Integral Vector Theorems

29.3

Introduction

Various theorems exist relating integrals involving vectors. Those involving line, surface and volume integrals are introduced here.

They are the multivariable calculus equivalent of the fundamental theorem of calculus for single variables ("integration and differentiation are the reverse of each other").

Use of these theorems can often make evaluation of certain vector integrals easier. This Section introduces the main theorems which are Gauss' divergence theorem, Stokes' theorem and Green's theorem.

Prerequisites

Before starting this Section you should ...

- be able to find the gradient of a scalar field and the divergence and curl of a vector field
- be familiar with the integration of vector functions

Learning Outcomes

On completion you should be able to ...

- use vector integral theorems to facilitate vector integration

1. Stokes' theorem

This is a theorem that equates a line integral to a surface integral. For any vector field $\underline{F}$ and a contour C which bounds an area S,

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

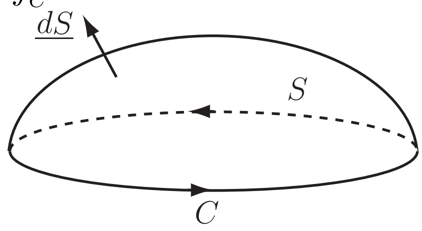

Figure 16: A surface for Stokes' theorem

Notes

(a) $\underline{dS}$ is a vector perpendicular to the surface S and $\underline{dr}$ is a line element along the contour C. The sense of $\underline{dS}$ is linked to the direction of travel along C by a right hand screw rule.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a line integral along a contour composed of several distinct parts (e.g. a square or other figure).

(d) $\underline{\nabla} \times \underline{F}$ is a vector field representing the curl of the vector field $\underline{F}$ and may, alternatively, be written as curl $\underline{F}$.

Justification of Stokes' theorem

Imagine that the surface S is divided into a set of infinitesimally small rectangles $ABCD$ where the axes are adjusted so that AB and CD lie parallel to the new x-axis i.e. $AB = \delta x$ and BC and AD lie parallel to the new y-axis i.e. $BC = \delta y$.

Now, $\displaystyle\oint_C \underline{F} \cdot \underline{dr}$ is calculated, where C is the boundary of a typical such rectangle.

The contributions along AB, BC, CD and DA are

$$\underline{F}(x, y, 0) \cdot \underline{\delta x} = F_x(x, y, z)\delta x,$$

$$\underline{F}(x + \delta x, y, 0) \cdot \underline{\delta y} = F_y(x + \delta x, y, z)\delta y,$$

$$\underline{F}(x, y + \delta y, 0) \cdot (-\underline{\delta x}) = -F_x(x, y + \delta y, z)\delta x$$

$$\underline{F}(x, y, 0) \cdot (-\underline{\delta x}) = -F_y(x, y, z)\delta y.$$

Thus,

$$\begin{aligned}
\oint_C \underline{F} \cdot \underline{dr} &\approx (F_x(x, y, z) - F_x(x, y + \delta y, z))\delta x + (F_y(x + \delta x, y, z) - F_y(x, y, z))\delta y \\
&\approx \frac{\partial F_y}{\partial x}\delta x\delta y - \frac{\partial F_x}{\partial y}\delta x\delta y \\
&\approx (\underline{\nabla} \times \underline{F})_z \delta S \\
&= (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}
\end{aligned}$$

as $\underline{dS}$ is perpendicular to the x- and y- axes.

Thus, for each small rectangle, $\oint_C \underline{F} \cdot \underline{dr} \approx (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$

When the contributions over all the small rectangles are summed, the line integrals along the inner parts of the rectangles cancel and all that remains is the line integral around the outside of the surface S. The surface integrals sum. Hence, the theorem applies for the area S bounded by the contour C. While the above does not constitute a formal proof of Stokes' theorem, it does give an appreciation of the origin of the theorem.

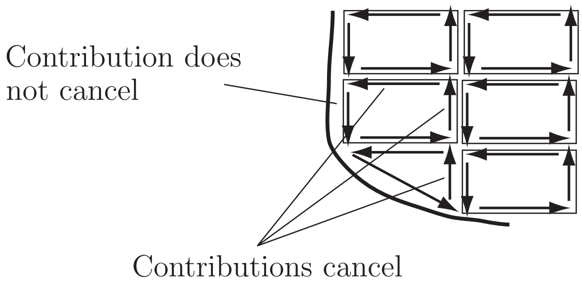

Figure 17: Line integral cancellation and non-cancellation

Key Point 8

Stokes' Theorem

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$$

The closed contour integral of the scalar product of a vector function with the vector along the contour is equal to the integral of the scalar product of the curl of that vector function and the unit normal, over the corresponding surface.

Example 32

Verify Stokes' theorem for the vector function $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ and the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

If $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ then $\underline{\nabla} \times \underline{F} = (z+1)\underline{i} + (-1-2y)\underline{k} = \underline{i} + (-1-2y)\underline{k}$ (as $z = 0$).
Note that $\underline{dS} = dxdy\underline{k}$ so that $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (-1-2y)dydx$

Thus
$$\begin{aligned} \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{x=0}^1 \int_{y=0}^1 (-1-2y)dydx \\ &= \int_{x=0}^1 \Big[(-y-y^2)\Big]_{y=0}^1 dx = \int_{x=0}^1 (-2)dx \\ &= \Big[-2x\Big]_0^1 = -2+0 = -2 \end{aligned}$$

To evaluate $\oint_C \underline{F} \cdot \underline{dr}$, we must consider the four sides separately.
When $y = 0$, $\underline{F} = -x\underline{j}$ and $\underline{dr} = dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = 0$ i.e. the contribution of this side to the integral is zero.
When $x = 1$, $\underline{F} = y^2\underline{i} - \underline{j}$ and $\underline{dr} = dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = -dy$ so the contribution to the integral is
$$\int_{y=0}^1 (-dy) = \Big[-y\Big]_0^1 = -1.$$
When $y = 1$, $\underline{F} = \underline{i} - x\underline{j}$ and $\underline{dr} = -dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = -dx$ so the contribution to the integral is
$$\int_{x=0}^1 (-dx) = \Big[-x\Big]_0^1 = -1.$$
When $x = 0$, $\underline{F} = y^2\underline{i}$ and $\underline{dr} = -dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = 0$ so the contribution to the integral is zero.

