Linear Multistep Methods

32.2

Introduction

In the previous Section we saw two methods (Euler and trapezium) for approximating the solutions of certain initial value problems. In this Section we will see that those two methods are special cases of a more general collection of techniques called linear multistep methods. Techniques for determining the properties of these methods will be presented.

Another class of approximations, called Runge-Kutta methods, will also be discussed briefly. These are not linear multistep methods, but the two are sometimes used in conjunction.

Prerequisites

Before starting this Section you should . . .

- review Section 32.1

Learning Outcomes

On completion you should be able to . . .

- implement linear multistep methods to carry out time steps of numerical methods
- evaluate the zero stability of linear multistep methods
- establish the order of linear multistep methods
- implement a Runge-Kutta method

1. General linear multistep methods

Euler's method and the trapezium method are both special cases of a wider class of so-called **linear multistep** methods. The following Key Point gives the most general situation that we will look at.

Key Point 8

The general k-step linear multistep method is given by

$$\alpha_k y_{n+k} + \cdots + \alpha_1 y_{n+1} + \alpha_0 y_n = h\left(\beta_k f_{n+k} + \cdots + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

or equivalently

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = h \sum_{j=0}^{k} \beta_j \; f_{n+j}.$$

It is always the case that $\alpha_k \neq 0$. Also, at least one of α_0 and β_0 will be non-zero.

A linear multistep method is defined by the choice of the quantities

$$k, \quad \alpha_0, \quad \alpha_1, \quad \ldots, \quad \alpha_k, \quad \beta_0, \quad \beta_1, \quad \ldots, \quad \beta_k$$

- If $\beta_k = 0$ the method is called **explicit**. (Because at each step, when we are trying to find the newest y_{n+k}, there is no appearance of this unknown on the right-hand side.)
- If $\beta_k \neq 0$ the method is called **implicit**. (Because y_{n+k} now appears on both sides of the equation (on the right-hand side it appears through $f_{n+k} = f((n+k)h, y_{n+k})$, and we cannot, in general, rearrange to get an explicit formula for y_{n+k}.)

The next Example shows one such choice.

Example 5

Write down the linear multistep scheme defined by the choices $k = 1$, $\alpha_0 = -1$, $\alpha_1 = 1$, $\beta_0 = \beta_1 = \frac{1}{2}$.

Solution

Here $k = 1$ so that

$$\alpha_1 y_{n+1} + \alpha_0 y_n = h(\beta_1 f_{n+1} + \beta_0 f_n)$$

and substituting the values in for the four coefficients gives

$$y_{n+1} - y_n = h\left(\tfrac{1}{2} f_{n+1} + \tfrac{1}{2} f_n\right)$$

which, as we know, is the trapezium method.

Task

Write down the linear multistep scheme defined by the choices $k = 1$, $\alpha_0 = -1$, $\alpha_1 = 1$, $\beta_0 = 1$ and $\beta_1 = 0$.

Your solution

Answer

Here $k = 1$ and we have

$$\alpha_1 y_{n+1} + \alpha_0 y_n = h(\beta_1 f_{n+1} + \beta_0 f_n)$$

and substituting the values in for the four coefficients gives

$$y_{n+1} - y_n = h f_n$$

which, as we know, is Euler's method.

Task

Write down the linear multistep scheme defined by the choices $k = 2$, $\alpha_0 = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_2 = 0$, $\beta_1 = \frac{3}{2}$ and $\beta_0 = -\frac{1}{2}$.

Your solution

Answer

Here $k = 2$ (so we are looking at a 2-step scheme) and we have

$$\alpha_2 y_{n+1} + \alpha_1 y_{n+1} + \alpha_0 y_n = h(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n)$$

Substituting the values in for the six coefficients gives

$$y_{n+2} - y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n)$$

which is an example of a scheme that is explicit (because $\beta_k = \beta_2$ is zero).

In the preceding Section we saw several examples implementing the Euler and trapezium methods.

