

Integral Vector Calculus

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Learning outcomes

In this Workbook you will learn how to integrate functions involving vectors. You will learn how to evaluate line integrals i.e. where a scalar or a vector is summed along a line or contour. You will be able to evaluate surface and volume integrals where a function involving vectors is summed over a surface or volume. You will learn about some theorems relating to line, surface or volume integrals viz Stokes' theorem, Gauss' divergence theorem and Green's theorem.

Line Integrals

29.1



Introduction

HELM workbook 28 considered the differentiation of scalar and vector fields. Here we consider how to integrate such fields along a line. Firstly, integrals involving scalars along a line will be considered. Subsequently, line integrals involving vectors will be considered. These can give scalar or vector answers depending on the form of integral involved. Of particular interest are the integrals of conservative vector fields.



Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the basic techniques of integration
- be familiar with the operators div, grad and curl



Learning Outcomes

On completion you should be able to ...

- integrate a scalar or vector quantity along a line

1. Line integrals

HELM 28 was concerned with evaluating an integral over **all** points within a rectangle or other shape (or over a cuboid or other volume). In a related manner, an integral can take place over a line or curve running through a two-dimensional (or three-dimensional) region. Line integrals may involve scalar or vector fields. Those involving scalar fields are dealt with first.

Line integrals in two dimensions

A line integral in two dimensions may be written as

$$\int_C F(x, y) dw$$

There are three main features determining this integral:

$F(x, y)$: This is the scalar function to be integrated e.g. $F(x, y) = x^2 + 4y^2$.

C : This is the curve along which integration takes place. e.g. $y = x^2$ or $x = \sin y$ or $x = t - 1$; $y = t^2$ (where x and y are expressed in terms of a parameter t).

dw : This gives the variable of the integration. Three main cases are dx , dy and ds . Here 's' is arc length and so indicates position along the curve C .

$$ds \text{ may be written as } ds = \sqrt{(dx)^2 + (dy)^2} \text{ or } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

A fourth case is when $F(x, y) dw$ has the form: $F_1 dx + F_2 dy$. This is a combination of the cases dx and dy .

The integral $\int_C F(x, y) ds$ represents the area beneath the surface $z = F(x, y)$ but above the curve C .

The integrals $\int_C F(x, y) dx$ and $\int_C F(x, y) dy$ represent the projections of this area onto the xz and yz planes respectively.

A particular case of the integral $\int_C F(x, y) ds$ is the integral $\int_C 1 ds$. This is a means of calculating the length along a curve i.e. an arc length.

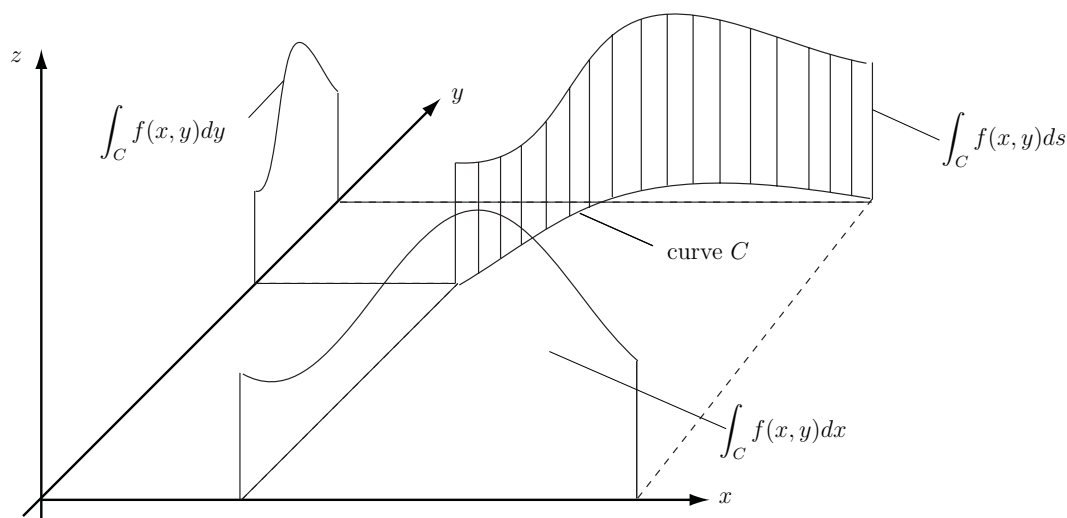


Figure 1: Representation of a line integral and its projections onto the xz and yz planes

The technique for evaluating a line integral is to express all quantities in the integral in terms of a single variable. If the integral is with respect to 'x' or 'y', then the curve 'C' and the function 'F' may be expressed in terms of the relevant variable. If the integral is with respect to ds , normally all quantities are expressed in terms of x . If x and y are given in terms of a parameter t , then t is used as the variable.



Example 1

Find $\int_C x(1+4y) dx$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$.

Solution

As this integral concerns only points along C and the integration is carried out with respect to x , y may be replaced by x^2 . The limits on x will be 0 to 1. So the integral becomes

$$\begin{aligned} \int_C x(1+4y) dx &= \int_{x=0}^1 x(1+4x^2) dx = \int_{x=0}^1 (x+4x^3) dx \\ &= \left[\frac{x^2}{2} + x^4 \right]_0^1 = \left(\frac{1}{2} + 1 \right) - (0) = \frac{3}{2} \end{aligned}$$



Example 2

Find $\int_C x(1+4y) dy$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same as Example 1 other than dx being replaced by dy .

Solution

As this integral concerns only points along C and the integration is carried out with respect to y , everything may be expressed in terms of y , i.e. x may be replaced by $y^{1/2}$. The limits on y will be 0 to 1. So the integral becomes

$$\begin{aligned} \int_C x(1+4y) dy &= \int_{y=0}^1 y^{1/2}(1+4y) dy = \int_{y=0}^1 (y^{1/2} + 4y^{3/2}) dy \\ &= \left[\frac{2}{3}y^{3/2} + \frac{8}{5}y^{5/2} \right]_0^1 = \left(\frac{2}{3} + \frac{8}{5} \right) - (0) = \frac{34}{15} \end{aligned}$$

**Example 3**

Find $\int_C x(1+4y) ds$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same integral and curve as the previous two examples but the integration is now carried out with respect to s , the arc length parameter.

Solution

As this integral is with respect to x , all parts of the integral can be expressed in terms of x . Along

$$y = x^2, \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (2x)^2} dx = \sqrt{1 + 4x^2} dx$$

So, the integral is

$$\int_C x(1+4y) ds = \int_{x=0}^1 x(1+4x^2) \sqrt{1+4x^2} dx = \int_{x=0}^1 x(1+4x^2)^{3/2} dx$$

This can be evaluated using the transformation $u = 1 + 4x^2$ so $du = 8x dx$ i.e. $x dx = \frac{du}{8}$.
When $x = 0$, $u = 1$ and when $x = 1$, $u = 5$.

Hence,

$$\begin{aligned} \int_{x=0}^1 x(1+4x^2)^{3/2} dx &= \frac{1}{8} \int_{u=1}^5 u^{3/2} du \\ &= \frac{1}{8} \times \frac{2}{5} \left[u^{5/2} \right]_1^5 = \frac{1}{20} [5^{5/2} - 1] \approx 2.745 \end{aligned}$$

Note that the results for Examples 1,2 and 3 are all different: Example 3 is the area between a curve and a surface above; Examples 1 and 2 give projections of this area onto other planes.