The integral $\oint_C \underline{F} \cdot \underline{dr}$ is the sum of the contributions i.e. $0 - 1 - 1 + 0 = -2$.

Thus $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -2$ i.e. Stokes' theorem has been verified.

Example 33

Using cylindrical polar coordinates verify Stokes' theorem for the function $\underline{F} = \rho^2\underline{\hat{\phi}}$ the circle $\rho = a, z = 0$ and the surface $\rho \leq a, z = 0$.

Solution

Firstly, find $\oint_C \underline{F} \cdot \underline{dr}$. This can be done by integrating along the contour $\rho = a$ from $\phi = 0$ to $\phi = 2\pi$. Here $\underline{F} = a^2\underline{\hat{\phi}}$ (as $\rho = a$) and $\underline{dr} = a\ d\phi\ \underline{\hat{\phi}}$ (remembering the scale factor) so $\underline{F}\cdot\underline{dr} = a^3 d\phi$ and hence

$$\oint_C \underline{F} \cdot \underline{dr} = \int_0^{2\pi} a^3 d\phi = 2\pi a^3$$

As $\underline{F} = \rho^2\underline{\hat{\phi}}$, $\underline{\nabla} \times \underline{F} = 3\rho\underline{\hat{z}}$ and $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 3\rho$ as $\underline{dS} = \underline{\hat{z}}$.
Thus

$$\begin{aligned}\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} 3\rho \times \rho d\rho d\phi = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} 3\rho^2 d\rho d\phi \\ &= \int_{\phi=0}^{2\pi}\left[\rho^3\right]_{\rho=0}^{a} d\phi = \int_0^{2\pi} a^3 d\phi = 2\pi a^3\end{aligned}$$

Hence

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 2\pi a^3$$

Example 34

Find the closed line integral $\oint_C \underline{F}\cdot\underline{dr}$ for the vector field $\underline{F} = y^2\underline{i}+(x^2-z)\underline{j}+2xy\underline{k}$ and for the contour $ABCDEFGHA$ in Figure 18.

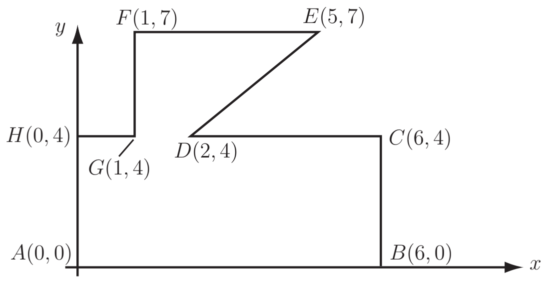

Figure 18: Closed contour $ABCDEFGHA$

Solution

To find the line integral directly would require eight line integrals i.e. along AB, BC, CD, DE, EF, FG, GH and HA. It is easier to carry out a surface integral to find $\displaystyle\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ which is equal to the required line integral $\displaystyle\oint_C \underline{F} \cdot \underline{dr}$ by Stokes' theorem.

As $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$, $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 - z & 2xy \end{vmatrix} = (2x+1)\underline{i} - 2y\underline{j} + (2x - 2y)\underline{k}$

As the contour lies in the x-y plane, the unit normal is $\underline{k}$ and $\underline{dS} = dxdy\underline{k}$

Hence $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (2x - 2y)dxdy$.

To work out $\displaystyle\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$, it is necessary to divide the area inside the contour into two smaller areas i.e. the rectangle $ABCDGH$ and the trapezium $DEFG$. On $ABCDGH$, the integral is

$$\begin{aligned} \int_{y=0}^{4}\int_{x=0}^{6} (2x - 2y)dxdy &= \int_{y=0}^{4} \Big[x^2 - 2xy\Big]_{x=0}^{6} dy = \int_{y=0}^{4} (36 - 12y)dy \\ &= \Big[36y - 6y^2\Big]_0^4 = 36 \times 4 - 6 \times 16 - 0 = 48 \end{aligned}$$

On $DEFG$, the integral is

$$\begin{aligned} \int_{y=4}^{7}\int_{x=1}^{y-2} (2x - 2y)dxdy &= \int_{y=4}^{7} \Big[x^2 - 2xy\Big]_{x=1}^{y-2} dy = \int_{y=4}^{7} (-y^2 + 2y + 3)dy \\ &= \Big[-\frac{1}{3}y^3 + y^2 + 3y\Big]_4^7 = -\frac{343}{3} + 49 + 21 + \frac{64}{3} - 16 - 12 = -51 \end{aligned}$$

So the full integral is, $\displaystyle\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 48 - 51 = -3$.

$\therefore$ By Stokes' theorem, $\displaystyle\oint_C \underline{F} \cdot \underline{dr} = -3$

From Stokes' theorem, it can be seen that surface integrals of the form $\displaystyle\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ depend only on the contour bounding the surface and not on the internal part of the surface.

Task

Verify Stokes' theorem for the vector field $\underline{F} = x^2\underline{i} + 2xy\underline{j} + z\underline{k}$ and the triangle with vertices at $(0,0,0)$, $(3,0,0)$ and $(3,1,0)$.

