Hyperbolic PDEs 32.5

Introduction

In the preceding Section we looked at parabolic partial differential equations. Another class of PDE modelling initial value problems are of the hyperbolic type.

In this Section we will concentrate on the wave equation, which was introduced in HELM 25.

Prerequisites

Before starting this Section you should ...

- revise those aspects of HELM 25 which deal with the wave equation
- familiarise yourself with difference methods for approximating first and second derivatives
- be familiar with the numerical methods used for parabolic equations

Learning Outcomes

On completion you should be able to ...

- obtain simple numerical solutions of the wave equation

1. The (one-dimensional) wave equation

The wave equation is a PDE which (as its name suggests) models wave-like phenomena. It is a model of waves on water, of sound waves, of waves of reactant in chemical reactions and so on. For the purposes of most of the following examples we may think of the application in hand as that of being a length of string tightly stretched between two points. Let $u = u(x,t)$ be the displacement from rest of the string at time t and distance x from one end. Oscillations in the string may be modelled by the **wave equation**

$$u_{tt} = c^2 u_{xx} \qquad (0 < x < \ell, \quad t > 0)$$

where ℓ is the length of the string, $t = 0$ is some initial time and $c > 0$ is a constant (the **wave speed**) dependent on the material properties of the string. (Further discussion of the constant c is given in HELM 25.2.)

The wave equation is hyperbolic, as we can readily verify on recalling the definitions at the beginning of Section 32.4. Extra information is needed to specify the initial value problem. The initial position and initial velocity are given as

$$\left.\begin{array}{rcl} u(x,0) & = & f(x) \\ u_t(x,0) & = & g(x) \end{array}\right\} \quad 0 \le x \le \ell$$

Finally, we need boundary conditions specifying how the ends of the string are held. For example

$$u(0,t) = u(\ell,t) = 0 \qquad (t > 0)$$

models the situation where the string is fixed at each end.
(We will suppose that $f(0) = f(\ell) = 0$ so that there is no apparent conflict at the ends of the string at the initial time.)

2. Numerical solutions

The approach we will adopt is similar to that seen in Section 32.4 where we looked at parabolic equations. We use the notation

$$u_j^n$$

to denote an approximation to u evaluated at $x = j \times \delta x$, $t = n \times \delta t$. Approximating the derivatives in the PDE

$$u_{tt} = c^2 u_x$$

by central differences we obtain the numerical difference equation

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\delta t)^2} = c^2 \ \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}.$$

Multiplying through by $(\delta t)^2$ this can be rearranged to give

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

in which $\mu = \dfrac{c\delta t}{\delta x}$ is called the **Courant number.**

The equation above gives u_j^{n+1} in terms of u-approximations at earlier time-steps (that is, all the appearances of u on the right-hand side have a superscript smaller than $n + 1$).

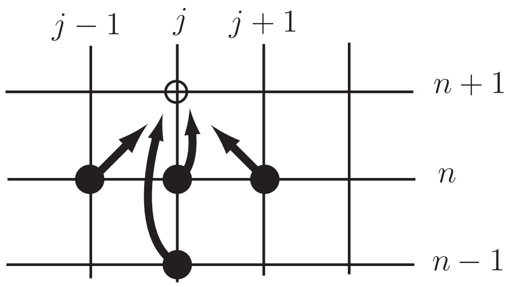

Figure 9

Thinking of the numerical stencil graphically we have the situation shown above. We may think of the values on the right-hand side of the equation "pointing to" a new value on the left-hand side.

Key Point 22

Timesteps (other than the first one) are carried out by using the numerical stencil

$$\underbrace{u_j^{n+1}}_{\substack{\uparrow \\ \text{"new" approximation} \\ \text{at } (n+1)^{\text{th}} \text{ time-step}}} = \underbrace{2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}_{\substack{\uparrow \\ \text{"old" approximations at} \\ \text{earlier time-steps}}}$$

(We will deal with how to carry out the *first* time-step shortly.)

The time-stepping process has much in common with the corresponding procedure for parabolic problems. The following Example will help establish the general idea.

