Orthogonal Curvilinear Coordinates

28.3

Introduction

The derivatives div, grad and curl from Section 28.2 can be carried out using coordinate systems other than the rectangular Cartesian coordinates. This Section shows how to calculate these derivatives in other coordinate systems. Two coordinate systems - cylindrical polar coordinates and spherical polar coordinates - will be illustrated.

Prerequisites

Before starting this Section you should . . .

- be able to find the gradient, divergence and curl of a field in Cartesian coordinates
- be familiar with polar coordinates

Learning Outcomes

On completion you should be able to . . .

- find the divergence, gradient or curl of a vector or scalar field expressed in terms of orthogonal curvilinear coordinates

1. Orthogonal curvilinear coordinates

The results shown in Section 28.2 have been given in terms of the familiar Cartesian (x, y, z) coordinate system. However, other coordinate systems can be used to better describe some physical situations. A set of coordinates $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$ where the directions at any point indicated by u, v and w are orthogonal (perpendicular) to each other is referred to as a set of **orthogonal curvilinear coordinates**. With each coordinate is associated a **scale factor** h_u, h_v or h_w respectively where $h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$ (with similar expressions for h_v and h_w). The scale factor gives a measure of how a change in the coordinate changes the position of a point.

Two commonly-used sets of orthogonal curvilinear coordinates are **cylindrical polar coordinates** and **spherical polar coordinates**. These are similar to the plane polar coordinates introduced in HELM 17.2 but represent extensions to three dimensions.

Cylindrical polar coordinates

This corresponds to plane polar (ρ, ϕ) coordinates with an added z-coordinate directed out of the xy plane. Normally the variables ρ and ϕ are used instead of r and θ to give the three coordinates ρ, ϕ and z. A cylinder has equation $\rho =$ constant.
The relationship between the coordinate systems is given by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

(i.e. the same z is used by the two coordinate systems). See Figure 20(a).

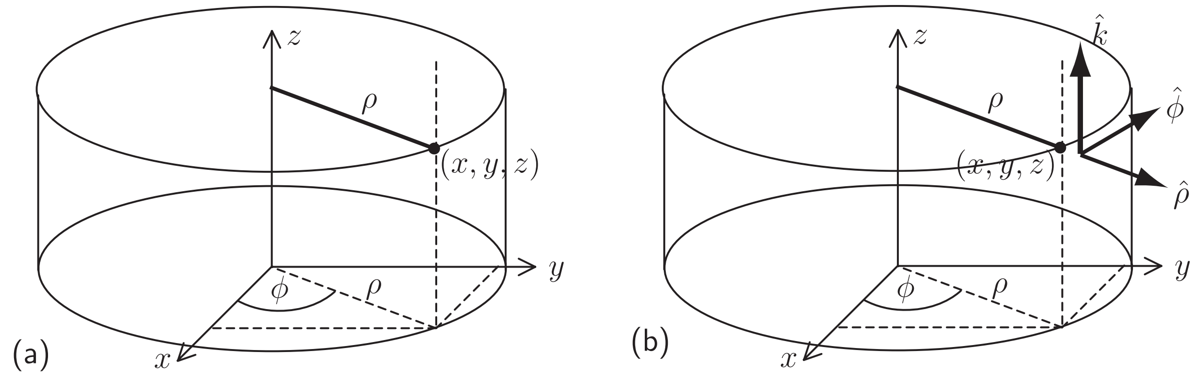

Figure 20: Cylindrical polar coordinates

The scale factors h_ρ, h_ϕ and h_z are given as follows

$$h_\rho = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = \sqrt{(\cos\phi)^2 + (\sin\phi)^2 + 0} = 1$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-\rho\sin\phi)^2 + (\rho\cos\phi)^2 + 0} = \rho$$

$$h_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{(0^2 + 0^2 + 1^2)} = 1$$

Spherical polar coordinates

In this system a point is referred to by its distance from the origin r and two angles ϕ and θ. The angle θ is the angle between the positive z-axis and the line from the origin to the point. The angle ϕ is the angle from the x-axis to the projection of the point in the xy plane.