The next Example deals with the explicit 2-step that was the subject of the Task above.

Example 6

A numerical scheme has been used to approximate the solution of

$$\frac{dy}{dt} = t + y, \qquad y(0) = 3$$

and has produced the following estimates, to 6 decimal places,

$$y(0.4) \approx 4.509822, \quad y(0.45) \approx 4.755313$$

Now use the 2-step, explicit linear multistep scheme

$$y_{n+2} - y_{n+1} = h\,(1.5 f_{n+1} - 0.5 f_n)$$

to approximate $y(0.5)$.

Solution

Evidently the value $h = 0.05$ will serve our purposes and we seek $y_{10} \approx y(0.5)$. The values we will need to use in our implementation of the 2-step scheme are $y_9 = 4.755313$ and

$$f_9 = f(0.45, y_9) = 5.205313 \qquad f_8 = f(0.4, y_8) = 4.909822$$

to 6 decimal places since $f(t, y) = t + y$. It follows that

$$\begin{aligned} y_{10} &= y_9 + 0.05 \times (1.5f_9 - 0.5f_8) \\ &= 5.022966 \end{aligned}$$

And we conclude that $y(0.5) \approx 5.022966$, where this approximation has been given to 6 decimal places.

Notice that in this implementation of a 2-step method we needed to use the values of the *two* y values preceding the one currently being sought. Both y_8 and y_9 were used in finding y_{10}.

Similarly, a k-step method will use, in general, k previous y values at each time step.

This means that there is an issue to be resolved in implementing methods that are 2- or higher-step, because when we start we are only given *one* starting value y_0. This issue will be dealt with towards the end of this Section. The following exercise involves a 2-step method, but (like the example above) it does not encounter the difficulty relating to starting values as it assumes that the numerical procedure is already underway.

Task

A numerical scheme has been used to approximate the solution of

$$\frac{dy}{dt} = t/y \qquad y(0) = -2$$

and has produced the following estimates, to 6 decimal places,

$$y(0.24) \approx -2.013162, \quad y(0.26) \approx -2.015546$$

Now use the 2-step, explicit linear multistep scheme

$$y_{n+2} - \tfrac{1}{2}y_{n+1} - \tfrac{1}{2}y_n = \tfrac{3}{2}hf_{n+1}$$

to approximate $y(0.28)$.

Your solution

Answer
Evidently the value $h = 0.02$ will serve our purposes and we seek $y_{14} \approx y(0.28)$. The values we will need to use in our implementation of the 2-step scheme are $y_{13} = -2.015546$, $y_{12} = -2.013162$ and

$$f_{13} = f(0.26, y_{13}) = -0.128997$$

to 6 decimal places since $f(t, y) = t/y$. It follows that

$$\begin{aligned} y_{14} &= \tfrac{1}{2}y_{13} + \tfrac{1}{2}y_{12} + 0.02 \times \tfrac{3}{2}f_{13} \\ &= -2.018224 \end{aligned}$$

And we conclude that $y(0.28) \approx -2.018224$, to 6 decimal places.

Zero stability

We now begin to classify linear multistep methods. Some choices of the coefficients give rise to schemes that work well, and some do not. One property that is required if we are to obtain reliable approximations is that the scheme be **zero stable**. A scheme that is zero stable will not produce approximations which grow unrealistically with t.

We define the **first characteristic polynomial**

$$\rho(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots \alpha_k z^k$$

where the α_i are the coefficients of the linear multistep method as defined in Key Point 8 (page 21). This polynomial appears in the definition of zero stability given in the following Key Point.

Key Point 10

The linear multistep scheme

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = h \sum_{j=0}^{k} \beta_j \; f_{n+j}.$$

is said to be **zero stable** if the zeros of the first characteristic polynomial are such that

1. none is larger than 1 in magnitude
2. any zero equal to 1 in magnitude is simple (that is, not repeated)

The **second characteristic polynomial** is defined in terms of the coefficients on the right-hand side (the β_j), but its use is beyond the scope of this Workbook.