First find the normal vector $\underline{dS}$:

Your solution

Answer

$dxdy\underline{k}$

Then find the vector $\underline{\nabla} \times \underline{F}$:

Your solution

Answer

$2y\underline{k}$

Now evaluate the double integral $\displaystyle\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ over the triangle:

Your solution

Answer

1

Finally find the integral $\int \underline{F} \cdot \underline{dr}$ along the 3 sides of the triangle and so verify that the two sides of the Stokes' theorem are equal:

Your solution

Answer

$9 + 3 - 11 = 1$, Both sides of Stokes' theorem have value 1.

Exercises

1. Using plane-polar coordinates (or cylindrical polar coordinates with $z = 0$), verify Stokes' theorem for the vector field $\underline{F} = \rho\underline{\hat{\rho}} + \rho \cos\left(\dfrac{\pi\rho}{2}\right)\underline{\hat{\phi}}$ and the semi-circle $\rho \leq 1$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.
2. Verify Stokes' theorem for the vector field $\underline{F} = 2x\underline{i} + (y^2 - z)\underline{j} + xz\underline{k}$ and the contour around the rectangle with vertices at $(0, -2, 0)$,$(2, -2, 0)$, $(2, 0, 1)$ and $(0, 0, 1)$.
3. Verify Stokes' theorem for the vector field $\underline{F} = -y\underline{i} + x\underline{j} + z\underline{k}$
   (a) Over the triangle $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.
   (b) Over the triangle $(1, 0, 0), (1, 1, 0), (1, 1, 1)$.
4. Use Stokes' theorem to evaluate the integral

   $$\oint_C \underline{F} \cdot \underline{dr} \quad \text{where} \quad \underline{F} = \left(\sin(\frac{1}{x} + 1) + 5y\right)\underline{i} + (2x - e^{y^2})\underline{j}$$

   and C is the contour starting at $(0, 0)$ and going to $(5, 0)$, $(5, 2)$, $(6, 2)$, $(6, 5)$, $(3, 5)$, $(3, 2)$, $(0, 2)$ and returning to $(0, 0)$.

Answers

1. Both integrals give 0,

2. Both integrals give 1

3. (a) Both integrals give 1 (b) Both integrals give 0 (as $\underline{\nabla} \times \underline{F}$ is perpendicular to $\underline{dS}$)

4. -57, $[\underline{\nabla} \times \underline{F} = -3\,\underline{k}]$.

2. Gauss' theorem

This is sometimes known as the **divergence theorem** and is similar in form to Stokes' theorem but equates a surface integral to a volume integral. Gauss' theorem states that for a volume V, bounded by a closed surface S, any 'well-behaved' vector field $\underline{F}$ satisfies

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\, dV$$

Notes:

(a) $\underline{dS}$ is a unit normal pointing outwards from the interior of the volume V.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a surface integral over a surface composed of several distinct parts (e.g. a cube).

(d) $\underline{\nabla} \cdot \underline{F}$ is a scalar field representing the divergence of the vector field $\underline{F}$ and may, alternatively, be written as div $\underline{F}$.

(e) Gauss' theorem can be justified in a manner similar to that used for Stokes' theorem (i.e. by proving it for a small volume element, then summing up the volume elements and allowing the internal surface contributions to cancel.)

Key Point 9

Gauss' Theorem

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV$$

The closed surface integral of the scalar product of a vector function with the unit normal (or flux of a vector function through a surface) is equal to the integral of the divergence of that vector function over the corresponding volume.

Example 35

Verify Gauss' theorem for the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ and the function $\underline{F} = x\underline{i} + z\underline{j}$

Solution

To find $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$, the integral must be evaluated for all six faces of the cube and the results summed.

On the face $x = 0$, $\underline{F} = z\underline{j}$ and $\underline{dS} = -\underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dydz = 0$$

On the face $x = 1$, $\underline{F} = \underline{i} + z\underline{j}$ and $\underline{dS} = \underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 1\, dydz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 1\, dydz = 1$$

On the face $y = 0$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = -\underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = -z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = -\int_0^1 \int_0^1 z\, dxdz = -\frac{1}{2}$$

On the face $y = 1$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = \underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = z\, dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 z\, dxdz = \frac{1}{2}$$

On the face $z = 0$, $\underline{F} = x\underline{i}$ and $\underline{dS} = -\underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\, dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dxdy = 0$$

On the face $z = 1$, $\underline{F} = x\underline{i} + \underline{j}$ and $\underline{dS} = \underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\, dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\, dxdy = 0$$

Thus, summing over all six faces, $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + 0 = 1.$

To find $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\, dV$ note that $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial}{\partial x}x + \dfrac{\partial}{\partial y}z = 1 + 0 = 1.$

So $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\, dV = \int_0^1 \int_0^1 \int_0^1 1\, dxdydz = 1.$

So $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\, dV = 1$ hence verifying Gauss' theorem.

Note: The volume integral needed just one triple integral, but the surface integral required six double integrals. Reducing the number of integrals is often the motivation for using Gauss' theorem.