A useful analogy is of latitude, longitude and height on Earth.

- The variable r plays the role of height (but height measured above the centre of Earth rather than from the surface).
- The variable θ plays the role of latitude but is modified so that $\theta = 0$ represents the North Pole, $\theta = 90^\circ = \dfrac{\pi}{2}$ represents the equator and $\theta = 180^\circ = \pi$ represents the South Pole.
- The variable ϕ plays the role of longitude.

A sphere has equation $r =$ constant.
The relationship between the coordinate systems is given by

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta.$$ See Figure 21.

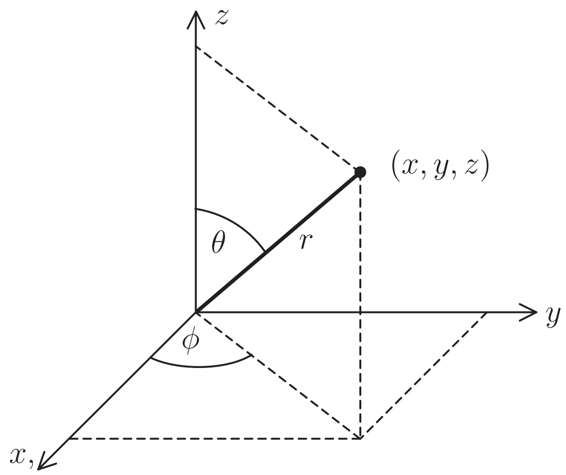

Figure 21: Spherical polar coordinates

The scale factors h_r, h_θ and h_ϕ are given by

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2} = r$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0} = r\sin\theta$$

2. Vector derivatives in orthogonal coordinates

Given an orthogonal coordinate system u, v, w with unit vectors $\underline{\hat{u}}$, $\underline{\hat{v}}$ and $\underline{\hat{w}}$ and scale factors, h_u, h_v and h_w, it is possible to find the derivatives $\underline{\nabla} f$, $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$.

It is found that

$$\text{grad } f = \underline{\nabla} f = \frac{1}{h_u}\frac{\partial f}{\partial u}\underline{\hat{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\underline{\hat{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\underline{\hat{w}}$$

If $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{div } \underline{F} = \underline{\nabla} \cdot \underline{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right]$$

Also if $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\underline{\hat{u}} & h_v\underline{\hat{v}} & h_w\underline{\hat{w}} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Key Point 6

In orthogonal curvilinear coordinates, the vector derivatives $\underline{\nabla} f$, $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$ include the scale factors h_u, h_v and h_w.

3. Cylindrical polar coordinates

In cylindrical polar coordinates (ρ, ϕ, z), the three unit vectors are $\underline{\hat{\rho}}$, $\underline{\hat{\phi}}$ and $\underline{\hat{z}}$ (see Figure 20(b) on page 38) with scale factors

$$h_\rho = 1,\ h_\phi = \rho,\ h_z = 1.$$

The quantities ρ and ϕ are related to x and y by $x = \rho\cos\phi$ and $y = \rho\sin\phi$. The unit vectors are $\underline{\hat{\rho}} = \cos\phi\underline{i} + \sin\phi\underline{j}$ and $\underline{\hat{\phi}} = -\sin\phi\underline{i} + \cos\phi\underline{j}$. In cylindrical polar coordinates,

$$\text{grad } f = \underline{\nabla} f = \frac{\partial f}{\partial \rho}\underline{\hat{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} + \frac{\partial f}{\partial z}\underline{\hat{z}}$$

The scale factor ρ is necessary in the ϕ-component because the derivatives with respect to ϕ are distorted by the distance from the axis $\rho = 0$.