Example 7

Find the roots of the first characteristic polynomial for each of the examples below and determine whether or not the method is zero stable.

(a) $y_{n+1} - y_n = hf_n$

(b) $y_{n+1} - 2y_n = hf_n$

(c) $y_{n+2} + 3y_{n+1} - 4y_n = h\,(2f_{n+2} + f_{n+1} + 2f_n)$

(d) $y_{n+2} - y_{n+1} = \frac{3}{2}hf_{n+1}$

(e) $y_{n+2} - 2y_{n+1} + y_n = h(f_{n+2} - f_n)$

(f) $y_{n+2} + 2y_{n+1} + 5y_n = h\,(f_{n+2} - f_{n+1} + 2f_n)$

Solution

(a) In this case $\rho(z) = z - 1$ and the single zero of ρ is $z = 1$. This is a simple (that is, not repeated) root with magnitude equal to 1, so the method is zero stable.

(b) $\rho(z) = z - 2$ which has one zero, $z = 2$. This has magnitude $2 > 1$ and therefore the method is not zero stable.

(c) $\rho(z) = z^2 + 3z - 4 = (z - 1)(z + 4)$. One root is $z = -4$ which has magnitude greater than 1 and the method is therefore not zero stable.

Solution (contd.)

(d) Here $\alpha_2 = 1$, $\alpha_1 = -1$ and $\alpha_0 = 0$, therefore

$$\rho(z) = z^2 - z = z(z-1)$$

which has two zeros, $z = 0$ and $z = 1$. These both have magnitude less than or equal to 1 and there is no repeated zero with magnitude equal to 1, so the method is zero stable.

(e) $\rho(z) = z^2 - 2z + 1 = (z-1)^2$. Here $z = 1$ is not a simple root, it is repeated and, since it has magnitude equal to 1, the method is not zero stable.

(f) $\rho(z) = z^2 + 2z + 5$ and the roots of $\rho(z) = 0$ can be found from the quadratic formula. In this case the roots are complex and are equal to Zero-stability requires that the absolute values have magnitude less than or equal to 1. Consequently we conclude that the method is not zero stable.

Task

Find the roots of the first characteristic polynomial for the linear multistep scheme

$$y_{n+2} - 2y_{n+1} + y_n = h\left(f_{n+2} + 2f_{n+1} + f_n\right)$$

and hence determine whether or not the scheme is zero stable.

Your solution

Answer
The first characteristic polynomial is

$$\rho(z) = \alpha_2 z^2 + \alpha_1 z + \alpha_0 = z^2 - 2z + 1$$

and the roots of $\rho(z) = 0$ are both equal to 1. In the case of roots that are equal, zero-stability requires that the absolute value has magnitude less than 1. Consequently we conclude that the method is not zero stable.

At this stage, the notion of zero stability is rather abstract, so let us try using a **zero unstable** method and see what happens. We consider the simple test problem

$$\frac{dy}{dt} = -y, \qquad y(0) = 1$$

which we know to have analytic solution $y(t) = e^{-t}$, a quantity which decays with increasing t. Implementing the zero unstable scheme

$$y_{n+1} - 2y_n = hf_n$$

on a spreadsheet package with $h = 0.05$ gives the following results

n	$t = nh$	$y_n \approx y(nh)$
0	0.00	1.00000
1	0.05	1.95000
2	0.10	3.80250
3	0.15	7.41488
4	0.20	14.45901
5	0.25	28.19506
6	0.30	54.98037
7	0.35	107.21172
8	0.40	209.06286
9	0.45	407.67258
10	0.50	794.96153
11	0.55	1550.17499
12	0.60	3022.84122
13	0.65	5894.54039
14	0.70	11494.35376
15	0.75	22413.98982

where 5 decimal places have been given for y_n. The dramatic growth in the values of y_n is due to the zero instability of the method. (There are in fact other things than zero instability wrong with the scheme $y_{n+1} - 2y_n = hf_n$, but it is the zero instability that is causing the large numbers.)

