC is a circular arc with the dimension shown. Because BC is also on a bridge, safety regulations require that the rear of one train must have passed point C before the front of the next train passes point B. Trains are 30 m long.



   If the maximum sideways force on a train can be no more than 0.1 of its weight, find the shortest time it can take for a train to travel from B to C. Hence find the minimum time between the front of one train passing point B and its rear end passing point C. Recommend a minimum distance between trains.

Answers

1. (a) According to Equation (2.2) on page 35, during travel round the bend the sideways force on the train is given by

$$Mr\omega^2 = \frac{Mv^2}{r}$$

The weight of the train is Mg. Given that $\dfrac{Mv^2}{r} \leq 0.1Mg$, the maximum possible speed, $v_{\max}$, is given by $v_{\max} = \sqrt{0.1rg}$.

Using $r = 200$ m and $g = 9.81$ m s^{-2}, this implies that $v_{\max} = 14.0$ m s^{-1} to 3 s.f.

(b) Given initial speed is 30 m s^{-1}, and final speed is 14 m s^{-1} and maximum braking force is 0.2 Mg, implying acceleration is $-0.2g$. Then, using the formula '$v^2 = u^2 + 2as$', where u is initial speed, v its final speed, a is acceleration and s is distance travelled, gives

$$20g = 900 - 0.4gs \qquad \text{or} \qquad s = 179.358 \text{ m}.$$

This suggests that braking should begin about 180 m before the start of the bend.

(c) Assumptions include constant maximum braking, negligible thinking time and no skidding.

2. The shortest time on the circular bend will be taken when the train is moving at the maximum possible speed.

This will occur when $\dfrac{Mv^2}{r} = 0.1Mg$. If $r = 144$ m and $g = 9.81$ m s^{-2}, this implies $v_{\max} = 11.885$ m s^{-1}.

The length of BC is $144\dfrac{\pi}{3} = 150.796$ m.

The time taken for any point on the train to move from B to C is $\dfrac{150.796}{11.885} = 12.687$ s.

So, given that the length of each train is 30 m, to make sure that the rear of one train has passed C before the front of the next train arrives at B, a minimum time between the trains of $(12.687 + \dfrac{30}{11.885})$ s $= 15.212$ s is required. After including a small safety margin, each train should be 16 s apart. Assuming that the trains are moving at a constant speed of 11.885 m s^{-1}, this implies that they should be at least 190 m apart.