Example 18

Given that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 1$ with boundary conditions $u(0,t) = u(1,t) = 0$ $(t > 0)$ with wave speed $c = 1.2$.
The numerical method $u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$ where $\mu = c\ \delta t/\delta x$, is implemented using $\delta x = 0.25$ and $\delta t = 0.1$.
Suppose that, after 5 time-steps, the following data forms part of the numerical solution:

$$\begin{array}{lcl}
u_0^4 &=& 0.0000 \\
u_1^4 &=& 0.9242 \\
u_2^4 &=& -0.0020 \\
u_3^4 &=& -0.9624 \\
u_4^4 &=& 0.0000
\end{array}
\qquad
\begin{array}{lcl}
u_0^5 &=& 0.0000 \\
u_1^5 &=& 0.7110 \\
u_2^5 &=& -0.0059 \\
u_3^5 &=& -0.7409 \\
u_4^5 &=& 0.0000
\end{array}$$

Carry out the next time-step so as to find an approximation to u at $t = 6\delta t$.

Solution

In this case $\mu = 1.2 \times 0.1/0.25 = 0.48$ and the required time-step is carried out as follows:

$$\begin{array}{rcl}
u_0^6 &=& 0 \quad \text{from the boundary condition} \\
u_1^6 = 2u_1^5 - u_1^4 + \mu^2(u_2^5 - 2u_1^5 + u_0^5) &=& -0.1689 \\
u_2^6 = 2u_2^5 - u_2^4 + \mu^2(u_3^5 - 2u_2^5 + u_1^5) &=& -0.0140 \\
u_3^6 = 2u_3^5 - u_3^4 + \mu^2(u_4^5 - 2u_3^5 + u_2^5) &=& -0.1794 \\
u_4^6 &=& 0 \quad \text{from the boundary condition}
\end{array}$$

to 4 decimal places and these are the approximations to $u(0, 6\delta t)$, $u(0.25, 6\delta t)$, $u(0.5, 6\delta t)$, $u(0.75, 6\delta t)$ and $u(1, 6\delta t)$, respectively.

The diagram below shows the numerical results that appeared in the example above. It can be seen that the example was a (rather coarse) model of a standing wave with two antinodes.

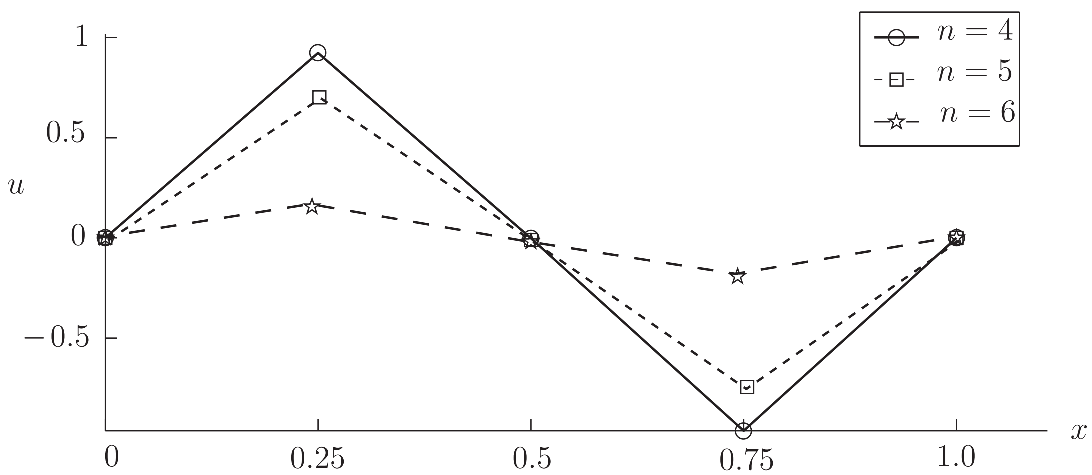

Figure 10

Task

Suppose that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 1$. It is given that u satisfies boundary conditions $u(0,t) = u(1,t) = 0$ $(t > 0)$ and initial conditions that need not be stated for the purposes of this question. The application is such that the wave speed $c = 1.2$.
The numerical method $u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$ where $\mu = c\ \delta t/\delta x$, is implemented using $\delta x = 0.25$ and $\delta t = 0.2$.
Suppose that, after 8 time-steps, the following data forms part of the numerical solution:

$$\begin{array}{rclcrcl} u_0^7 &=& 0.0000 & \quad & u_0^8 &=& 0.0000 \\ u_1^7 &=& 0.6423 & & u_1^8 &=& 0.4640 \\ u_2^7 &=& 0.8976 & & u_2^8 &=& 0.6792 \\ u_3^7 &=& 0.6789 & & u_3^8 &=& 0.4668 \\ u_4^7 &=& 0.0000 & & u_4^8 &=& 0.0000 \end{array}$$