If $\underline{F} = F_\rho \hat{\underline{\rho}} + F_\phi \hat{\underline{\phi}} + F_z \hat{\underline{z}}$ then

$$\text{div } \underline{F} = \underline{\nabla} \cdot \underline{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right]$$

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.$$

Example 20

Working in cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = \rho^2 + z^2$

Solution

If $f = \rho^2 + z^2$ then $\dfrac{\partial f}{\partial \rho} = 2\rho, \dfrac{\partial f}{\partial \phi} = 0$ and $\dfrac{\partial f}{\partial z} = 2z$ so $\underline{\nabla} f = 2\rho\hat{\underline{\rho}} + 2z\hat{\underline{z}}$.

Example 21

Working in cylindrical polar coordinates find

(a) $\underline{\nabla} f$ for $f = \rho^3 \sin\phi$

(b) Show that the result for (a) is consistent with that found working in Cartesian coordinates.

Solution

(a) If $f = \rho^3 \sin\phi$ then $\dfrac{\partial f}{\partial \rho} = 3\rho^2 \sin\phi$, $\dfrac{\partial f}{\partial \phi} = \rho^3 \cos\phi$ and $\dfrac{\partial f}{\partial z} = 0$ and hence,

$$\underline{\nabla} f = 3\rho^2 \sin\phi \hat{\underline{\rho}} + \rho^2 \cos\phi \hat{\underline{\phi}}.$$

(b) $f = \rho^3 \sin\phi = \rho^2 \rho \sin\phi = (x^2 + y^2)y = x^2 y + y^3$ so $\underline{\nabla} f = 2xy\underline{i} + (x^2 + 3y^2)\underline{j}$.
Using cylindrical polar coordinates, from (a) we have

$$\begin{aligned}
\underline{\nabla} f &= 3\rho^2 \sin\phi \hat{\underline{\rho}} + \rho^2 \cos\phi \hat{\underline{\phi}} \\
&= 3\rho^2 \sin\phi(\cos\phi \underline{i} + \sin\phi \underline{j}) + \rho^2 \cos\phi(-\sin\phi \underline{i} + \cos\phi \underline{j}) \\
&= \left[3\rho^2 \sin\phi\cos\phi - \rho^2 \sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2 \sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} \\
&= \left[2\rho^2 \sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2 \sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} = 2xy\underline{i} + (3y^2 + x^2)\underline{j}
\end{aligned}$$

So the results using Cartesian and cylindrical polar coordinates are consistent.

Example 22

Find $\underline{\nabla} \cdot \underline{F}$ for $\underline{F} = F_\rho \underline{\hat{\rho}} + F_\phi \underline{\hat{\phi}} + F_z \underline{\hat{z}} = \rho^3 \underline{\hat{\rho}} + \rho z \underline{\hat{\phi}} + \rho z \sin\phi \underline{\hat{z}}$. Show that the results are consistent with those found using Cartesian coordinates.

Solution

Here, $F_\rho = \rho^3$, $F_\phi = \rho z$ and $F_z = \rho z \sin\phi$ so

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right] \\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^4) + \frac{\partial}{\partial \phi}(\rho z) + \frac{\partial}{\partial z}(\rho^2 z \sin\phi)\right] \\
&= \frac{1}{\rho}\left[4\rho^3 + 0 + \rho^2 \sin\phi\right] \\
&= 4\rho^2 + \rho\sin\phi
\end{aligned}$$