Consistency and order

A scheme that is zero stable will produce approximations that do not grow in size in a way that is not present in the exact, analytic solution. Zero stability is a required property, but it is not enough on its own. There remains the issue of whether the approximations are close to the exact values.

The **truncation error** of the general linear multistep method is a measure of how well the differential equation and the numerical method agree with each other. It is defined by

$$\tau_j = \frac{1}{\beta}\left(\frac{c_0}{h}y(jh) + c_1 y'(jh) + c_2 h y''(jh) + c_3 h^2 y'''(jh) + \dots\right) = \frac{1}{\beta h}\sum_{p=0}^{\infty} c_p h^p y^{(p)}(jh)$$

where $\beta = \sum \beta_j$ is a normalising factor.

It is the first few terms in this expression that will matter most in what follows, and it helps us that there are formulae for the coefficients which appear

$$c_0 = \sum \alpha_j, \quad c_1 = \sum (j\alpha_j - \beta_j), \quad c_2 = \sum\left(\frac{j^2}{2}\alpha_j - j\beta_j\right), \quad c_3 = \sum\left(\frac{j^3}{3!}\alpha_j - \frac{j^2}{2}\beta_j\right)$$

and so on, the general formula for $p \geq 2$ is $c_p = \sum\left(\frac{j^p}{(p)!}\alpha_j - \frac{j^{p-1}}{(p-1)!}\beta_j\right)$.

Recall that the truncation error is intended to be a measure of how well the differential equation and its approximation agree with each other. We say that the numerical method is **consistent** with the differential equation if τ_j tends to zero as $h \to 0$. The following Key Point says this in other words.

Key Point 11

The linear multistep scheme is said to be **consistent** if $c_0 = 0$ and $c_1 = 0$.

Example 8

Show that Euler's method ($y_{n+1} = y_n + hf_n$) is consistent.

Solution

In this case $\alpha_1 = 1$, $\alpha_0 = -1$, $\beta_1 = 0$ and $\beta_0 = 1$. It follows that

$$c_0 = \sum \alpha_j = 1 - 1 = 0 \qquad \text{and} \qquad c_1 = \sum j\alpha_j - \beta_j = 1\alpha_1 - (\beta_0 + \beta_1) = 1 - (1 + 0) = 0$$

and therefore Euler's method is consistent.

Task

Show that the trapezium method ($y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n)$) is consistent.

Your solution

Answer

In this case $\alpha_1 = 1$, $\alpha_0 = -1$, $\beta_1 = \frac{1}{2}$ and $\beta_0 = \frac{1}{2}$. It follows that

$$c_0 = \sum \alpha_j = 1 - 1 = 0 \qquad \text{and} \qquad c_1 = \sum j\alpha_j - \beta_j = 1\alpha_1 - (\beta_0 + \beta_1) = 1 - (\tfrac{1}{2} + \tfrac{1}{2}) = 0$$

and therefore the trapezium method is consistent.

Task

Determine the consistency (or otherwise) of the following 2-step linear multistep schemes

(a) $y_{n+2} - 2y_{n+1} + y_n = h(f_{n+2} - f_n)$

(b) $y_{n+2} - y_{n+1} = h(f_{n+1} - 2f_n)$

(c) $y_{n+2} - y_{n+1} = h(2f_{n+2} - f_{n+1})$

Your solution

Answer

(a) $c_0 = \alpha_2 + \alpha_1 + \alpha_0 = 1 - 2 + 1 = 0$, $c_1 = 2\alpha_2 + 1 \times \alpha_1 + 0 \times \alpha_0 - (\beta_2 + \beta_1 + \beta_0) = 2(1) + 1(-2) + 0 - (1 - 1) = 0$. Therefore the method is consistent.