Carry out the next time-step so as to find an approximation to u at $t = 9\delta t$.

Your solution

Answer

In this case $\mu = 1.2 \times 0.2/0.25 = 0.96$ and the required time-step is carried out as follows:

$$\begin{array}{rcl} u_0^9 &=& 0 \quad \text{from the boundary condition} \\ u_1^9 = 2u_1^8 - u_1^7 + \mu^2(u_2^8 - 2u_1^8 + u_0^8) &=& 0.0564 \\ u_2^9 = 2u_2^8 - u_2^7 + \mu^2(u_3^8 - 2u_2^8 + u_1^8) &=& 0.0667 \\ u_3^9 = 2u_3^8 - u_3^7 + \mu^2(u_4^8 - 2u_3^8 + u_2^8) &=& 0.0202 \\ u_4^9 &=& 0 \quad \text{from the boundary condition} \end{array}$$

to 4 decimal places and these are the approximations to $u(0, 9\delta t)$, $u(0.25, 9\delta t)$, $u(0.5, 9\delta t)$, $u(0.75, 9\delta t)$ and $u(1, 9\delta t)$, respectively.

The above Task concerns a stretched string oscillating in such a way that at the 9$^{\text{th}}$ time-step the string is approximately flat. The motion continues with u taking negative values. Figure 11 below uses data calculated above, and also data for the next two time-steps so as to show subsequent progress of the solution.

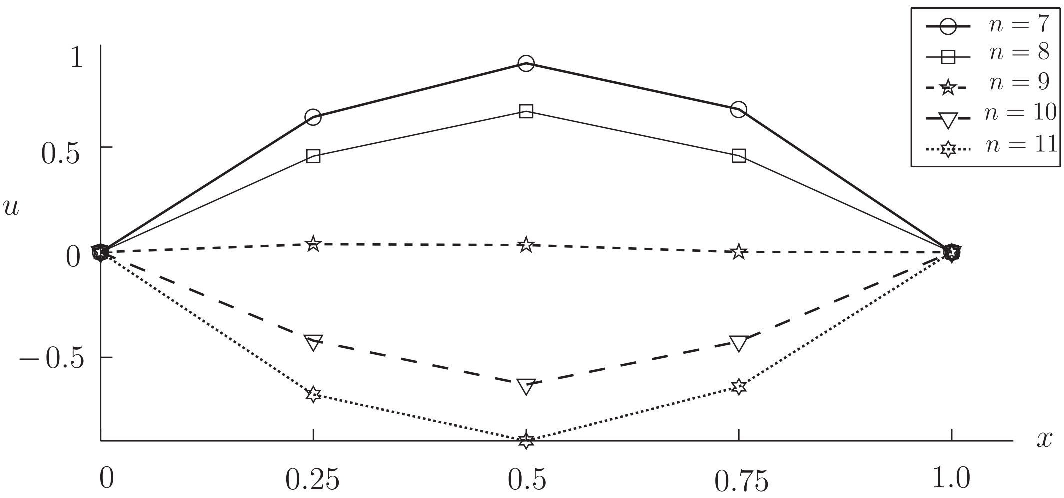

Figure 11

3. The first time-step

In the Example and Task above we have seen how time-steps can be carried out using the numerical stencil

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

but there remains one issue which, so far, we have neglected. How do we carry out the *first* time-step?

Initial conditions

The initial time-step must use information from the two initial conditions

$$\left.\begin{array}{rcl} u(x,0) & = & f(x) \\ u_t(x,0) & = & g(x) \end{array}\right\} \quad 0 \leq x \leq \ell$$

The first initial condition is easy enough to interpret. It gives u_j^n in the case where $n = 0$. In fact

$$u_j^0 = f_j$$

where f_j is simply shorthand for $f(j \times \delta x)$.