Converting to Cartesian coordinates,

$$\begin{aligned}
\underline{F} &= F_\rho \underline{\hat{\rho}} + F_\phi \underline{\hat{\phi}} + F_z \underline{\hat{z}} = \rho^3 \underline{\hat{\rho}} + \rho z \underline{\hat{\phi}} + \rho z \sin\phi \underline{\hat{z}} \\
&= \rho^3(\cos\phi \underline{i} + \sin\phi \underline{j}) + \rho z(-\sin\phi \underline{i} + \cos\phi \underline{j}) + \rho z \sin\phi \underline{k} \\
&= (\rho^3\cos\phi - \rho z \sin\phi)\underline{i} + (\rho^3 \sin\phi + \rho z \cos\phi)\underline{j} + \rho z \sin\phi \underline{k} \\
&= \left[\rho^2(\rho\cos\phi) - \rho\sin\phi z\right]\underline{i} + \left[\rho^2(\rho\sin\phi) + \rho\cos\phi z\right]\underline{j} + \rho\sin\phi z\underline{k} \\
&= \left[(x^2+y^2)x - yz\right]\underline{i} + \left[(x^2+y^2)y + xz\right]\underline{j} + yz\underline{k} \\
&= (x^3 + xy^2 - yz)\underline{i} + (x^2y + y^3 + xz)\underline{j} + yz\underline{k}
\end{aligned}$$

So

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{\partial}{\partial x}(x^3 + xy^2 - yz) + \frac{\partial}{\partial y}(x^2y + y^3 + xz) + \frac{\partial}{\partial z}(yz) \\
&= (3x^2 + y^2) + (x^2 + 3y^2) + y = 4x^2 + 4y^2 + y \\
&= 4(x^2 + y^2) + y \\
&= 4\rho^2 + \rho\sin\phi
\end{aligned}$$

So $\underline{\nabla} \cdot \underline{F}$ is the same in both coordinate systems.

Example 23

Find $\underline{\nabla} \times \underline{F}$ for $\underline{F} = \rho^2 \hat{\underline{\rho}} + z \sin\phi \hat{\underline{\phi}} + 2z\cos\phi \hat{\underline{z}}$.

Solution

$$\underline{\nabla} \times \underline{F} = \frac{1}{\rho} \begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho} \begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \rho^2 & \rho z\sin\phi & 2z\cos\phi \end{vmatrix}$$

$$\begin{aligned}
&= \frac{1}{\rho}\left[\hat{\underline{\rho}}\left[\frac{\partial}{\partial \phi}(2z\cos\phi) - \frac{\partial}{\partial z}(\rho z\sin\phi)\right] + \rho\hat{\underline{\phi}}\left[\frac{\partial}{\partial z}\rho^2 - \frac{\partial}{\partial \rho}(2z\cos\phi)\right] + \hat{\underline{z}}\left[\frac{\partial}{\partial \rho}(\rho z\sin\phi) - \frac{\partial}{\partial \phi}\rho^2\right]\right] \\
&= \frac{1}{\rho}\left[\hat{\underline{\rho}}(-2z\sin\phi - \rho\sin\phi) + \rho\hat{\underline{\phi}}(0) + \hat{\underline{z}}(z\sin\phi)\right] \\
&= -\frac{(2z\sin\phi + \rho\sin\phi)}{\rho}\hat{\underline{\rho}} + \frac{z\sin\phi}{\rho}\hat{\underline{z}}
\end{aligned}$$

Engineering Example 2

Divergence of a magnetic field

Introduction

A magnetic field $\underline{B}$ must satisfy $\underline{\nabla} \cdot \underline{B} = 0$. An associated current is given by:

$$\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$$

Problem in words

For the magnetic field (in cylindrical polar coordinates ρ, ϕ, z)

$$\underline{B} = B_0 \frac{\rho}{1+\rho^2}\hat{\underline{\phi}} + \alpha\hat{\underline{z}}$$

show that the divergence of $\underline{B}$ is zero and find the associated current.

Mathematical statement of problem

We must

(a) show that $\underline{\nabla} \cdot \underline{B} = 0$ (b) find the current $\underline{I} = \dfrac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$

Mathematical analysis

(a) Express $\underline{B}$ as (B_ρ, B_ϕ, B_z); then

$$\begin{aligned}
\underline{\nabla} \cdot \underline{B} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial}{\partial \phi}(B_\phi) + \frac{\partial}{\partial z}(\rho B_z)\right] \\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(0) + \frac{\partial}{\partial \phi}\left(B_0 \frac{\rho}{1+\rho^2}\right) + \rho\frac{\partial}{\partial z}(\alpha)\right] \\
&= \frac{1}{\rho}[0+0+0] = 0 \qquad \text{as required.}
\end{aligned}$$