Example 36

Use Gauss' theorem to evaluate the surface integral $\int\int_S \underline{F} \cdot \underline{dS}$ where $\underline{F}$ is the vector field $x^2y\underline{i} + 2xy\underline{j} + z^3\underline{k}$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

Note that to carry out the surface integral directly will involve, as in Example 35, the evaluation of six double integrals. However, by Gauss' theorem, the same result comes from the volume integral $\int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$. As $\underline{\nabla} \cdot \underline{F} = 2xy + 2x + 3z^2$, we have the triple integral

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_0^1 (2xy + 2x + 3z^2)\, dxdydz \\
= & \int_0^1 \int_0^1 \left[x^2y + x^2 + 3xz^2 \right]_{x=0}^1 dydz = \int_0^1 \int_0^1 (y + 1 + 3z^2) dydz \\
= & \int_0^1 \left[\frac{1}{2}y^2 + y + 3yz^2 \right]_{y=0}^1 dz = \int_0^1 (\frac{1}{2} + 1 + 3z^2) dz = \int_0^1 (\frac{3}{2} + 3z^2) dz \\
= & \left[\frac{3}{2}z + z^3 \right]_0^1 = \frac{5}{2}
\end{aligned}$$

The six double integrals would also sum to $\frac{5}{2}$ but this approach would require much more effort.

Engineering Example 5

Gauss' law

Introduction

From Gauss' theorem, it is possible to derive a result which can be used to gain insight into situations arising in Electrical Engineering. Knowing the electric field on a closed surface, it is possible to find the electric charge within this surface. Alternatively, in a sufficiently symmetrical situation, it is possible to find the electric field produced by a given charge distribution.
Gauss' theorem states

$$\int\int_S \underline{F} \cdot \underline{dS} = \int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$$

If $\underline{F} = \underline{E}$, the electric field, it can be shown that,

$$\underline{\nabla} \cdot \underline{F} = \underline{\nabla} \cdot \underline{E} = \frac{q}{\varepsilon_0}$$

where q is the amount of charge per unit volume, or charge density, and ε_0 is the permittivity of free space: $\varepsilon_0 = 10^{-9}/36\pi$ F m^{-1} $\approx$ 8.84$\times 10^{-12}$ F m^{-1}. Gauss' theorem becomes in this case

$$\int\int_S \underline{E} \cdot \underline{dS} = \int\int\int_V \underline{\nabla} \cdot \underline{E}\, dV = \int\int\int_V \frac{q}{\varepsilon_0}\, dV = \frac{1}{\varepsilon_0}\int\int\int_V q\, dV = \frac{Q}{\varepsilon_0}$$

i.e.

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

which is known as Gauss' law. Here Q is the total charge inside the surface S.

Note: this is one of the important Maxwell's Laws.

Problem in words

A point charge lies at the centre of a cube. Given the electric field, find the magnitude of the charge, using Gauss' law .

Mathematical statement of problem

Consider the cube $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$, $-\frac{1}{2} \leq z \leq \frac{1}{2}$ where the dimensions are in metres. A point charge Q lies at the centre of the cube. If the electric field on the top face $(z = \frac{1}{2})$ is given by

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

find the charge Q from Gauss' law .

$$\left[\text{Hint :} \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \left(x^2 + y^2 + \frac{1}{4}\right)^{-\frac{3}{2}}\, dy\, dx = \frac{4\pi}{3}\right]$$

Mathematical analysis

From Gauss' law

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

so

$$Q = \varepsilon_0 \int\int_S \underline{E} \cdot \underline{dS} = 6\varepsilon_0 \int\int_{S(top)} \underline{E} \cdot \underline{dS}$$

since, using the symmetry of the six faces of the cube, it is possible to integrate over just one of them (here the top face is chosen) and multiply by 6. On the top face

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + \frac{1}{2}\underline{k}}{\left(x^2 + y^2 + \frac{1}{4}\right)^{\frac{3}{2}}}$$

and

$$\begin{aligned} \underline{dS} &= \text{(element of surface area)} \times \text{(unit normal)} \\ &= dx\, dy\, \underline{k} \end{aligned}$$

So

$$\begin{aligned} \underline{E} \cdot \underline{dS} &= 10 \frac{\frac{1}{2}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}} \, dy \, dx \\ &= 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}} \, dy \, dx \end{aligned}$$

Now

$$\begin{aligned} \int\int_{S(top)} \underline{E} \cdot \underline{dS} &= \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}} \, dy \, dx \\ &= 5 \times \frac{4\pi}{3} \quad \text{(using the hint)} \\ &= \frac{20\pi}{3} \end{aligned}$$

So, from Gauss' law,

$$Q = 6\varepsilon_0 \times \frac{20\pi}{3} = 40\pi\varepsilon_0 \approx 10^{-9}\text{C}$$

Interpretation

Gauss' law can be used to find a charge from its effects elsewhere.

The form of $\underline{E} = 10\dfrac{x\underline{i}+y\underline{j}+\frac{1}{2}\underline{k}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}$ comes from the fact that $\underline{E}$ is radial and equals $10\dfrac{\underline{r}}{r^3} = 10\dfrac{\hat{\underline{r}}}{r^2}$

Example 37

Verify Gauss' theorem for the vector field $\underline{F} = y^2\underline{j} - xz\underline{k}$ and the triangular prism with vertices at $(0,0,0)$, $(2,0,0)$, $(0,0,1)$, $(0,4,0)$, $(2,4,0)$ and $(0,4,1)$ (see Figure 19).