(b) $c_0 = 1 - 1 + 0 = 0$, $c_1 = 2 - 1 - (1 - 2) = 2$ so the method is inconsistent.

(c) This method is consistent, because $c_0 = 1 - 1 = 0$ and $c_1 = 2 - 1 - (2 - 1) = 0$.

(Notice also that the first characteristic polynomial $\rho(z)$, defined on page 6 of this Section, evaluated at $z = 1$ is equal to $\alpha_0 + \alpha_1 + \cdots + \alpha_k = c_0$. It follows that a consistent scheme must always have $z = 1$ as one of the roots of its $\rho(z)$.)

Assuming that the method is consistent, the **order** of the scheme tells us how quickly the truncation error tends to zero as $h \to 0$. For example, if $c_0 = 0$, $c_1 = 0$, $c_2 = 0$ and $c_3 \neq 0$ then the first non-zero term in τ_j will be the one involving h^2 and the linear multistep method is called **second-order**. This means that if h is small then τ_j is dominated by the h^2 term (because the h^3 and subsequent terms will be tiny in comparison) and halving h will cause τ_j to decrease by a factor of approximately $\frac{1}{4}$. The decrease is only approximately known because the h^3 and other terms will have a small effect. We summarise the general situation in the following Key Point.

Key Point 12

A linear multistep method is said to be of **order** p if

$c_0 = c_1 = c_2 = \cdots = c_p = 0$ and $c_{p+1} \neq 0$

Combining the last two Key Points gives us another way of describing consistency: "A linear multistep method is consistent if it is at least first order".

Example 9

Find the order of

(a) Euler's method

(b) The trapezium method.

Solution

(a) We have already found that $c_0 = c_1 = 0$ so the first quantity to calculate is

$$c_2 = \sum \left(\frac{j^2}{2}\alpha_j - j\beta_j \right) = \tfrac{1}{2}\alpha_1 - \beta_1 = \tfrac{1}{2}$$

which is not zero and therefore Euler's method is of order 1. (Or, in other words, Euler's method is first order.)

(b) We have already found that $c_0 = c_1 = 0$ so the first quantity to calculate is

$$c_2 = \sum \left(\frac{j^2}{2}\alpha_j - j\beta_j \right) = \tfrac{1}{2}\alpha_1 - \beta_1 = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

this is equal to zero, so we must calculate the next coefficient

$$c_3 = \sum \left(\frac{j^3}{3!}\alpha_j - \frac{j^2}{2}\beta_j \right) = \tfrac{1}{6}\alpha_1 - \tfrac{1}{2}\beta_1 = \tfrac{1}{6} - \tfrac{1}{4} = -\tfrac{1}{12}$$

which is not zero. Hence the trapezium method is of order 2 (that is, it is second order).

This finally explains some of the results we saw in the first Section of this Workbook. We saw that the errors incurred by the Euler and trapezium methods, for a particular test problem, were roughly proportional to h and h^2 respectively. This behaviour is dictated by the first non-zero term in the truncation error which is the one involving c_2h for Euler and the one involving c_3h^2 for trapezium.

We now apply the method to another linear multistep scheme.

Example 10

Find the order of the 4-step, explicit linear multistep scheme

$$y_{n+4} - y_{n+3} = \frac{h}{24}\Big(55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n\Big)$$

Solution

In the established notation we have $\alpha_4 = 1$, $\alpha_3 = -1$, $\alpha_2 = 0$, $\alpha_1 = 0$ and $\alpha_0 = 0$. The β terms similarly come from the coefficients on the right hand side (remembering the denominator of 24). Now

$$c_0 = \sum \alpha_j = 0 \qquad \text{and} \qquad c_1 = \sum j\alpha_j - \beta_j = 0$$

from which we conclude that the method is consistent.
We also find that

$$c_2 = \sum \tfrac{1}{2}j^2\alpha_j - j\beta_j = 0, \qquad c_3 = \sum \tfrac{1}{6}j^3\alpha_j - \tfrac{1}{2}j^2\beta_j = 0,$$
$$c_4 = \sum \tfrac{1}{24}j^4\alpha_j - \tfrac{1}{6}j^3\beta_j = 0 \qquad c_5 = \sum \tfrac{1}{120}j^5\alpha_j - \tfrac{1}{24}j^4\beta_j = 0.348611 \quad \text{to 6 d.p.}$$

(The exact value of c_5 is $\frac{251}{720}$.)