**Example 4**

Find $\int_C xy dx$ where, on C , x and y are given in terms of a parameter t by

$$x = 3t^2, \quad y = t^3 - 1 \text{ for } t \text{ varying from } 0 \text{ to } 1.$$

Solution

Everything can be expressed in terms of t , the parameter. Here $x = 3t^2$ so $dx = 6t dt$. The limits on t are $t = 0$ and $t = 1$. The integral becomes

$$\begin{aligned} \int_C xy dx &= \int_{t=0}^1 3t^2 (t^3 - 1) 6t dt = \int_{t=0}^1 (18t^6 - 18t^3) dt \\ &= \left[\frac{18}{7} t^7 - \frac{18}{4} t^4 \right]_0^1 = \frac{18}{7} - \frac{9}{2} - 0 = -\frac{27}{14} \end{aligned}$$



Key Point 1

A line integral is normally evaluated by expressing all variables in terms of one variable.

In general

$$\int_C f(x, y) ds \neq \int_C f(x, y) dy \neq \int_C f(x, y) dx$$



For $F(x, y) = 2x + y^2$, find (i) $\int_C F(x, y) dx$, (ii) $\int_C F(x, y) dy$,
(iii) $\int_C F(x, y) ds$ where C is the line $y = 2x$ from $(0, 0)$ to $(1, 2)$.

Express each integral as a simple integral with respect to a single variable and hence evaluate each integral:

Your solution

Answer

$$(i) \int_{x=0}^1 (2x + 4x^2) dx = \frac{7}{3}, \quad (ii) \int_{y=0}^2 (y + y^2) dy = \frac{14}{3}, \quad (iii) \int_{x=0}^1 (2x + 4x^2)\sqrt{5} dx = \frac{7}{3}\sqrt{5}$$



Find (i) $\int_C F(x, y) dx$, (ii) $\int_C F(x, y) dy$, (iii) $\int_C F(x, y) ds$ where $F(x, y) = 1$ and C is the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $(1, \frac{1}{2})$ to $(2, 2 - \frac{1}{4}\ln 2)$.

Your solution**Answer**

$$(i) \int_1^2 1 dx = 1, \quad (ii) \int_{1/2}^{2-(1/4)\ln 2} 1 dy = \frac{3}{2} - \frac{1}{4}\ln 2, \quad (iii) y = \frac{1}{2}x^2 - \frac{1}{4}\ln x \Rightarrow \frac{dy}{dx} = x - \frac{1}{4x}$$

$$\therefore \int 1 ds = \int_1^2 \sqrt{1 + (x - \frac{1}{4x})^2} dx = \int_1^2 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} dx = \int_1^2 (x + \frac{1}{4x}) dx = \frac{3}{2} + \frac{1}{4}\ln 2.$$



Find (i) $\int_C F(x, y) dx$, (ii) $\int_C F(x, y) dy$, (iii) $\int_C F(x, y) ds$ where $F(x, y) = \sin 2x$ and C is the curve $y = \sin x$ from $(0, 0)$ to $(\frac{\pi}{2}, 1)$.

Your solution**Answer**

$$(i) \int_0^{\pi/2} \sin 2x dx = 1, \quad (ii) \int_0^{\pi/2} 2 \sin x \cos^2 x dx = \frac{2}{3}$$

$$(iii) \int_0^{\pi/2} \sin 2x \sqrt{1 + \cos^2 x} dx = \frac{2}{3}(2\sqrt{2} - 1), \text{ using the substitution } u = 1 + \cos^2 x.$$

2. Line integrals of scalar products

Integrals of the form $\int_C \underline{F} \cdot d\underline{r}$ occur in applications such as the following.

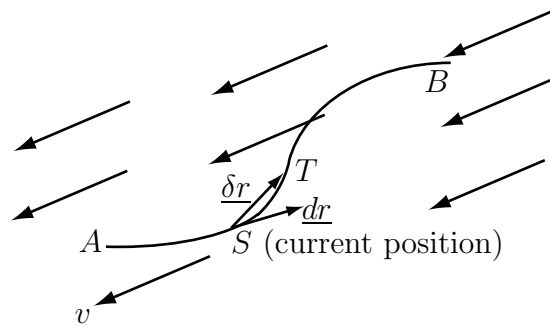


Figure 2: Schematic for cyclist travelling from A to B into a head wind

Consider a cyclist riding along the road from A to B (Figure 2). Suppose it is necessary to find the total work the cyclist has to do in overcoming a wind of velocity \underline{v} .

On moving from S to T , along an element $d\underline{r}$ of road, the work done is given by 'Force \times distance' $= |\underline{F}| \times |d\underline{r}| \cos \theta$ where \underline{F} , the force, is directly proportional to \underline{v} , but in the opposite direction, and $|d\underline{r}| \cos \theta$ is the component of the distance travelled in the direction of the wind.

So, the work done travelling $d\underline{r}$ is $-k\underline{v} \cdot d\underline{r}$. Letting $d\underline{r}$ become infinitesimally small, the work done becomes $-k\underline{v} \cdot d\underline{r}$ and the total work is $-k \int_A^B \underline{v} \cdot d\underline{r}$.

This is an example of the integral along a line, of the scalar product of a vector field, with a vector element of the line. The term **scalar line integral** is often used for integrals of this form. The vector $d\underline{r}$ may be considered to be $dx \underline{i} + dy \underline{j} + dz \underline{k}$.

Multiplying out the scalar product, the 'scalar line integral' of the vector \underline{F} along contour C , is given by $\int_C \underline{F} \cdot d\underline{r}$ and equals $\int_C \{F_x dx + F_y dy + F_z dz\}$ in three dimensions, and $\int_C \{F_x dx + F_y dy\}$ in two dimensions, where F_x, F_y, F_z are the components of \underline{F} .

If the contour C has its start and end points in the same positions i.e. it represents a closed contour, the symbol \oint_C rather than \int_C is used, i.e. $\oint_C \underline{F} \cdot d\underline{r}$.

As before, to evaluate the line integral, express the path and the function \underline{F} in terms of either x, y and z , or in terms of a parameter t . Note that t often represents time.



Example 5

Find $\int_C \{2xy dx - 5x dy\}$ where C is the curve $y = x^3$ $0 \leq x \leq 1$.

[This is the integral $\int_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = 2xy\underline{i} - 5x\underline{j}$ and $d\underline{r} = dx \underline{i} + dy \underline{j}$.]

Solution

It is possible to split this integral into two different integrals and express the first term as a function of x and the second term as a function of y . However, it is also possible to express everything in terms of x . Note that on C , $y = x^3$ so $dy = 3x^2 dx$ and the integral becomes

$$\begin{aligned} \int_C \{2xy dx - 5x dy\} &= \int_{x=0}^1 (2x x^3 dx - 5x 3x^2 dx) = \int_0^1 (2x^4 - 15x^3) dx \\ &= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4 \right]_0^1 = \frac{2}{5} - \frac{15}{4} - 0 = -\frac{67}{20} \end{aligned}$$

**Key Point 2**

An integral of the form $\int_C \underline{F} \cdot \underline{dr}$ may be expressed as $\int_C \{F_x dx + F_y dy + F_z dz\}$. Knowing the expression for the path C , every term in the integral can be further expressed in terms of one of the variables x , y or z or in terms of a parameter t and hence integrated.

If an integral is two-dimensional there are no terms involving z .

The integral $\int_C \underline{F} \cdot \underline{dr}$ evaluates to a scalar.