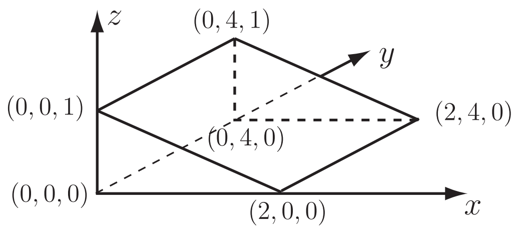

Figure 19: The triangular prism defined by six vertices

Solution

As $\underline{F} = y^2\underline{j} - xz\underline{k}$, $\underline{\nabla} \cdot \underline{F} = 0 + 2y - x = 2y - x$.
Thus

$$\begin{aligned}
\iiint_V \underline{\nabla} \cdot \underline{F} dV &= \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=0}^{1-x/2} (2y-x)dzdydx \\
&= \int_{x=0}^{2}\int_{y=0}^{4}\left[2yz - xz\right]_{z=0}^{1-x/2} dydx = \int_{x=0}^{2}\int_{y=0}^{4}(2y - xy - x + \frac{1}{2}x^2)dydx \\
&= \int_{x=0}^{2}\left[y^2 - \frac{1}{2}xy^2 - xy + \frac{1}{2}x^2y\right]_{y=0}^{4} dx \quad = \int_{x=0}^{2}(16 - 12x + 2x^2)dx \\
&= \left[16x - 6x^2 + \frac{2}{3}x^3\right]_0^2 = \frac{40}{3}
\end{aligned}$$

To work out $\iint_S \underline{F} \cdot \underline{dS}$, it is necessary to consider the contributions from the five faces separately.
On the front face, $y = 0$, $\underline{F} = -xz\underline{k}$ and $\underline{dS} = -\underline{j}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the back face, $y = 4$, $\underline{F} = 16\underline{j} - xz\underline{k}$ and $\underline{dS} = \underline{j}$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{z=0}^{1-x/2} 16dzdx = \int_{x=0}^{2}\left[\ 16z\right]_{z=0}^{1-x/2} dx = \int_{x=0}^{2} 16(1 - x/2)dx = \left[\ 16x - 4x^2\right]_0^2 = 16.$$

On the left face, $x = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{i}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the bottom face, $z = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{k}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the top right (sloping) face, $z = 1 - x/2$, $\underline{F} = y^2\underline{j} + (\frac{1}{2}x^2 - x)\underline{k}$ and the unit normal $\underline{\hat{n}} = \frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}$
Thus $\underline{dS} = \left[\frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}\right] dydw$ where dw measures the distance along the slope for a constant y.
As $dw = \frac{\sqrt{5}}{2}dx$, $\underline{dS} = \left[\frac{1}{2}\underline{i} + \underline{k}\right] dydx$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{y=0}^{4}(\frac{1}{2}x^2 - x)dydx = \int_{x=0}^{2}(2x^2 - 4x)dx = \left[\frac{2}{3}x^3 - 2x^2\right]_0^2 = -\frac{8}{3}.$$

Adding the contributions, $\iint_S \underline{F} \cdot \underline{dS} = 0 + 16 + 0 + 0 - \frac{8}{3} = \frac{40}{3}$.

Thus $\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV = \frac{40}{3}$ hence verifying Gauss' divergence theorem.

Engineering Example 6

Field strength around a charged line

Problem in words

Find the electric field strength at a given distance from a uniformly charged line.

Mathematical statement of problem

Determine the electric field at a distance r from a uniformly charged line (charge per unit length ρ_L). You may assume from symmetry that the field points directly away from the line.

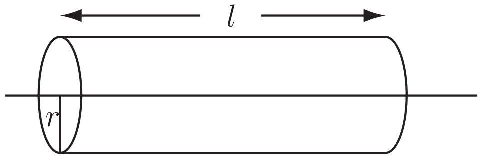

Figure 20: Field strength around a line charge

Mathematical analysis

Imagine a cylinder a distance r from the line and of length l (see Figure 20). From Gauss' law

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

As the charge per unit length is ρ_L, then the right-hand side equals $\rho_L l/\varepsilon_0$. On the left-hand side, the integral can be expressed as the sum

$$\int\int_S \underline{E} \cdot \underline{dS} = \int\int_{S(ends)} \underline{E} \cdot \underline{dS} \; + \int\int_{S(curved)} \underline{E} \cdot \underline{dS}$$

Looking first at the circular ends of the cylinder, the fact that the field lines point radially away from the charged line implies that the electric field is in the plane of these circles and has no normal component. Therefore $\underline{E} \cdot \underline{dS}$ will be zero for these ends.
Next, over the curved surface of the cylinder, the electric field is normal to it, and the symmetry of the problem implies that the strength of the electric field will be constant (here denoted by E). Therefore the integral = Total curved surface area × Field strength $= 2\pi r l E$.

So, by Gauss' law

$$\int\int_{S(ends)} \underline{E} \cdot \underline{dS} \; + \int\int_{S(curved)} \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

or

$$0 \; + \; 2\pi r l E = \frac{\rho_L l}{\varepsilon_0}$$

Interpretation

Hence, the field strength E is given by $\quad E = \dfrac{\rho_L}{2\pi\varepsilon_0 r}$

Engineering Example 7

Field strength on a cylinder

Problem in words

Given the electric field $\underline{E}$ on the surface of a cylinder, use Gauss' law to find the charge per unit length.

Mathematical statement of problem

On the surface of a long cylinder of radius a and length l, the electric field is given by

$$\underline{E} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{(a + b\cos\theta)\,\underline{\hat{r}} - b\sin\theta\,\underline{\hat{\theta}}}{(a^2 + 2ab\cos\theta + b^2)}$$

(using cylindrical polar co-ordinates) due to a line of charge a distance b ($< a$) from the centre of the cylinder. Using Gauss' law , find the charge per unit length.