Because c_5 is the first non-zero coefficient we conclude that the method is of order 4.

So the scheme in Example 10 has the property that the truncation error will tend to zero proportional to h^4 (approximately) as $h \to 0$. This is a good thing, as it says that the error will decay to zero very quickly, when h is decreased.

Task

Find the order of the 2-step linear multistep scheme

$$y_{n+2} - y_{n+1} = \frac{h}{12}\Big(f_{n+2} + 8f_{n+1} - f_n\Big)$$

Your solution

Answer

In the established notation we have $\alpha_2 = 1$, $\alpha_1 = -1$ and $\alpha_0 = 0$. Also $\beta_2 = \frac{5}{12}$, $\beta_1 = \frac{2}{3}$ and $\beta_0 = -\frac{1}{12}$. Now

$$c_0 = \sum \alpha_j = 1 - 1 + 0 = 0 \qquad \text{and} \qquad c_1 = \sum j\alpha_j - \beta_j = 2\alpha_2 + \alpha_1 - (\beta_2 + \beta_1 + \beta_0) = 0$$

from which we conclude that the method is consistent.
We also find that

$$\begin{aligned} c_2 &= \sum \tfrac{1}{2}j^2\alpha_j - j\beta_j = \tfrac{1}{2}(4\alpha_2 + \alpha_1) - (2\beta_2 + \beta_1) = 0 \\ c_3 &= \sum \tfrac{1}{6}j^3\alpha_j - \tfrac{1}{2}j^2\beta_j = \tfrac{1}{6}(8\alpha_2 + \alpha_1) - \tfrac{1}{2}(4\beta_2 + \beta_1) = 0 \\ c_4 &= \sum \tfrac{1}{24}j^4\alpha_j - \tfrac{1}{6}j^3\beta_j = \tfrac{1}{24}(16\alpha_2 + \alpha_1) - \tfrac{1}{6}(8\beta_2 + \beta_1) = -\tfrac{1}{24} \end{aligned}$$

so that the method is of order 3.

Convergence

The key result concerning linear multistep methods is given in the following Key Point.

Key Point 13

The numerical approximation to the initial value problem converges to the actual solution as $h \to 0$ if

1. the scheme is zero stable
2. the scheme is consistent

The proof of this result lies beyond the scope of this Workbook. It is worth pointing out that this is not the whole story. The convergence result is useful, but only deals with h as it tends to zero. In practice we use a finite, non-zero value of h and there are ways of determining how big an h it is possible to "get away with" for a particular linear multistep scheme applied to a particular initial value problem.

If, when implementing the methods described above, it is found that the numerical approximations behave in an unexpected way (for example, if the numbers are very large when they should not be, or if decreasing h does not seem to lead to results that converge) then one topic to look for in further reading is that of "absolute stability".

2. An example of a Runge-Kutta method

A full discussion of the so-called Runge-Kutta methods is not required here, but we do need to touch on them to resolve a remaining issue in the implementation of linear multistep schemes.

The problem with linear multistep methods is that a zero-stable, 1-step method can never be better than second order (you need not worry about why this is true, it was proved in the latter half of the last century by a man called Dahlquist). We have seen methods of higher order than 2, but they were all at least 2-step methods. And the problem with 2-step methods is that we need 2 starting values to implement them and we are only ever given 1 starting value: the initial condition $y(0)$.