**Example 6**

Three paths from $(0, 0)$ to $(1, 2)$ are defined by

- (a) $C_1 : y = 2x$
- (b) $C_2 : y = 2x^2$
- (c) $C_3 : y = 0$ from $(0, 0)$ to $(1, 0)$ and $x = 1$ from $(1, 0)$ to $(1, 2)$

Sketch each path, and along each path find $\int \underline{F} \cdot \underline{dr}$, where $\underline{F} = y^2 \underline{i} + xy \underline{j}$.

Solution

(a) $\int \underline{F} \cdot \underline{dr} = \int \{y^2 dx + xy dy\}$. Along $y = 2x$, $\frac{dy}{dx} = 2$ so $dy = 2dx$. Then

$$\begin{aligned} \int_{C_1} \underline{F} \cdot \underline{dr} &= \int_{x=0}^1 \{(2x)^2 dx + x(2x)(2dx)\} \\ &= \int_0^1 (4x^2 + 4x^2) dx = \int_0^1 8x^2 dx = \left[\frac{8}{3}x^3 \right]_0^1 = \frac{8}{3} \end{aligned}$$

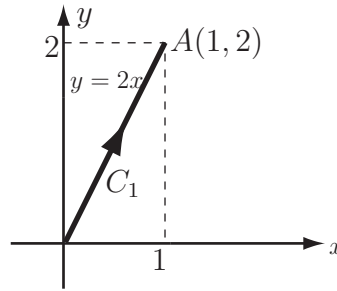


Figure 3(a): Integration along path C_1

(b) $\int \underline{F} \cdot \underline{dr} = \int \{y^2 dx + xy dy\}$. Along $y = 2x^2$, $\frac{dy}{dx} = 4x$ so $dy = 4x dx$. Then

$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_{x=0}^1 \{(2x^2)^2 dx + x(2x^2)(4x dx)\} = \int_0^1 12x^4 dx = \left[\frac{12}{5}x^5 \right]_0^1 = \frac{12}{5}$$

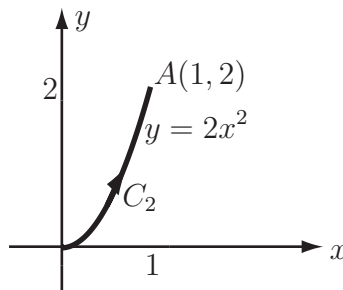


Figure 3(b): Integration along path C_2

Note that the answer is different to part (a), i.e., the line integral depends upon the path taken.

(c) As the contour C_3 , has two distinct parts with different equations, it is necessary to break the full contour OA into the two parts, namely OB and BA where B is the point $(1, 0)$. Hence

$$\int_{C_3} \underline{F} \cdot \underline{dr} = \int_O^B \underline{F} \cdot \underline{dr} + \int_B^A \underline{F} \cdot \underline{dr}$$

Solution (contd.)

Along OB , $y = 0$ so $dy = 0$. Then

$$\int_O^B \underline{F} \cdot \underline{dr} = \int_{x=0}^1 (0^2 dx + x \times 0 \times 0) = \int_0^1 0 dx = 0$$

Along AB , $x = 1$ so $dx = 0$. Then

$$\int_A^B \underline{F} \cdot \underline{dr} = \int_{y=0}^2 (y^2 \times 0 + 1 \times y \times dy) = \int_0^2 y dy = \left[\frac{1}{2} y^2 \right]_0^2 = 2.$$

Hence $\int_{C_3} \underline{F} \cdot \underline{dr} = 0 + 2 = 2$

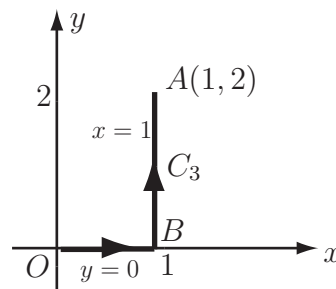


Figure 3(c): Integration along path C_3

Once again, the result is path dependent.

**Key Point 3**

In general, the value of a line integral depends on the path of integration as well as upon the end points.



Example 7

Find $\int_A^O \underline{F} \cdot d\underline{r}$, where $\underline{F} = y^2 \underline{i} + xy \underline{j}$ (as in Example 6) and the path C_4 from A to O is the straight line from $(1, 2)$ to $(0, 0)$, that is the reverse of C_1 in Example 6(a).

Deduce $\oint_C \underline{F} \cdot d\underline{r}$, the integral around the closed path C formed by the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$ and the line $y = 2x$ from $(1, 2)$ to $(0, 0)$.

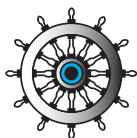
Solution

Reversing the path interchanges the limits of integration, which results in a change of sign for the value of the integral.

$$\int_A^O \underline{F} \cdot d\underline{r} = - \int_O^A \underline{F} \cdot d\underline{r} = -\frac{8}{3}$$

The integral along the parabola (calculated in Example 6(b)) evaluates to $\frac{12}{5}$, then

$$\oint_C \underline{F} \cdot d\underline{r} = \int_{C_2} \underline{F} \cdot d\underline{r} + \int_{C_4} \underline{F} \cdot d\underline{r} = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15} \approx -0.267$$



Example 8

Consider the vector field

$$\underline{F} = y^2 z^3 \underline{i} + 2xyz^3 \underline{j} + 3xy^2 z^2 \underline{k}$$

Let C_1 and C_2 be the curves from $O = (0, 0, 0)$ to $A = (1, 1, 1)$, given by

$$C_1 : x = t, \quad y = t, \quad z = t \quad (0 \leq t \leq 1)$$

$$C_2 : x = t^2, \quad y = t, \quad z = t^2 \quad (0 \leq t \leq 1)$$

- Evaluate the scalar integral of the vector field along each path.
- Find the value of $\oint_C \underline{F} \cdot d\underline{r}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O .

Solution

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

Now by substituting for $x = y = z = t$ in \underline{F} we have

$$\underline{F} = t^5\mathbf{i} + 2t^5\mathbf{j} + 3t^5\mathbf{k}$$

Hence $\underline{F} \cdot \frac{d\mathbf{r}}{dt} = t^5 + 2t^5 + 3t^5 = 6t^5$. The values of $t = 0$ and $t = 1$ correspond to the start and end point of C_1 and so these are the required limits of integration. Now

$$\int_{C_1} \underline{F} \cdot d\mathbf{r} = \int_0^1 \underline{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 6t^5 dt = \left[t^6 \right]_0^1 = 1$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + 2t\mathbf{k}$. Substituting $x = t^2$, $y = t$ and $z = t^2$ in \underline{F} we have

$$\underline{F} = t^8\mathbf{i} + 2t^9\mathbf{j} + 3t^8\mathbf{k} \text{ and } \underline{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^9 + 2t^9 + 6t^9 = 10t^9$$

$$\int_{C_2} \underline{F} \cdot d\mathbf{r} = \int_0^1 10t^9 dt = \left[t^{10} \right]_0^1 = 1$$

(b) For the closed path C

$$\oint_C \underline{F} \cdot d\mathbf{r} = \int_{C_1} \underline{F} \cdot d\mathbf{r} - \int_{C_2} \underline{F} \cdot d\mathbf{r} = 1 - 1 = 0$$

(Note: A line integral round a closed path is not necessarily zero - see Example 7.)

Further points on Example 8

Vector Field	Path	Line Integral
\underline{F}	C_1	1
\underline{F}	C_2	1
\underline{F}	closed	0

Note that the value of the line integral of \underline{F} is 1 for both paths C_1 and C_2 . In fact, this result would hold for any path from $(0, 0, 0)$ to $(1, 1, 1)$.

The field \underline{F} is an example of a **conservative vector field**; these are discussed in detail in the next subsection.