Hint:- $\displaystyle\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,d\theta = \frac{2\pi}{a}$

Mathematical analysis

Consider a cylindrical section - as in the previous example, there are no contributions from the ends of the cylinder since the electric field has no normal component here. However, on the curved surface

$$\underline{dS} = a\,d\theta\,dz\,\underline{\hat{r}}$$

so

$$\underline{E}\cdot\underline{dS} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,a\,d\theta\,dz$$

Integrating over the curved surface of the cylinder

$$\begin{aligned}
\int\!\!\int_S \underline{E}\cdot\underline{dS} &= \int_{z=0}^{l}\int_{\theta=0}^{\theta=2\pi} \frac{a\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,d\theta\,dz \\
&= \frac{a\rho_L l}{2\pi\varepsilon_0}\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,d\theta \\
&= \frac{\rho_L l}{\varepsilon_0} \quad \text{using the given result for the integral.}
\end{aligned}$$

Then, if Q is the total charge inside the cylinder, from Gauss' law

$$\frac{\rho_L l}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \text{so} \quad \rho_L = \frac{Q}{l} \quad \text{as one would expect.}$$

Interpretation

Therefore the charge per unit length on the line of charge is given by ρ_L (i.e. the charge per unit length is constant).

Task

Verify Gauss' theorem for the vector field $\underline{F} = x\underline{i} - y\underline{j} + z\underline{k}$ and the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

(a) Find the vector $\underline{\nabla} \cdot \underline{F}$.

(b) Evaluate the integral $\displaystyle\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} \underline{\nabla} \cdot \underline{F} dxdydz.$

(c) For each side, evaluate the normal vector $\underline{dS}$ and the surface integral $\displaystyle\int\int_S \underline{F} \cdot \underline{dS}.$

(d) Show that the two sides of the statement of Gauss' theorem are equal.

Your solution

Answer

(a) $1 - 1 + 1 = 1$

(b) 1

(c) $-dxdy\underline{k},\ 0;\quad dxdy\underline{k},\ \ 1;\quad -dxdz\underline{j},\ 0;\quad dxdz\underline{j},\ \ -1;\quad -dydz\underline{i},\ 0;\quad dydz\underline{i},\ \ 1$

(d) Both sides are equal to 1.

Exercises

1. Verify Gauss' theorem for the vector field $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and the cuboid $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 4$.

2. Verify Gauss' theorem, using cylindrical polar coordinates, for the vector field $\underline{F} = \rho^{-2}\underline{\hat{\rho}}$ over the cylinder $0 \leq \rho \leq r_0$, $-1 \leq z \leq 1$ for

   (a) $r_0 = 1$

   (b) $r_0 = 2$

3. If S is the surface of the tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, find the surface integral
$\displaystyle\iint_S (x\underline{i} + yz\underline{j}) \cdot \underline{dS}$

   (a) directly

   (b) by using Gauss' theorem

   Hint :- When evaluating directly, show that the unit normal on the sloping face is $\frac{1}{\sqrt{3}}(\underline{i}+\underline{j}+\underline{k})$ and that $\underline{dS} = (\underline{i} + \underline{j} + \underline{k})dxdy$

Answers

1. Both sides are 156,

2. Both sides equal (a) 4π, (b) 2π,

3. (a) $\dfrac{5}{24}$ [only contribution is from the sloping face] (b) $\dfrac{5}{24}$ [by volume integral of $(1 + z)$].

3. Green's Identities (3D)

Like Gauss' theorem, Green's identities relate surface integrals to volume integrals. However, Green's identities are concerned with two scalar fields $u(x,y,z)$ and $w(x,y,z)$. Two statements of Green's identities are as follows

$$\iint_S (u\underline{\nabla} w)\cdot \underline{dS} = \iiint_V \left\{\underline{\nabla} u\cdot\underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV \qquad [1]$$

and

$$\iint_S \{u\underline{\nabla} w - v\underline{\nabla} u\}\cdot \underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV \qquad [2]$$

Proof of Green's identities

Green's identities can be derived from Gauss' theorem and a vector derivative identity.

Vector identity (1) from subsection 6 of 28.2 states that $\underline{\nabla}\cdot(\phi\underline{A}) = (\underline{\nabla}\phi)\cdot\underline{A} + \phi(\underline{\nabla}\cdot\underline{A})$.
Letting $\phi = u$ and $\underline{A} = \underline{\nabla} w$ in this identity,

$$\underline{\nabla}\cdot(u\underline{\nabla} w) = (\underline{\nabla} u)\cdot(\underline{\nabla} w) + u(\underline{\nabla}\cdot(\underline{\nabla} w)) = (\underline{\nabla} u)\cdot(\underline{\nabla} w) + u\underline{\nabla}^2 w$$

Gauss' theorem states

$$\iint_S \underline{F}\cdot\underline{dS} = \iiint_V \underline{\nabla}\cdot\underline{F} dV$$

Now, letting $\underline{F} = u\underline{\nabla} w$,

$$\begin{aligned}\iint_S (u\underline{\nabla} w)\cdot\underline{dS} &= \iiint_V \underline{\nabla}\cdot(u\underline{\nabla} w) dV\\ &= \iiint_V \left\{(\underline{\nabla} u)\cdot(\underline{\nabla} w) + u\underline{\nabla}^2 w\right\} dV\end{aligned}$$

This is Green's identity [1].
Reversing the roles of u and w,

$$\iint_S (w\underline{\nabla} u)\cdot\underline{dS} = \iiint_V \left\{(\underline{\nabla} w)\cdot(\underline{\nabla} u) + w\underline{\nabla}^2 u\right\} dV$$

Subtracting the last two equations yields Green's identity [2].

Key Point 10

Green's Identities

[1] $$\iint_S (u\underline{\nabla} w)\cdot \underline{dS} = \iiint_V \left\{\underline{\nabla} u\cdot\underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV$$

[2] $$\iint_S \{u\underline{\nabla} w - v\underline{\nabla} u\}\cdot \underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV$$

Example 38

Verify Green's first identity for $u = (x - x^2)y$, $w = xy + z^2$ and the unit cube, $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

As $w = xy + z^2$, $\underline{\nabla} w = y\underline{i} + x\underline{j} + 2z\underline{k}$. Thus $u\underline{\nabla} w = (xy - x^2y)(y\underline{i} + x\underline{j} + 2z\underline{k})$ and the surface integral is of this quantity (scalar product with $\underline{dS}$) integrated over the surface of the unit cube.