(b) To find the current evaluate

$$\begin{aligned}
\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B}) &= \frac{1}{\mu_0}\frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix} = \begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & B_0\dfrac{\rho^2}{1+\rho^2} & \alpha \end{vmatrix} \\
&= \frac{1}{\mu_0 \rho}\left[0\hat{\underline{\rho}} + 0\rho\hat{\underline{\phi}} + B_0\frac{\partial}{\partial \rho}\left(\frac{\rho^2}{1+\rho^2}\right)\hat{\underline{z}}\right] \\
&= \frac{1}{\mu_0 \rho}B_0\left[\frac{2\rho}{(1+\rho^2)^2}\right]\hat{\underline{z}} = \frac{2B_0}{\mu_0(1+\rho^2)^2}\,\hat{\underline{z}}
\end{aligned}$$

Interpretation

The magnetic field is in the form of a helix with the current pointing along its axis (Fig 22). Such an arrangement is often used for the magnetic containment of charged particles in a fusion reactor.

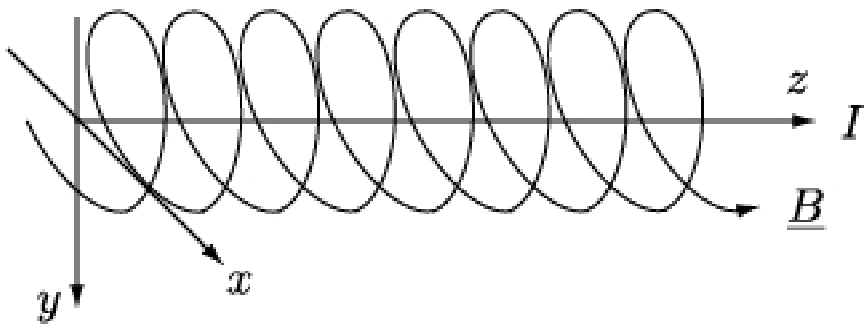

Figure 22: The magnetic field forms a helix

Example 24

A magnetic field $\underline{B}$ is given by $\underline{B} = \rho^{-2}\underline{\hat{\phi}} + k\underline{\hat{z}}$. Find $\underline{\nabla} \cdot \underline{B}$ and $\underline{\nabla} \times \underline{B}$.

Solution

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(0) + \frac{\partial}{\partial \phi}(\rho^{-2}) + \frac{\partial}{\partial z}(k\rho)\right] = \frac{1}{\rho}[0+0+0] = 0$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho^{-1} & k \end{vmatrix}$$

$$= -\frac{1}{\rho^3}\,\underline{\hat{z}}$$

All magnetic fields satisfy $\underline{\nabla} \cdot \underline{B} = 0$ i.e. an absence of magnetic monopoles.
Note that there is a class of magnetic fields known as potential fields that satisfy $\underline{\nabla} \times \underline{B} = \underline{0}$

Task

Using cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = \rho^2 z \sin\phi$

Your solution

Answer

$$\frac{\partial}{\partial \rho}[\rho^2 z \sin\phi]\underline{\hat{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial \phi}[\rho^2 z \sin\phi]\underline{\hat{\phi}} + \frac{\partial}{\partial z}[\rho^2 z \sin\phi]\underline{\hat{z}} = 2\rho z \sin\phi\underline{\hat{\rho}} + \rho z\cos\phi\underline{\hat{\phi}} + \rho^2\sin\phi\underline{\hat{z}}$$

Task

Using cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = z\sin 2\phi$

Your solution

Answer

$$\frac{\partial}{\partial \rho}[z\sin 2\phi]\underline{\hat{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial \phi}[z\sin 2\phi]\underline{\hat{\phi}} + \frac{\partial}{\partial z}[z\sin 2\phi]\underline{\hat{z}} = \frac{2}{\rho}z\cos 2\phi\underline{\hat{\phi}} + \sin 2\phi\underline{\hat{z}}$$