In $\int_C \underline{F} \cdot d\mathbf{r}$, the vector field \underline{F} may be the gradient of a scalar field or the curl of a vector field.



Consider the vector field

$$\underline{G} = x\underline{i} + (4x - y)\underline{j}$$

Let C_1 and C_2 be the curves from $O = (0, 0, 0)$ to $A = (1, 1, 1)$, given by

$$C_1 : x = t, \quad y = t, \quad z = t \quad (0 \leq t \leq 1)$$

$$C_2 : x = t^2, \quad y = t, \quad z = t^2 \quad (0 \leq t \leq 1)$$

- (a) Evaluate the scalar integral $\int_C \underline{G} \cdot d\underline{r}$ of each vector field along each path.
- (b) Find the value of $\oint_C \underline{G} \cdot d\underline{r}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O .

Your solution

Answer

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

Substituting for $x = y = z = t$ in \underline{G} we have

$$\underline{G} = t\mathbf{i} + 3t\mathbf{j} \text{ and } \underline{G} \cdot \frac{d\mathbf{r}}{dt} = t + 3t = 4t$$

The limits of integration are $t = 0$ and $t = 1$, then

$$\int_{C_1} \underline{G} \cdot d\mathbf{r} = \int_0^1 \underline{G} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 4t dt = \left[2t^2 \right]_0^1 = 2$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + 2t\mathbf{k}$.

Substituting $x = t^2$, $y = t$ and $z = t^2$ in \underline{G} we have

$$\underline{G} = t^2\mathbf{i} + (4t^2 - t)\mathbf{j} \text{ and } \underline{G} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 + 4t^2 - t$$

$$\int_{C_2} \underline{G} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 4t^2 - t) dt = \left[\frac{1}{2}t^4 + \frac{4}{3}t^3 - \frac{1}{2}t^2 \right]_0^1 = \frac{4}{3}$$

(b) For the closed path C
$$\oint_C \underline{G} \cdot d\mathbf{r} = \int_{C_1} \underline{G} \cdot d\mathbf{r} - \int_{C_2} \underline{G} \cdot d\mathbf{r} = 2 - \frac{4}{3} = \frac{2}{3}$$

(Note: The value of the integral around the closed path is non-zero, unlike Example 8.)


Example 9

Find $\int_C \{\nabla(x^2y)\} \cdot d\mathbf{r}$ where C is the contour $y = 2x - x^2$ from $(0, 0)$ to $(2, 0)$.

Here, ∇ refers to the gradient operator, i.e. $\nabla\phi \equiv \text{grad } \phi$

Solution

Note that $\nabla(x^2y) = 2xy\mathbf{i} + x^2\mathbf{j}$ so the integral is $\int_C \{2xy dx + x^2 dy\}$.

On $y = 2x - x^2$, $dy = (2 - 2x) dx$ so the integral becomes

$$\begin{aligned} \int_C \{2xy dx + x^2 dy\} &= \int_{x=0}^2 \{2x(2x - x^2) dx + x^2(2 - 2x) dx\} \\ &= \int_0^2 (6x^2 - 4x^3) dx = \left[2x^3 - x^4 \right]_0^2 = 0 \end{aligned}$$



Example 10

Two paths from $(0, 0)$ to $(4, 2)$ are defined by

(a) $C_1 : y = \frac{1}{2}x \quad 0 \leq x \leq 4$

(b) $C_2 :$ The straight line $y = 0$ from $(0, 0)$ to $(4, 0)$ followed by

$C_3 :$ The straight line $x = 4$ from $(4, 0)$ to $(4, 2)$

For each path find $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = 2x\underline{i} + 2y\underline{j}$.

Solution

(a) For the straight line $y = \frac{1}{2}x$ we have $dy = \frac{1}{2}dx$

Then,
$$\int_{C_1} \underline{F} \cdot \underline{dr} = \int_{C_1} 2x \, dx + 2y \, dy = \int_0^4 \left(2x + \frac{x}{2}\right) dx = \int_0^4 \frac{5x}{2} dx = 20$$

(b) For the straight line from $(0, 0)$ to $(4, 0)$ we have
$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_0^4 2x \, dx = 16$$

For the straight line from $(4, 0)$ to $(4, 2)$ we have
$$\int_{C_3} \underline{F} \cdot \underline{dr} = \int_0^2 2y \, dy = 4$$

Adding these two results gives
$$\int_C \underline{F} \cdot \underline{dr} = 16 + 4 = 20$$



Evaluate $\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$ along each of the following paths

(a) $C_1 :$ from $(1, 1)$ to $(2, 4)$ along the straight line $y = 3x - 2$:

(b) $C_2 :$ from $(1, 1)$ to $(2, 4)$ along the parabola $y = x^2$:

(c) $C_3 :$ along the straight line $x = 1$ from $(1, 1)$ to $(1, 4)$ then along the straight line $y = 4$ from $(1, 4)$ to $(2, 4)$.

Your solution

Answer

(a) $\int_1^2 (10x - 4) dx = 11,$

(b) $\int_1^2 (x + x^2 + 2x^3) dx = \frac{34}{3},$ (this differs from (a) showing path dependence)

(c) $\int_1^4 (1 + y) dy + \int_1^2 (x - 4) dx = 8$



For the function \underline{F} and paths in the last Task, deduce $\oint \underline{F} \cdot \underline{dr}$ for the closed paths

- (a) C_1 followed by the reverse of C_2 .
- (b) C_2 followed by the reverse of C_3 .
- (c) C_3 followed by the reverse of C_1 .

Your solution**Answer**

(a) $-\frac{1}{3},$ (b) $\frac{10}{3},$ (c) $-3.$ (note that all these are non-zero.)

Exercises

1. Consider $\int_C \underline{F} \cdot d\underline{r}$, where $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. Find the value of the line integral along each of the paths from $(0, 0)$ to $(1, 4)$.

(a) $y = 4x$ (b) $y = 4x^2$ (c) $y = 4x^{1/2}$ (d) $y = 4x^3$

2. Consider the vector field $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$ and the two curves between $(0, 0, 0)$ and $(1, -1, 2)$ defined by

$C_1 : x = t^2, y = -t, z = 2t$ for $0 \leq t \leq 1$.

$C_2 : x = t - 1, y = 1 - t, z = 2t - 2$ for $1 \leq t \leq 2$.

(a) Find $\int_{C_1} \underline{F} \cdot d\underline{r}$, $\int_{C_2} \underline{F} \cdot d\underline{r}$

(b) Find $\oint_C \underline{F} \cdot d\underline{r}$ where C is the closed path from $(0, 0, 0)$ to $(1, -1, 2)$ along C_1 and back to $(0, 0, 0)$ along C_2 .

3. Consider the vector field $\underline{G} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$ and the two curves between $(0, 0, 0)$ and $(1, -1, 2)$ defined by

$C_1 : x = t^2, y = -t, z = 2t$ for $0 \leq t \leq 1$.

$C_2 : x = t - 1, y = 1 - t, z = 2t - 2$ for $1 \leq t \leq 2$.

(a) Find $\int_{C_1} \underline{G} \cdot d\underline{r}$, $\int_{C_2} \underline{G} \cdot d\underline{r}$

(b) Find $\oint_C \underline{G} \cdot d\underline{r}$ where C is the closed path from $(0, 0, 0)$ to $(1, -1, 2)$ along C_1 and back to $(0, 0, 0)$ along C_2 .