One way out of this "Catch 22" is to use a **Runge-Kutta method** to generate the extra starting value(s) we need. Runge-Kutta methods are not linear multistep methods and do not suffer from the problem mentioned above. There is no such thing as a free lunch, of course, and Runge-Kutta methods are generally more expensive in effort to implement than linear multistep methods because of the number of evaluations of f required at each time step.

The following Key Point gives a statement of what is, perhaps, the most popular Runge-Kutta method (sometimes called "RK4").

Key Point 14

Runge Kutta method (RK4)

Consider the usual initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(0) = y_0.$$

$$\begin{array}{rrcl} \text{Calculate} & K_1 & = & f(nh, y_n) \\ \text{then} & K_2 & = & f((n+\frac{1}{2})h, y_n + \frac{1}{2}hK_1) \\ \text{then} & K_3 & = & f((n+\frac{1}{2})h, y_n + \frac{1}{2}hK_2) \\ \text{then} & K_4 & = & f((n+1)h, y_n + hK_3) \\ \text{finally} & y_{n+1} & = & y_n + \dfrac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{array}$$

Notice that each calculation is explicit, all of the right-hand sides in the formulae in the Key Point above involve known quantities.

Example 11

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = \cos(y) \qquad y(0) = 3$$

Carry out one time step of the Runge-Kutta method RK4 with a step size of $h = 0.1$ so as to obtain an approximation to $y(0.1)$.

Solution

The iteration must be carried out in four stages. We start by calculating

$$K_1 = f(0, y_0) = f(0, 3) = -0.989992$$

a value we now use in finding

$$K_2 = f(\tfrac{1}{2}h, y_0 + \tfrac{1}{2}hK_1) = f(0.05, 2.950500) = -0.981797$$

This value K_2 is now used in our evaluation of

$$K_3 = f(\tfrac{1}{2}h, y_0 + \tfrac{1}{2}hK_2) = f(0.05, 2.950910) = -0.981875$$

which, in turn, is used in

$$K_4 = f(h, y_0 + hK_3) = f(0.1, 2.901812) = -0.971390$$

All four of these values are then used to complete the iteration

$$\begin{aligned} y_1 &= y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ &= 3 + \frac{0.1}{6}(-0.989992 + 2 \times -0.981797 + 2 \times -0.981875 - 0.971390) \\ &= 2.901855 \qquad \text{to 6 decimal places.} \end{aligned}$$

Task

Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = y(1-y) \qquad y(0) = 0.7$$

Carry out one time step of the Runge-Kutta method RK4 with a step size of $h = 0.1$ so as to obtain an approximation to $y(0.1)$.

Your solution

Answer

The time step must be carried out in four stages. We start by calculating

$$K_1 = f(0, y_0) = f(0, 0.7) = 0.210000$$

a value we now use in finding

$$K_2 = f(\tfrac{1}{2}h, y_0 + \tfrac{1}{2}hK_1) = f(0.05, 0.710500) = 0.205690$$

This value K_2 is now used in our evaluation of

$$K_3 = f(\tfrac{1}{2}h, y_0 + \tfrac{1}{2}hK_2) = f(0.05, 0.710284) = 0.205780$$

which, in turn, is used in

$$K_4 = f(h, y_0 + hK_3) = f(0.1, 0.720578) = 0.201345$$

All four of these values are then used to complete the time step

$$\begin{aligned} y_1 &= y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ &= 0.7 + \frac{0.1}{6}(0.210000 + 2 \times 0.205690 + 2 \times 0.205780 + 0.201345) \\ &= 0.720571 \ \text{ to 6 d.p.} \end{aligned}$$

Exercises

1. Assuming the notation established earlier, write down the linear multistep scheme corresponding to the choices $k = 2$, $\alpha_0 = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_0 = \frac{-1}{12}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{5}{12}$.