Task

Find $\underline{\nabla} \cdot \underline{F}$ for $\underline{F} = \rho\cos\phi\underline{\hat{\rho}} - \rho\sin\phi\underline{\hat{\phi}} + \rho z\underline{\hat{z}}$

i.e. $F_\rho = \rho\cos\phi, F_\phi = -\rho\sin\phi, F_z = \rho z$

(a) First find the derivatives $\dfrac{\partial}{\partial \rho}[\rho F_\rho],\ \dfrac{\partial}{\partial \phi}[F_\phi],\ \dfrac{\partial}{\partial z}[\rho F_z]$:

Your solution

Answer

$2\rho\cos\phi, \quad -\rho\cos\phi, \quad \rho^2$

(b) Now combine these to find $\underline{\nabla} \cdot \underline{F}$:

Your solution

Answer

$$\begin{aligned}
\underline{\nabla}\cdot\underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{\partial}{\partial\phi}(F_\phi)+\frac{\partial}{\partial z}(\rho F_z)\right]\\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho^2\cos\phi)+\frac{\partial}{\partial\phi}(-\rho\sin\phi)+\frac{\partial}{\partial z}(\rho^2 z)\right]\\
&= \frac{1}{\rho}\left[2\rho\cos\phi-\rho\cos\phi+\rho^2\right]\\
&= \cos\phi+\rho
\end{aligned}$$

Task

Find $\underline{\nabla}\times\underline{F}$ for $\underline{F}=F_\rho\underline{\hat{\rho}}+F_\phi\underline{\hat{\phi}}+F_z\underline{\hat{z}}=\rho^3\underline{\hat{\rho}}+\rho z\underline{\hat{\phi}}+\rho z\sin\phi\underline{\hat{z}}$. Show that the results are consistent with those found using Cartesian coordinates.

(a) Find the curl $\underline{\nabla}\times\underline{F}$:

Your solution

Answer

$$\frac{1}{\rho}\begin{vmatrix}\underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}}\\[2mm] \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\[2mm] \rho^3 & \rho^2 z & \rho z\sin\phi\end{vmatrix} = (z\cos\phi-\rho)\underline{\hat{\rho}}-z\sin\phi\underline{\hat{\phi}}+2z\underline{\hat{z}}$$

(b) Find $\underline{F}$ in Cartesian coordinates:

Your solution

Answer

Use $\underline{\hat{\rho}}=\cos\phi\underline{i}+\sin\phi\underline{j}$, $\underline{\hat{\phi}}=-\sin\phi\underline{i}+\cos\phi\underline{j}$ to get $\underline{F}=(x^3+xy^2-yz)\underline{i}+(x^2y+y^3+xz)\underline{j}+yz\underline{k}$

(c) Hence find $\underline{\nabla}\times\underline{F}$ in Cartesian coordinates:

Your solution

Answer

$(z-x)\underline{i}-y\underline{j}+2z\underline{k}$

(d) Using $\underline{\hat{\rho}} = \cos\phi\underline{i} + \sin\phi\underline{j}$ and $\underline{\hat{\phi}} = -\sin\phi\underline{i} + \cos\phi\underline{j}$, show that the solution to part (a) is equal to the solution for part (c):

Your solution

Answer

$(z\cos\phi-\rho)\ \underline{\hat{\rho}} - z\sin\phi\ \underline{\hat{\phi}} + 2z\ \underline{\hat{z}} = (z\cos\phi-\rho)(\cos\phi\ \underline{i}+\sin\phi\ \underline{j}) - z\sin\phi(-\sin\phi\ \underline{i}+\cos\phi\ \underline{j}) + 2z\ \underline{k}$

$= [z\cos^2\phi - \rho\cos\phi + z\sin^2\phi]\ \underline{i} + [z\cos\phi\sin\phi - \rho\sin\phi - z\sin\phi\cos\phi]\ \underline{j} + 2z\ \underline{k}$