4. Find $\int_C \underline{F} \cdot d\underline{r}$ along $y = 2x$ from $(0, 0)$ to $(2, 4)$ for

(a) $\underline{F} = \nabla(x^2y)$

(b) $\underline{F} = \nabla \times (\frac{1}{2}x^2y^2\underline{k})$ [Here $\nabla \times \underline{f}$ represents the curl of \underline{f}]

Answers

1. All are 12, and in fact the integral would be 12 for **any** path from $(0, 0)$ to $(1, 4)$.

2 (a) 2, $\frac{5}{3}$ (b) $\frac{1}{3}$.

3 (a) 0, 0 (b) 0.

4. (a) $\int_C 2xy \, dx + x^2 \, dy = 16$, (b) $\int_C x^2y \, dx - xy^2 \, dy = -24$.

3. Conservative vector fields

For some line integrals in the previous section, the value of the integral depended only on the vector field \underline{F} and the start and end points of the line but not on the actual path between the start and end points. However, for other line integrals, the result depended on the actual details of the path of the line.

Vector fields are classified according to whether the line integrals are path dependent or path independent. Those vector fields for which *all* line integrals between *all* pairs of points are path independent are called **conservative vector fields**.

There are five properties of a conservative vector field (P1 to P5 below). It is impossible to check the value of every line integral over every path, but it *is* possible to use any one of these five properties (particularly property P3 below) to determine whether or not a vector field is conservative. These properties are also used to simplify calculations with conservative vector fields over non-closed paths.

P1 The line integral $\int_A^B \underline{F} \cdot d\underline{r}$ depends only on the end points A and B and is independent of the actual path taken.

P2 The line integral around any closed curve is zero. That is $\oint_C \underline{F} \cdot d\underline{r} = 0$ for all C .

P3 The curl of a conservative vector field \underline{F} is zero i.e. $\nabla \times \underline{F} = \underline{0}$.

P4 For any conservative vector field \underline{F} , it is possible to find a scalar field ϕ such that $\nabla\phi = \underline{F}$. Then, $\int_C \underline{F} \cdot d\underline{r} = \phi(B) - \phi(A)$ where A and B are the start and end points of contour C .

[This is sometimes called the Fundamental Theorem of Line Integrals and is comparable with the Fundamental Theorem of Calculus.]

P5 All gradient fields are conservative. That is, $\underline{F} = \nabla\phi$ is a conservative vector field for any scalar field ϕ .



Example 11

Consider the following vector fields.

1. $\underline{F}_1 = y^2\underline{i} + xy\underline{j}$ (Example 6) 2. $\underline{F}_2 = 2x\underline{i} + 2y\underline{j}$ (Example 10)

3. $\underline{F}_3 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$ (Example 8)

4. $\underline{F}_4 = x\underline{i} + (4x - y)\underline{j}$ (Task on page 14)

Determine which of these vector fields are conservative where possible by referring to the answers given in the solution. For those that are conservative find a scalar field ϕ such that $\underline{F} = \nabla\phi$ and use property P4 to verify the values of the line integrals.

Solution

- Two different values were obtained for line integrals over the paths C_1 and C_2 . Hence, by P1, \underline{F}_1 is not conservative. [It is also possible to reach this conclusion from P3 by finding that $\nabla \times \underline{F} = -y\underline{k} \neq \underline{0}$.]

Solution (contd.)

2. For the closed path consisting of C_2 and C_3 from $(0,0)$ to $(4,2)$ and back to $(0,0)$ along C_1 we obtain the value $20 + (-20) = 0$. This alone does not mean that \underline{F}_2 is conservative as there could be other paths giving different values. So by using P3

$$\nabla \times \underline{F}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-0) = \underline{0}$$

As $\nabla \times \underline{F}_2 = \underline{0}$, P3 gives that \underline{F}_2 is a conservative vector field.

Now, find a ϕ such that $\underline{F}_2 = \nabla\phi$. Then $\frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} = 2x\underline{i} + 2y\underline{j}$.

Thus

$$\left. \begin{array}{l} \frac{\partial\phi}{\partial x} = 2x \Rightarrow \phi = x^2 + f(y) \\ \frac{\partial\phi}{\partial y} = 2y \Rightarrow \phi = y^2 + g(x) \end{array} \right\} \Rightarrow \phi = x^2 + y^2 (+ \text{constant})$$

Using P4: $\int_{(0,0)}^{(4,2)} \underline{F}_2 \cdot d\underline{r} = \int_{(0,0)}^{(4,2)} (\nabla\phi) \cdot d\underline{r} = \phi(4,2) - \phi(0,0) = (4^2 + 2^2) - (0^2 + 0^2) = 20$.

3. The fact that line integrals along two different paths between the same start and end points have the same value is consistent with \underline{F}_3 being a conservative field according to P1. So too is the fact that the integral around a closed path is zero according to P2. However, neither fact can be used to *conclude* that \underline{F}_3 is a conservative field. This can be done by showing that $\nabla \times \underline{F}_3 = \underline{0}$.

$$\text{Now, } \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\underline{i} - (3y^2z^2 - 3y^2z^2)\underline{j} + (2yz^3 - 2yz^3)\underline{k} = \underline{0}.$$

As $\nabla \times \underline{F}_3 = \underline{0}$, P3 gives that \underline{F}_3 is a conservative field.

To find ϕ that satisfies $\nabla\phi = \underline{F}_3$, it is necessary to satisfy

$$\left. \begin{array}{l} \frac{\partial\phi}{\partial x} = y^2z^3 \rightarrow \phi = xy^2z^3 + f(y,z) \\ \frac{\partial\phi}{\partial y} = 2xyz^3 \rightarrow \phi = xy^2z^3 + g(x,z) \\ \frac{\partial\phi}{\partial z} = 3xy^2z^2 \rightarrow \phi = xy^2z^3 + h(x,y) \end{array} \right\} \rightarrow \phi = xy^2z^3$$

Using P4: $\int_{(0,0,0)}^{(1,1,1)} \underline{F}_3 \cdot d\underline{r} = \phi(1,1,1) - \phi(0,0,0) = 1 - 0 = 1$ in agreement with Example 8(a).

Solution (contd.)

4. As the integral along C_1 is 2 and the integral along C_2 (same start and end points but different intermediate points) is $\frac{4}{3}$, F_4 is **not** a conservative field using P1.

Note that $\nabla \times \underline{F}_4 = 4\underline{k} \neq \underline{0}$ so, using P3, this is an independent conclusion that \underline{F}_4 is **not** conservative.

**Engineering Example 1****Work done moving a charge in an electric field****Introduction**

If a charge, q , is moved through an electric field, \underline{E} , from A to B , then the work required is given by the line integral

$$W_{AB} = -q \int_A^B \underline{E} \cdot d\underline{r}$$

Problem in words

Compare the work done in moving a charge through the electric field around a point charge in a vacuum via two different paths.

Mathematical statement of problem

An electric field \underline{E} is given by

$$\begin{aligned} \underline{E} &= \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \\ &= \frac{Q}{4\pi\epsilon_0(x^2 + y^2 + z^2)} \times \frac{x\underline{i} + y\underline{j} + z\underline{k}}{\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{Q(x\underline{i} + y\underline{j} + z\underline{k})}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{\frac{3}{2}}} \end{aligned}$$

where \underline{r} is the position vector with magnitude r and unit vector \hat{r} , and $\frac{1}{4\pi\epsilon_0}$ is a combination of constants of proportionality, where $\epsilon_0 = 10^{-9}/36\pi \text{ F m}^{-1}$.