On the three faces $x = 0$, $x = 1$, $y = 0$, the vector $u\underline{\nabla} w = \underline{0}$ and so the contribution to the surface integral is zero.

On the face $y = 1$, $u\underline{\nabla} w = (x - x^2)(\underline{i} + x\underline{j} + 2z\underline{k})$ and $\underline{dS} = dxdz\underline{j}$ so $(u\underline{\nabla} w)\cdot\underline{dS} = (x^2 - x^3)dxdz$ and the contribution to the integral is

$$\int_{x=0}^{1}\int_{z=0}^{1}(x^2 - x^3)dzdx = \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

On the face $z = 0$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j})$ and $\underline{dS} = -dxdz\underline{k}$ so $(u\underline{\nabla} w)\cdot\underline{dS} = 0$ and the contribution to the integral is zero.

On the face $z = 1$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j} + 2\underline{k})$ and $\underline{dS} = dxdy\underline{k}$ so $(u\underline{\nabla} w)\cdot\underline{dS} = 2y(x - x^2)dxdy$ and the contribution to the integral is

$$\int_{x=0}^{1}\int_{y=0}^{1} 2y(x - x^2)dydx = \int_{x=0}^{1}\left[y^2(x - x^2)\right]_{y=0}^{1} dx = \int_0^1 (x - x^2)dx = \frac{1}{6}.$$

Thus, $\displaystyle\iint_S (u\underline{\nabla} w)\cdot\underline{dS} = 0 + 0 + 0 + \frac{1}{12} + 0 + \frac{1}{6} = \frac{1}{4}.$

Now evaluate $\displaystyle\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV.$

Note that $\underline{\nabla} u = (1 - 2x)y\underline{i} + (x - x^2)\underline{j}$ and $\underline{\nabla}^2 w = 2$ so

$$\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w = (1 - 2x)y^2 + (x - x^2)x + 2(x - x^2)y = x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2$$

and the integral

$$\begin{aligned}
\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV &= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(x^2 - x^3 + 2xy - 2x^2y + y^2 - 2xy^2)dxdydz \\
&= \int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^3}{3} - \frac{x^4}{4} + x^2y - \frac{2}{3}x^3y + xy^2 - x^2y^2\right]_{x=0}^{1} dydz \\
&= \int_{z=0}^{1}\int_{y=0}^{1}\left(\frac{1}{12} + \frac{y}{3}\right)dydz = \int_{z=0}^{1}\left[\frac{y}{12} + \frac{y^2}{6}\right]_{y=0}^{1} dz \\
&= \int_{z=0}^{1}\left(\frac{1}{4}\right)dz = \left[\frac{z}{4}\right]_{z=0}^{1} = \frac{1}{4}
\end{aligned}$$

Hence $\displaystyle\iint_S (u\underline{\nabla} w)\cdot\underline{dS} = \iiint_V \left[\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right] dV = \frac{1}{4}$ and Green's first identity is verified.

Green's theorem in the plane

This states that

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

S is a 2-D surface with perimeter C; $P(x,y)$ and $Q(x,y)$ are scalar functions.

This should not be confused with Green's identities.

Justification of Green's theorem in the plane

Green's theorem in the plane can be derived from Stokes' theorem.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Now let $\underline{F}$ be the vector field $P(x,y)\underline{i} + Q(x,y)\underline{j}$ i.e. there is no dependence on z and there are no components in the $z-$ direction. Now

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underline{k}$$

and $\underline{dS} = dxdy\underline{k}$ giving $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dxdy$.

Thus Stokes' theorem becomes

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_C \underline{F} \cdot \underline{dr}$$

and Green's theorem in the plane follows.

Key Point 11

Green's Theorem in the Plane

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

This relates a line integral around a closed path C with a double integral over the region S enclosed by C. It is effectively a two-dimensional form of Stokes' theorem.

Example 39

Evaluate the line integral $\oint_C \left[(4x^2+y-3)dx+(3x^2+4y^2-2)dy\right]$ around the rectangle $0 \le x \le 3$, $0 \le y \le 1$.

Solution

The integral could be obtained by evaluating four line integrals but it is easier to note that $[(4x^2+y-3)dx+(3x^2+4y^2-2)dy]$ is of the form $Pdx+Qdy$ with $P = 4x^2+y-3$ and $Q = 3x^2+4y^2-2$. It is thus of a suitable form for Green's theorem in the plane.

Note that $\dfrac{\partial Q}{\partial x} = 6x$ and $\dfrac{\partial P}{\partial y} = 1$.

Green's theorem in the plane becomes

$$\begin{aligned}\oint_C \{(4x^2+y-3)dx+(3x^2+4y^2-2)dy\} &= \int_{y=0}^{1}\int_{x=0}^{3}(6x-1)\,dxdy \\ &= \int_{y=0}^{1}\left[3x^2-x\right]_{x=0}^{3} dy = \int_{y=0}^{1} 24\ dy = 24\end{aligned}$$

Example 40

Verify Green's theorem in the plane for the integral $\oint_C \left[4zdy + (y^2-2)dz\right]$ and the triangular contour starting at the origin $O = (0,0,0)$ and going to $A = (0,2,0)$ and $B = (0,0,1)$ before returning to the origin.

Solution

The whole of the contour is in the plane $x = 0$ and Green's theorem in the plane becomes

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$$

(a) Firstly evaluate $\oint_C \left\{4zdy + (y^2-2)dz\right\}$.