The second initial condition, the one involving g, gives information about $u_t = \dfrac{\partial u}{\partial t}$ at $t = 0$. We can approximate the t-derivative of u at $t = 0$ and $x = j \times \delta x$ by a central difference to write

$$\frac{u_j^1 - u_j^{-1}}{2\delta t} = g_j$$

in which g_j is shorthand for $g(j \times \delta x)$.

This last expression involves u_j^{-1} which, if it has a meaning at all, refers to u at time $t = -\delta t$, that is, *before* the initial time $t = 0$. One way to think of u_j^{-1} is simply as an artificial quantity which proves useful later on. The equation above, rearranged for u_j^{-1} is

$$u_j^{-1} = u_j^1 - 2\delta t \times g_j$$

Key Point 23

A central difference used to approximate the first derivative in the condition defining initial speed gives rise to the following useful equation

$$u_j^{-1} = u_j^1 - 2\delta t \times g_j$$

The first time-step

To carry out the first time-step we put $n = 0$ in the numerical stencil

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right),$$

to give

$$u_j^1 = 2u_j^0 - u_j^{-1} + \mu^2 \left(u_{j+1}^0 - 2u_j^0 + u_{j-1}^0 \right).$$

Those terms on the right-hand side with a 0 superscript are known *via* the function f since we know that $u_j^0 = f_j$. Hence

$$u_j^1 = 2f_j - u_j^{-1} + \mu^2 \left(f_{j+1} - 2f_j + f_{j-1} \right).$$

And the u_j^{-1} term is dealt with using the Key Point above to give

$$u_j^1 = 2f_j - u_j^1 + 2\delta t \times g_j + \mu^2 \left(f_{j+1} - 2f_j + f_{j-1} \right).$$

and therefore, moving the latest appearance of u_j^1 over to the left-hand side and dividing by 2,

$$\begin{aligned} u_j^1 &= f_j + \delta t \times g_j + \tfrac{1}{2}\mu^2 \left(f_{j+1} - 2f_j + f_{j-1} \right) \\ &= \tfrac{1}{2}\mu^2 \left(f_{j-1} + f_{j+1} \right) + (1 - \mu^2) f_j + \delta t \times g_j \end{aligned}$$

Key Point 24

The first time-step is carried out by using the initial data and can be summarised as

$$u_j^1 = \tfrac{1}{2}\mu^2 \left(f_{j-1} + f_{j+1} \right) + (1 - \mu^2) f_j + \delta t \times g_j$$

Example 19

Suppose that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 1$. It is given that u satisfies boundary conditions $u(0,t) = u(1,t) = 0$ $(t > 0)$ and initial conditions that may be summarised as

$$\begin{array}{lllllll} f_0 & = & 0.0000 & \qquad & g_0 & = & 0.0000 \\ f_1 & = & 0.6000 & & g_1 & = & 0.1000 \\ f_2 & = & 0.0000 & & g_2 & = & 0.2000 \\ f_3 & = & -0.5000 & & g_3 & = & 0.1000 \\ f_4 & = & 0.0000 & & g_4 & = & 0.0000 \end{array}$$

The application is such that the wave speed $c = 1$.
Carry out the first two time-steps of the numerical method

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

where $\mu = c\ \delta t/\delta x$ in which $\delta x = 0.25$ and $\delta t = 0.2$.

Solution

In this case $\mu = 1 \times 0.2/0.25 = 0.8$ and the first time-step is carried out as follows (to 4 d.p.):

$$\begin{array}{rcl} u_0^1 & = & 0 \quad \text{from the boundary condition} \\ u_1^1 = \frac{1}{2}\mu^2(f_0 + f_2) + (1-\mu^2)f_1 + \delta t g_1 & = & 0.2360 \\ u_2^1 = \frac{1}{2}\mu^2(f_1 + f_3) + (1-\mu^2)f_2 + \delta t g_2 & = & 0.0720 \\ u_3^1 = \frac{1}{2}\mu^2(f_2 + f_4) + (1-\mu^2)f_3 + \delta t g_3 & = & -0.0160 \\ u_4^1 & = & 0 \quad \text{from the boundary condition} \end{array}$$