Given that $Q = 10^{-8}\text{C}$, find the work done in bringing a charge of $q = 10^{-10}\text{C}$ from the point $A = (10, 10, 0)$ to the point $B = (1, 1, 0)$ (where the dimensions are in metres)

(a) by the direct straight line $y = x, z = 0$

(b) by the straight line pair via $C = (10, 1, 0)$

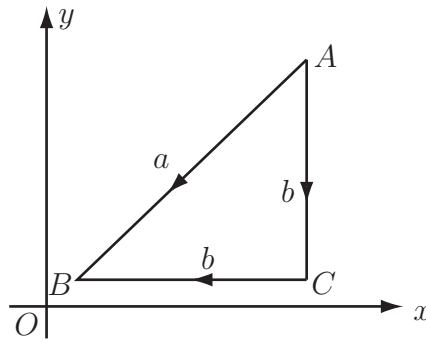


Figure 4: Two routes (*a* and *b*) along which a charge can move through an electric field

The path comprises two straight lines from $A = (10, 10, 0)$ to $B = (1, 1, 0)$ via $C = (10, 1, 0)$ (see Figure 4).

Mathematical analysis

(a) Here $Q/(4\pi\epsilon_0) = 90$ so

$$\underline{E} = \frac{90[x\mathbf{i} + y\mathbf{j}]}{(x^2 + y^2)^{\frac{3}{2}}}$$

as $z = 0$ over the region of interest. The work done

$$\begin{aligned} W_{AB} &= -q \int_A^B \underline{E} \cdot d\mathbf{r} \\ &= -10^{-10} \int_A^B \frac{90}{(x^2 + y^2)^{\frac{3}{2}}} [x\mathbf{i} + y\mathbf{j}] \cdot [dx\mathbf{i} + dy\mathbf{j}] \end{aligned}$$

Using $y = x$, $dy = dx$

$$\begin{aligned} W_{AB} &= -10^{-10} \int_{x=10}^1 \frac{90}{(2x^2)^{\frac{3}{2}}} \{x dx + x dx\} \\ &= -10^{-10} \int_{10}^1 \frac{90}{(2\sqrt{2})} x^{-3} 2x dx \\ &= \frac{90 \times -10^{-10}}{\sqrt{2}} \int_{10}^1 x^{-2} dx \\ &= \frac{9 \times -10^{-9}}{\sqrt{2}} \left[-x^{-1} \right]_{10}^1 \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} \left[x^{-1} \right]_{10}^1 \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} [1 - 0.1] \\ &= 5.73 \times 10^{-9} \text{ J} \end{aligned}$$

(b) The first part of the path is A to C where $x = 10$, $dx = 0$ and y goes from 10 to 1.

$$\begin{aligned}
 W_{AC} &= -q \int_A^C \underline{E} \cdot d\underline{r} \\
 &= -10^{-10} \int_{y=10}^1 \frac{90}{(100 + y^2)^{\frac{3}{2}}} [x\underline{i} + y\underline{j}] \cdot [0\underline{i} + dy\underline{j}] \\
 &= -10^{-10} \int_{10}^1 \frac{90y \, dy}{(100 + y^2)^{\frac{3}{2}}} \\
 &= -10^{-10} \int_{u=200}^{101} \frac{45 \, du}{u^{\frac{3}{2}}} \quad (\text{substituting } u = 100 + y^2, \, du = 2y \, dy) \\
 &= -45 \times 10^{-10} \int_{200}^{101} u^{-\frac{3}{2}} \, du \\
 &= -45 \times 10^{-10} \left[-2u^{-\frac{1}{2}} \right]_{200}^{101} \\
 &= 45 \times 10^{-10} \left(\frac{2}{\sqrt{101}} - \frac{2}{\sqrt{200}} \right) = 2.59 \times 10^{-10} \text{ J}
 \end{aligned}$$

The second part is C to B , where $y = 1$, $dy = 0$ and x goes from 10 to 1.

$$\begin{aligned}
 W_{CB} &= -10^{-10} \int_{x=10}^1 \frac{90}{(x^2 + 1)^{\frac{3}{2}}} [x\underline{i} + y\underline{j}] \cdot [dx\underline{i} + 0\underline{j}] \\
 &= -10^{-10} \int_{10}^1 \frac{90x \, dx}{(x^2 + 1)^{\frac{3}{2}}} \\
 &= -10^{-10} \int_{u=101}^2 \frac{45 \, du}{u^{\frac{3}{2}}} \quad (\text{substituting } u = x^2 + 1, \, du = 2x \, dx) \\
 &= -45 \times 10^{-10} \int_{101}^2 u^{-\frac{3}{2}} \, du \\
 &= -45 \times 10^{-10} \left[-2u^{-\frac{1}{2}} \right]_{101}^2 \\
 &= 45 \times 10^{-10} \left(\frac{2}{\sqrt{2}} - \frac{2}{\sqrt{101}} \right) = 5.468 \times 10^{-9} \text{ J}
 \end{aligned}$$

The sum of the two components W_{AC} and W_{CB} is $5.73 \times 10^{-9} \text{ J}$.

Therefore the work done over the two paths (a) and (b) is identical.

Interpretation

In fact, the work done is independent of the route taken as the electric field \underline{E} around a point charge in a vacuum is a **conservative** field.



Example 12

1. Show that $I = \int_{(0,0)}^{(2,1)} \{(2xy + 1)dx + (x^2 - 2y)dy\}$ is independent of the path taken.
2. Find I using property P1. (Page 19)
3. Find I using property P4. (Page 19)
4. Find $I = \oint_C \{(2xy + 1)dx + (x^2 - 2y)dy\}$ where C is
 - (a) the circle $x^2 + y^2 = 1$
 - (b) the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

Solution

1. The integral $I = \int_{(0,0)}^{(2,1)} \{(2xy + 1)dx + (x^2 - 2y)dy\}$ may be re-written $\int_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = (2xy + 1)\underline{i} + (x^2 - 2y)\underline{j}$.

$$\text{Now } \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 1 & x^2 - 2y & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$$

As $\underline{\nabla} \times \underline{F} = \underline{0}$, \underline{F} is a conservative field and I is independent of the path taken between $(0, 0)$ and $(2, 1)$.

2. As I is independent of the path taken from $(0, 0)$ to $(2, 1)$, it can be evaluated along **any** such path. One possibility is the straight line $y = \frac{1}{2}x$. On this line, $dy = \frac{1}{2}dx$. The integral I becomes

$$\begin{aligned} I &= \int_{(0,0)}^{(2,1)} \{(2xy + 1)dx + (x^2 - 2y)dy\} \\ &= \int_{x=0}^2 \left\{ \left(2x \times \frac{1}{2}x + 1\right)dx + (x^2 - x)\frac{1}{2}dx \right\} \\ &= \int_0^2 \left(\frac{3}{2}x^2 - \frac{1}{2}x + 1 \right) dx \\ &= \left[\frac{1}{2}x^3 - \frac{1}{4}x^2 + x \right]_0^2 = 4 - 1 + 2 - 0 = 5 \end{aligned}$$

Solution (contd.)

3. If $\underline{F} = \nabla\phi$ then

$$\left. \begin{aligned} \frac{\partial\phi}{\partial x} = 2xy + 1 &\rightarrow \phi = x^2y + x + f(y) \\ \frac{\partial\phi}{\partial y} = x^2 - 2y &\rightarrow \phi = x^2y - y^2 + g(x) \end{aligned} \right\} \rightarrow \phi = x^2y + x - y^2 + C.$$

These are consistent if $\phi = x^2y + x - y^2$ (plus a constant which may be omitted since it cancels).

$$\text{So } I = \phi(2, 1) - \phi(0, 0) = (4 + 2 - 1) - 0 = 5$$

4. As \underline{F} is a conservative field, all integrals around a closed contour are zero.

Exercises

1. Determine whether the following vector fields are conservative

- (a) $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$
 (b) $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$
 (c) $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$
 (d) $\underline{F} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$

2. Consider the integral $\int_C \underline{F} \cdot d\underline{r}$ with $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. From Exercise 1(b) \underline{F} is a conservative vector field. Find a scalar field ϕ so that $\nabla\phi = \underline{F}$. Use property P4 to evaluate the integral $\int_C \underline{F} \cdot d\underline{r}$ where C is an integral with start-point $(0, 0)$ and end point $(1, 4)$.