On OA, $z = 0$ and $dz = 0$. As the integrand is zero, the integral will also be zero.

On AB, $z = (1 - \frac{y}{2})$ and $dz = -\frac{1}{2}dy$. The integral is

$$\int_{y=2}^{0} \left((4-2y)dy - \frac{1}{2}(y^2-2)dy\right) = \int_2^0 (5 - 2y - \frac{1}{2}y^2)dy = \left[5y - y^2 - \frac{1}{6}y^3\right]_2^0 = -\frac{14}{3}$$

On BO, $y = 0$ and $dy = 0$. The integral is $\int_1^0 (-2)dz = \left[-2z\right]_1^0 = 2$.

Summing, $\oint_C \left(4zdy + (y^2-2)dz\right) = -\frac{8}{3}$

(b) Secondly evaluate $\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$

In this example, $P = 4z$ and $Q = y^2 - 2$. Thus $\frac{\partial P}{\partial z} = 4$ and $\frac{\partial Q}{\partial y} = 2y$. Hence,

$$\begin{aligned}
\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz &= \int_{y=0}^{2}\int_{z=0}^{1-y/2} (2y-4)\,dzdy \\
&= \int_{y=0}^{2} \left[2yz - 4z\right]_{z=0}^{1-y/2} dy = \int_{y=0}^{2} \left(-y^2 + 4y - 4\right) dy \\
&= \left[-\frac{1}{3}y^3 + 2y^2 - 4y\right]_0^2 = -\frac{8}{3}
\end{aligned}$$

Hence:

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz = -\frac{8}{3}$$ and Green's theorem in the plane is verified.

One very useful, special case of Green's theorem in the plane is when $Q = x$ and $P = -y$. The theorem becomes

$$\oint_C \{-ydx + xdy\} = \iint_S (1 - (-1))\, dxdy$$

The right-hand side becomes $\iint_S 2\, dxdy$ i.e. $2A$ where A is the area inside the contour C. Hence

$$A = \frac{1}{2}\oint_C \{xdy - ydx\}$$

This result is known as the **area theorem**. It gives us the area bounded by a curve C in terms of a line integral around C.

Example 41

Verify the area theorem for the segment of the circle $x^2 + y^2 = 4$ lying above the line $y = 1$.

Solution

Firstly, the area of the segment $ADBC$ can be found by subtracting the area of the triangle $OADB$ from the area of the sector $OACB$. The triangle has area $\frac{1}{2} \times 2\sqrt{3} \times 1 = \sqrt{3}$. The sector has area $\frac{\pi}{3} \times 2^2 = \frac{4}{3}\pi$. Thus segment $ADBC$ has area $\frac{4}{3}\pi - \sqrt{3}$.

Now, evaluate the integral $\oint_C \{xdy - ydx\}$ around the segment.

Along the line, $y = 1$, $dy = 0$ so the integral $\int_C \{xdy - ydx\}$ becomes $\int_{-\sqrt{3}}^{\sqrt{3}} (x \times 0 - 1 \times dx) = \int_{-\sqrt{3}}^{\sqrt{3}} (-dx) = -2\sqrt{3}$.

Along the arc of the circle, $y = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$ so $dy = -x(4 - x^2)^{-1/2}dx$. The integral $\int_C \{xdy - ydx\}$ becomes

$$\begin{aligned}
\int_{\sqrt{3}}^{-\sqrt{3}} \{-x^2(4 - x^2)^{-1/2} - (4 - x^2)^{1/2}\}dx &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4}{\sqrt{4 - x^2}} dx \\
&= \int_{-\pi/3}^{\pi/3} 4\frac{1}{2\cos\theta} 2\cos\theta\, d\theta \text{ (letting } x = 2\sin\theta\text{)} \\
&= \int_{-\pi/3}^{\pi/3} 4d\theta = \frac{8}{3}\pi
\end{aligned}$$

So, $\frac{1}{2}\oint_C \{xdy - ydx\} = \frac{1}{2}\left[\frac{8}{3}\pi - 2\sqrt{3}\right] = \frac{4}{3}\pi - \sqrt{3}$.

Hence both sides of the area theorem equal $\frac{4}{3}\pi - \sqrt{3}$ thus verifying the theorem.

Task

Verify Green's theorem in the plane when applied to the integral

$$\oint_C \{(5x + 2y - 7)dx + (3x - 4y + 5)dy\}$$

where C represents the perimeter of the trapezium with vertices at $(0, 0)$, $(3, 0)$, $(6, 1)$ and $(1, 1)$.

First let $P = 5x + 2y - 7$ and $Q = 3x - 4y + 5$ and find $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$:

Your solution

Answer

1

Now find $\displaystyle\int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ over the trapezium:

Your solution

Answer

4 (by elementary geometry)

Now find $\displaystyle\int (Pdx + Qdy)$ along the four sides of the trapezium, beginning with the line from $(0, 0)$ to $(3, 0)$, and then proceeding anti-clockwise.

Your solution

Answers $1.5,\ 66,\ -62.5,\ -1$ whose sum is 4.

Finally show that the two sides of the statement of Green's theorem are equal:

Your solution

Answer

Both sides are 4.

Exercises

1. Verify Green's identity [1] (page 73) for the functions $u = xyz$, $w = y^2$ and the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

2. Verify the area theorem for

   (a) The area above $y = 0$, but below $y = 1 - x^2$.

   (b) The segment of the circle $x^2 + y^2 = 1$, to the upper left of the line $y = 1 - x$.

Answers

1. Both integrals in [1] equal $\dfrac{1}{2}$

2. (a) both sides give a value of $\dfrac{4}{3}$, (b) both sides give a value of $\dfrac{\pi}{4} - \dfrac{1}{2}$.