The second time-step is as follows (to 4 d.p.):

$$\begin{array}{rcl} u_0^2 & = & 0 \quad \text{from the boundary condition} \\ u_1^2 = 2u_1^1 - u_1^0 + \mu^2(u_2^1 - 2u_1^1 + u_0^1) & = & -0.3840 \\ u_2^2 = 2u_2^1 - u_2^0 + \mu^2(u_3^1 - 2u_2^1 + u_1^1) & = & 0.1005 \\ u_3^2 = 2u_3^1 - u_3^0 + \mu^2(u_4^1 - 2u_3^1 + u_2^1) & = & 0.4309 \\ u_4^2 & = & 0 \quad \text{from the boundary condition} \end{array}$$

Task

Suppose that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 0.8$. It is given that u satisfies boundary conditions $u(0,t) = u(0.8,t) = 0$ $(t > 0)$ and initial conditions that may be summarised as

$$\begin{array}{lll} f_0 = 0.0000 & \quad & g_0 = 0.0000 \\ f_1 = 0.1703 & & g_1 = 0.4227 \\ f_2 = 0.2364 & & g_2 = 0.5417 \\ f_3 = 0.1703 & & g_3 = 0.4227 \\ f_4 = 0.0000 & & g_4 = 0.0000 \end{array}$$

The application is such that the wave speed $c = 1$.
Carry out the first two time-steps of the numerical method

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

where $\mu = c\ \delta t/\delta x$ in which $\delta x = 0.2$ and $\delta t = 0.11$.

Your solution

Answer
In this case $\mu = 1 \times 0.11/0.2 = 0.55$ and the first time-step is carried out as follows:

$$\begin{aligned}
u_0^1 &= 0 \quad \text{from the boundary condition} \\
u_1^1 = \tfrac{1}{2}\mu^2(f_0 + f_2) + (1-\mu^2)f_1 + \delta t g_1 &= 0.2010 \\
u_2^1 = \tfrac{1}{2}\mu^2(f_1 + f_3) + (1-\mu^2)f_2 + \delta t g_2 &= 0.2760 \\
u_3^1 = \tfrac{1}{2}\mu^2(f_2 + f_4) + (1-\mu^2)f_3 + \delta t g_3 &= 0.2010 \\
u_4^1 &= 0 \quad \text{from the boundary condition}
\end{aligned}$$

The second time-step is as follows:

$$\begin{aligned}
u_0^2 &= 0 \quad \text{from the boundary condition} \\
u_1^2 = 2u_1^1 - u_1^0 + \mu^2(u_2^1 - 2u_1^1 + u_0^1) &= 0.1936 \\
u_2^2 = 2u_2^1 - u_2^0 + \mu^2(u_3^1 - 2u_2^1 + u_1^1) &= 0.2702 \\
u_3^2 = 2u_3^1 - u_3^0 + \mu^2(u_4^1 - 2u_3^1 + u_2^1) &= 0.1936 \\
u_4^2 &= 0 \quad \text{from the boundary condition}
\end{aligned}$$

4. Stability

There is a stability constraint that is common to many methods for obtaining numerical solutions of the wave equation. Issues relating to stability of numerical methods can be extremely complicated, but the following Key Point is enough for our purposes.

Key Point 25

The numerical method seen in this Section requires that

$$\mu \leq 1 \qquad \text{that is,} \quad \frac{c\delta t}{\delta x} \leq 1$$

for solutions not to grow unrealistically with n.

This is called the CFL condition (named after an acronym of three mathematicians Courant, Friedrichs and Lewy).