3. For the following conservative vector fields \underline{F} , find a scalar field ϕ such that $\nabla\phi = \underline{F}$ and hence evaluate the $I = \int_C \underline{F} \cdot d\underline{r}$ for the contours C indicated.

- (a) $\underline{F} = (4x^3y - 2x)\underline{i} + (x^4 - 2y)\underline{j}$; any path from $(0, 0)$ to $(2, 1)$.
 (b) $\underline{F} = (e^x + y^3)\underline{i} + (3xy^2)\underline{j}$; closed path starting from any point on the circle $x^2 + y^2 = 1$.
 (c) $\underline{F} = (y^2 + \sin z)\underline{i} + 2xy\underline{j} + x \cos z\underline{k}$; any path from $(1, 1, 0)$ to $(2, 0, \pi)$.
 (d) $\underline{F} = \frac{1}{x}\underline{i} + 4y^3z^2\underline{j} + 2y^4zk$; any path from $(1, 1, 1)$ to $(1, 2, 3)$.

Answers

1. (a) No, (b) Yes, (c) No, (d) Yes

2. $x^3y^2 - y + C$, 12

3. (a) $x^4y - x^2 - y^2$, 11; (b) $e^x + xy^3$, 0; (c) $xy^2 + x \sin z$, -1; (d) $\ln x + y^4z^2$, 143

4. Vector line integrals

It is also possible to form less commonly used integrals of the types:

$$\int_C f(x, y, z) \underline{dr} \quad \text{and} \quad \int_C \underline{F}(x, y, z) \times \underline{dr}.$$

Each of these integrals evaluates to a vector.

Remembering that $\underline{dr} = dx \underline{i} + dy \underline{j} + dz \underline{k}$, an integral of the form $\int_C f(x, y, z) \underline{dr}$ becomes

$\int_C f(x, y, z) dx \underline{i} + \int_C f(x, y, z) dy \underline{j} + \int_C f(x, y, z) dz \underline{k}$. The first term can be evaluated by expressing y and z in terms of x . Similarly the second and third terms can be evaluated by expressing all terms as functions of y and z respectively. Alternatively, all variables can be expressed in terms of a parameter t . If an integral is two-dimensional, the term in z will be absent.



Example 13

Evaluate the integral $\int_C xy^2 \underline{dr}$ where C represents the contour $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Solution

This is a two-dimensional integral so the term in z will be absent.

$$\begin{aligned} I &= \int_C xy^2 \underline{dr} \\ &= \int_C xy^2 (dx \underline{i} + dy \underline{j}) \\ &= \int_C xy^2 dx \underline{i} + \int_C xy^2 dy \underline{j} \\ &= \int_{x=0}^1 x(x^2)^2 dx \underline{i} + \int_{y=0}^1 y^{1/2} y^2 dy \underline{j} \\ &= \int_0^1 x^5 dx \underline{i} + \int_0^1 y^{5/2} dy \underline{j} \\ &= \left[\frac{1}{6} x^6 \right]_0^1 \underline{i} + \left[\frac{2}{7} x^{7/2} \right]_0^1 \underline{j} \\ &= \frac{1}{6} \underline{i} + \frac{2}{7} \underline{j} \end{aligned}$$

**Example 14**

Find $I = \int_C x \underline{dr}$ for the contour C given parametrically by $x = \cos t$, $y = \sin t$, $z = t - \pi$ starting at $t = 0$ and going to $t = 2\pi$, i.e. the contour starts at $(1, 0, -\pi)$ and finishes at $(1, 0, \pi)$.

Solution

The integral becomes $\int_C x(dx \underline{i} + dy \underline{j} + dz \underline{k})$.

Now, $x = \cos t$, $y = \sin t$, $z = t - \pi$ so $dx = -\sin t dt$, $dy = \cos t dt$ and $dz = dt$. So

$$\begin{aligned} I &= \int_0^{2\pi} \cos t (-\sin t dt \underline{i} + \cos t dt \underline{j} + dt \underline{k}) \\ &= -\int_0^{2\pi} \cos t \sin t dt \underline{i} + \int_0^{2\pi} \cos^2 t dt \underline{j} + \int_0^{2\pi} \cos t dt \underline{k} \\ &= -\frac{1}{2} \int_0^{2\pi} \sin 2t dt \underline{i} + \frac{1}{2} \int_0^{2\pi} (1 + \cos 2t) dt \underline{j} + \left[\sin t \right]_0^{2\pi} \underline{k} \\ &= \frac{1}{4} \left[\cos 2t \right]_0^{2\pi} \underline{i} + \frac{1}{2} \left[t + \frac{1}{2} \sin 2t \right]_0^{2\pi} \underline{j} + 0 \underline{k} \\ &= 0 \underline{i} + \pi \underline{j} = \pi \underline{j} \end{aligned}$$

Integrals of the form $\int_C \underline{F} \times \underline{dr}$ can be evaluated as follows. If the vector field $\underline{F} = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$ and $\underline{dr} = dx \underline{i} + dy \underline{j} + dz \underline{k}$ then:

$$\begin{aligned} \underline{F} \times \underline{dr} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix} = (F_2 dz - F_3 dy) \underline{i} + (F_3 dx - F_1 dz) \underline{j} + (F_1 dy - F_2 dx) \underline{k} \\ &= (F_3 \underline{j} - F_2 \underline{k}) dx + (F_1 \underline{k} - F_3 \underline{i}) dy + (F_2 \underline{i} - F_1 \underline{j}) dz \end{aligned}$$

There are thus a maximum of six terms involved in one such integral; the exact details may dictate which method to use.



Example 15

Evaluate the integral $\int_C (x^2 \underline{i} + 3xy \underline{j}) \times \underline{dr}$ where C represents the curve $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

Solution

Note that the z components of both \underline{F} and \underline{dr} are zero.

$$\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & 3xy & 0 \\ dx & dy & 0 \end{vmatrix} = (x^2 dy - 3xy dx) \underline{k} \quad \text{and}$$

$$\int_C (x^2 \underline{i} + 3xy \underline{j}) \times \underline{dr} = \int_C (x^2 dy - 3xy dx) \underline{k}$$

Now, on C , $y = 2x^2$ $dy = 4x dx$ and

$$\begin{aligned} \int_C (x^2 \underline{i} + 3xy \underline{j}) \times \underline{dr} &= \int_C \{x^2 dy - 3xy dx\} \underline{k} \\ &= \int_{x=0}^1 \{x^2 \times 4x dx - 3x \times 2x^2 dx\} \underline{k} \\ &= \int_0^1 -2x^3 dx \underline{k} \\ &= - \left[\frac{1}{2} x^4 \right]_0^1 \underline{k} \\ &= -\frac{1}{2} \underline{k} \end{aligned}$$



Engineering Example 2

Force on a loop due to a magnetic field

Introduction

A current \underline{I} in a magnetic field \underline{B} is subject to a force \underline{F} given by

$$\underline{F} = I \underline{dr} \times \underline{B}$$

where the current can be regarded as having magnitude I and flowing (positive charge) in the direction given by the vector \underline{dr} . The force is known as the Lorentz force and is responsible for the workings of an electric motor. If current flows around a loop, the total force on the loop is given by the integral of \underline{F} around the loop, i.e.

$$\underline{F} = \oint (I \underline{dr} \times \underline{B}) = -I \oint (\underline{B} \times \underline{dr})$$

where the closed path of the integral represents one circuit of the loop.