2. A numerical scheme has been used to approximate the solution of

$$\frac{dy}{dt} = t^2 - y^2 \qquad y(0) = 2$$

and has given the following estimates, to 6 decimal places,

$$y(0.3) \approx 1.471433, \quad y(0.32) \approx 1.447892$$

Now use the 2-step, explicit linear multistep scheme

$$y_{n+2} - 1.6y_{n+1} + 0.6y_n = h\,(5f_{n+1} - 4.6f_n)$$

to approximate $y(0.34)$.

3. Find the roots of the first characteristic polynomial for the linear multistep scheme

$$5y_{n+2} + 3y_{n+1} - 2y_n = h\,(f_{n+2} + 2f_{n+1} + f_n)$$

and hence determine whether or not the scheme is zero stable.

4. Find the order of the 2-step linear multistep scheme

$$y_{n+2} + 2y_{n+1} - 3y_n = \frac{h}{10}\Big(f_{n+2} + 16f_{n+1} + 17f_n\Big)$$

(Would you recommend using this method?)

5. Suppose that $y = y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = 1/y^2 \qquad y(0) = 2$$

Carry out one time step of the Runge-Kutta method RK4 with a step size of $h = 0.4$ so as to obtain an approximation to $y(0.4)$.

Answers

1. $y_{n+2} - y_{n+1} = \frac{h}{12}\left(5f_{n+2} + 8f_{n+1} - f_n\right)$

2. Evidently the value $h = 0.02$ will serve our purposes and we seek $y_{17} \approx y(0.34)$. The values we will need to use in our implementation of the 2-step scheme are $y_{16} = 1.447892$, $y_{15} = 1.471433$ and $f_{16} = f(0.32, y_{16}) = -1.993991$ $f_{15} = f(0.3, y_{15}) = -2.075116$ since $f(t, y) = t^2 - y^2$. It follows that

$$y_{17} = 1.6y_{16} - 0.6y_{15} + 0.02 \times (5f_{16} - 4.6f_{15}) = 1.425279$$

And we conclude that $y(0.34) \approx 1.425279$, to 6 decimal places.

3. The first characteristic polynomial is $\rho(z) = \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 5z^2 + 3z - 2$ and the roots of $\rho(z) = 0$ can be found from the quadratic formula. In this case the roots are real and distinct and are equal to 0.4 and -1. In the case of roots that are distinct zero-stability requires that the absolute values have magnitude less than or equal to 1 . Consequently we conclude that the method is zero stable.

4. In the established notation we have $\alpha_2 = 1$, $\alpha_1 = 2$ and $\alpha_0 = -3$. The $beta$ terms similarly come from the coefficients on the right hand side (remembering the denominator of 10).

Now $c_0 = \sum \alpha_j = 0$ and $c_1 = \sum j\alpha_j - \beta_j = 0$

from which we conclude that the method is consistent.

We also find that $c_2 = \sum \frac{1}{2}j^2\alpha_j - j\beta_j = 0$ $c_3 = \sum \frac{1}{6}j^3\alpha_j - \frac{1}{2}j^2\beta_j = -0.533333$

so that the method is of order 2 . This method is not to be recommended however (check the zero stability).

5. Each time step must be carried out in four stages. We start by calculating

$$K_1 = f(0, y_0) = f(0, 2) = 0.250000$$

a value we now use in finding $K_2 = f(\frac{1}{2}h, y_0 + \frac{1}{2}hK_1) = f(0.2, 2.050000) = 0.237954$

This value K_2 is now used in our evaluation of

$$K_3 = f(\tfrac{1}{2}h, y_0 + \tfrac{1}{2}hK_2) = f(0.2, 2.047591) = 0.238514$$

which, in turn, is used in $K_4 = f(h, y_0 + hK_3) = f(0.4, 2.095406) = 0.227753$

All four of these values are then used to complete the time step

$$\begin{aligned} y_1 &= y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ &= 2 + \frac{0.4}{6}(0.250000 + 2 \times 0.237954 + 2 \times 0.238514 + 0.227753) = 2.095379 \end{aligned}$$