Exercises

1. Suppose that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 0.6$. It is given that u satisfies boundary conditions $u(0,t) = u(0.6,t) = 0$ $(t > 0)$ and initial conditions that need not be stated for the purposes of this question. The application is such that the wave speed $c = 1.4$.
   The numerical method
   $$u_j^{n+1} = 2u_j^n - u_j^{n+1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$
   where $\mu = c\ \delta t/\delta x$, is implemented using $\delta x = 0.15$ and $\delta t = 0.1$.
   Suppose that, after 7 time-steps, the following data forms part of the numerical solution:
   $$\begin{array}{lcl} u_0^6 &=& 0.0000 \\ u_1^6 &=& 0.1024 \\ u_2^6 &=& 0.1986 \\ u_3^6 &=& 0.2361 \\ u_4^6 &=& 0.0000 \end{array} \qquad \begin{array}{lcl} u_0^7 &=& 0.0000 \\ u_1^7 &=& 0.0997 \\ u_2^7 &=& 0.1730 \\ u_3^7 &=& 0.1169 \\ u_4^7 &=& 0.0000 \end{array}$$
   Carry out the next time-step so as to find an approximation to u at $t = 8\delta t$.

2. Suppose that $u = u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ in $t > 0$ and $0 < x < 1$. It is given that u satisfies boundary conditions $u(0,t) = u(1,t) = 0$ $(t > 0)$. The initial elevation may be summarised as
   $$\begin{array}{lcl} f_0 &=& 0.0000 \\ f_3 &=& -0.1209 \end{array} \qquad \begin{array}{lcl} f_1 &=& 0.7812 \\ f_4 &=& 0.0000 \end{array} \qquad \begin{array}{lcl} f_2 &=& 0.2465 \end{array}$$
   and the string is initially at rest (that is, $g(x) = 0$). The application is such that the wave speed $c = 1$.
   Carry out the first two time-steps of the numerical method
   $$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$
   where $\mu = c\ \delta t/\delta x$ in which $\delta x = 0.25$ and $\delta t = 0.2$.

Answers

1. In this case $\mu = 1.4 \times 0.1/0.15 = 0.93333$ and the required time-step is carried out as follows:

$$\begin{aligned} u_0^8 &= 0 \quad \text{from the boundary condition} \\ u_1^8 = 2u_1^7 - u_1^6 + \mu^2(u_2^7 - 2u_1^7 + u_0^7) &= 0.0740 \\ u_2^8 = 2u_2^7 - u_2^6 + \mu^2(u_3^7 - 2u_2^7 + u_1^7) &= 0.0347 \\ u_3^8 = 2u_3^7 - u_3^6 + \mu^2(u_4^7 - 2u_3^7 + u_2^7) &= -0.0552 \\ u_4^8 &= 0 \quad \text{from the boundary condition} \end{aligned}$$

to 4 decimal places and these are the approximations to $u(0, 8\delta t)$, $u(0.15, 8\delta t)$, $u(0.3, 8\delta t)$, $u(0.45, 8\delta t)$ and $u(0.6, 8\delta t)$, respectively.

2. In this case $\mu = 1 \times 0.2/0.25 = 0.8$ and the first time-step is carried out as follows:

$$\begin{aligned} u_0^1 &= 0 \quad \text{from the boundary condition} \\ u_1^1 = \tfrac{1}{2}\mu^2(f_0 + f_2) + (1 - \mu^2)f_1 + \delta t g_1 &= 0.3601 \\ u_2^1 = \tfrac{1}{2}\mu^2(f_1 + f_3) + (1 - \mu^2)f_2 + \delta t g_2 &= 0.3000 \\ u_3^1 = \tfrac{1}{2}\mu^2(f_2 + f_4) + (1 - \mu^2)f_3 + \delta t g_3 &= 0.0354 \\ u_4^1 &= 0 \quad \text{from the boundary condition} \end{aligned}$$

to 4 decimal places.

The second time-step is as follows:

$$\begin{aligned} u_0^2 &= 0 \quad \text{from the boundary condition} \\ u_1^2 = 2u_1^1 - u_1^0 + \mu^2(u_2^1 - 2u_1^1 + u_0^1) &= -0.3299 \\ u_2^2 = 2u_2^1 - u_2^0 + \mu^2(u_3^1 - 2u_2^1 + u_1^1) &= 0.2226 \\ u_3^2 = 2u_3^1 - u_3^0 + \mu^2(u_4^1 - 2u_3^1 + u_2^1) &= 0.3384 \\ u_4^2 &= 0 \quad \text{from the boundary condition} \end{aligned}$$

to 4 decimal places.