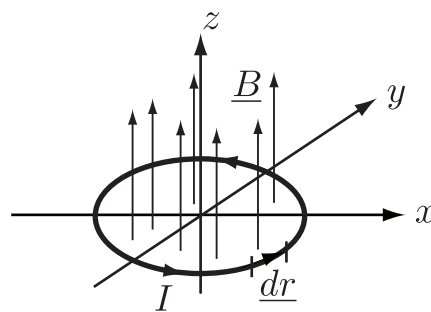


Figure 5: The magnetic field through a loop of current

Problem in words

A current of 1 amp flows around a circuit in the shape of the unit circle in the Oxy plane. A magnetic field of 1 tesla (T) in the positive z -direction is present. Find the total force on the circuit loop.

Mathematical statement of problem

Choose an origin at the centre of the circuit and use polar coordinates to describe the position of any point on the circuit and the length of a small element.

Calculate the line integral around the circuit to give the force required using the given values of current and magnetic field.

Mathematical analysis

The circuit is described parametrically by

$$x = \cos \theta \quad y = \sin \theta \quad z = 0$$

with

$$\begin{aligned} \underline{dr} &= -\sin \theta \, d\theta \, \underline{i} + \cos \theta \, d\theta \, \underline{j} \\ \underline{B} &= B \, \underline{k} \end{aligned}$$

since \underline{B} is constant. Therefore, the force on the circuit is given by

$$\underline{F} = -IB \oint \underline{k} \times \underline{dr} = - \oint \underline{k} \times \underline{dr} \quad (\text{since } I = 1 \text{ A and } B = 1 \text{ T})$$

where

$$\begin{aligned} \underline{k} \times \underline{dr} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ -\sin \theta \, d\theta & \cos \theta \, d\theta & 0 \end{vmatrix} \\ &= (-\cos \theta \, \underline{i} - \sin \theta \, \underline{j}) \, d\theta \end{aligned}$$

So

$$\begin{aligned} \underline{F} &= - \int_{\theta=0}^{2\pi} (-\cos \theta \, \underline{i} - \sin \theta \, \underline{j}) \, d\theta \\ &= \left[\sin \theta \, \underline{i} - \cos \theta \, \underline{j} \right]_{\theta=0}^{2\pi} \\ &= (0 - 0) \, \underline{i} - (1 - 1) \, \underline{j} = \underline{0} \end{aligned}$$

Hence there is no net force on the loop.

Interpretation

At any given point of the circle, the force on the point opposite is of the same magnitude but opposite direction, and so cancels, leaving a zero net force.

Tip: Use symmetry arguments to avoid detailed calculations whenever possible!

A scalar or vector involved in a vector line integral may itself be a vector derivative as this next Example illustrates.



Example 16

Find the vector line integral $\int_C (\nabla \cdot \underline{F}) \underline{dr}$ where \underline{F} is the vector $x^2 \underline{i} + 2xy \underline{j} + 2xz \underline{k}$ and C is the curve $y = x^2$, $z = x^3$ from $x = 0$ to $x = 1$ i.e. from $(0, 0, 0)$ to $(1, 1, 1)$. Here $\nabla \cdot \underline{F}$ is the (scalar) divergence of the vector \underline{F} .

Solution

As $\underline{F} = x^2 \underline{i} + 2xy \underline{j} + 2xz \underline{k}$, $\nabla \cdot \underline{F} = 2x + 2x + 2x = 6x$.

The integral

$$\begin{aligned} \int_C (\nabla \cdot \underline{F}) \underline{dr} &= \int_C 6x(dx \underline{i} + dy \underline{j} + dz \underline{k}) \\ &= \int_C 6x dx \underline{i} + \int_C 6x dy \underline{j} + \int_C 6x dz \underline{k} \end{aligned}$$

The first term is

$$\int_C 6x dx \underline{i} = \int_{x=0}^1 6x dx \underline{i} = \left[3x^2 \right]_0^1 \underline{i} = 3 \underline{i}$$

In the second term, as $y = x^2$ on C , dy may be replaced by $2x dx$ so

$$\int_C 6x dy \underline{j} = \int_{x=0}^1 6x \times 2x dx \underline{j} = \int_0^1 12x^2 dx \underline{j} = \left[4x^3 \right]_0^1 \underline{j} = 4 \underline{j}$$

In the third term, as $z = x^3$ on C , dz may be replaced by $3x^2 dx$ so

$$\int_C 6x dz \underline{k} = \int_{x=0}^1 6x \times 3x^2 dx \underline{k} = \int_0^1 18x^3 dx \underline{k} = \left[\frac{9}{2} x^4 \right]_0^1 \underline{k} = \frac{9}{2} \underline{k}$$

On summing, $\int_C (\nabla \cdot \underline{F}) \underline{dr} = 3 \underline{i} + 4 \underline{j} + \frac{9}{2} \underline{k}$.



Find the vector line integral $\int_C f \underline{dr}$ where $f = x^2$ and C is

- the curve $y = x^{1/2}$ from $(0, 0)$ to $(9, 3)$.
- the line $y = x/3$ from $(0, 0)$ to $(9, 3)$.

Your solution

Answer

$$(a) \int_0^9 (x^2 \underline{i} + \frac{1}{2} x^{3/2} \underline{j}) dx = 243 \underline{i} + \frac{243}{5} \underline{j}, \quad (b) \int_0^9 (x^2 \underline{i} + \frac{1}{3} x^2 \underline{j}) dx = 243 \underline{i} + 81 \underline{j}.$$



Evaluate the vector line integral $\int_C \underline{F} \times d\underline{r}$ when C represents the contour $y = 4 - 4x$, $z = 2 - 2x$ from $(0, 4, 2)$ to $(1, 0, 0)$ and \underline{F} is the vector field $(x - z) \underline{j}$.

Your solution

Answer

$$\int_0^1 \{(4 - 6x) \underline{i} + (2 - 3x) \underline{k}\} = \underline{i} + \frac{1}{2} \underline{k}$$

Exercises

1. Evaluate the vector line integral $\int_C (\nabla \cdot \underline{F}) \underline{dr}$ in the case where $\underline{F} = x\underline{i} + xy\underline{j} + xy^2\underline{k}$ and C is the contour described by $x = 2t$, $y = t^2$, $z = 1 - t$ for t starting at $t = 0$ and going to $t = 1$.
2. When C is the contour $y = x^3$, $z = 0$, from $(0, 0, 0)$ to $(1, 1, 0)$, evaluate the vector line integrals

(a) $\int_C \{\nabla(xy)\} \times \underline{dr}$

(b) $\int_C \{\nabla \times (x^2\underline{i} + y^2\underline{k})\} \times \underline{dr}$

Answers

1. $\int_C (1+x)(dx\underline{i} + dy\underline{j} + dz\underline{k}) = 4\underline{i} + \frac{7}{3}\underline{j} - 2\underline{k}$,

2. (a) $\underline{k} \int_C y dy - x dx = 0\underline{k} = \underline{0}$, (b) $\underline{k} \int_C 2y dy = 1\underline{k} = \underline{k}$