$= [z - \rho\cos\phi]\ \underline{i} - \rho\sin\phi\ \underline{j} + 2z\ \underline{k} = (z-x)\ \underline{i} - y\ \underline{j} + 2z\ \underline{k}$

Exercises

1. For $\underline{F} = \rho\underline{\hat{\rho}} + (\rho\sin\phi + z)\underline{\hat{\phi}} + \rho z\underline{\hat{z}}$, find $\underline{\nabla}\cdot\underline{F}$ and $\underline{\nabla}\times\underline{F}$.
2. For $f = \rho^2 z^2\cos 2\phi$, find $\underline{\nabla}\times(\underline{\nabla}f)$.

Answers

1. $2 + \cos\phi + \rho, \qquad -\underline{\hat{\rho}} - z\ \underline{\hat{\phi}} + (2\sin\phi + \dfrac{z}{\rho})\ \underline{\hat{z}}$

2. $\underline{0}$

4. Spherical polar coordinates

In spherical polar coordinates (r, θ, ϕ), the 3 unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$ and $\underline{\hat{\phi}}$ with scale factors $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. The quantities r, θ and ϕ are related to x, y and z by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. In spherical polar coordinates,

$$\text{grad } f = \underline{\nabla}f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\underline{\hat{\phi}}$$

If $\underline{F} = F_r\underline{\hat{r}} + F_\theta\underline{\hat{\theta}} + F_\phi\underline{\hat{\phi}}$

then

$$\text{div } \underline{F} = \underline{\nabla}\cdot\underline{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial\theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right]$$

$$\text{curl } \underline{F} = \underline{\nabla}\times\underline{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Example 25

In spherical polar coordinates, find $\underline{\nabla} f$ for

(a) $f = r$ (b) $f = \dfrac{1}{r}$ (c) $f = r^2 \sin(\phi + \theta)$

[Note: parts (a) and (b) relate to Exercises 2(a) and 2(c) on page 22.]

Solution

(a)

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\
&= \frac{\partial (r)}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial (r)}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial (r)}{\partial \phi}\underline{\hat{\phi}} \\
&= 1\underline{\hat{r}} = \underline{\hat{r}}
\end{aligned}$$

(b)

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\
&= \frac{\partial (\frac{1}{r})}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial (\frac{1}{r})}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial (\frac{1}{r})}{\partial \phi}\underline{\hat{\phi}} \\
&= -\frac{1}{r^2}\underline{\hat{r}}
\end{aligned}$$

(c)

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\
&= \frac{\partial (r\sin(\phi+\theta))}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial (r\sin(\phi+\theta))}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial (r^2\sin(\phi+\theta))}{\partial \phi}\underline{\hat{\phi}} \\
&= 2r\sin(\phi+\theta)\underline{\hat{r}} + \frac{1}{r}r^2\cos(\phi+\theta)\underline{\hat{\theta}} + \frac{1}{r\sin\theta}r^2\cos(\phi+\theta)\underline{\hat{\phi}} \\
&= 2r\sin(\phi+\theta)\underline{\hat{r}} + r\cos(\phi+\theta)\underline{\hat{\theta}} + \frac{r\cos(\phi+\theta)}{\sin\theta}\underline{\hat{\phi}}
\end{aligned}$$

Engineering Example 3

Electric potential

Introduction

There is a scalar quantity V, called the electric potential, which satisfies

$\underline{\nabla} V = -\underline{E}$ where $\underline{E}$ is the electric field.

It is often easier to handle scalar fields rather than vector fields. It is therefore convenient to work with V and then derive $\underline{E}$ from it.

Problem in words

Given the electric potential, find the electric field.

Mathematical statement of problem

For a point charge, Q, the potential V is given by

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Verify, using spherical polar coordinates, that $\underline{E} = -\underline{\nabla} V = \dfrac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}$

Mathematical analysis

In spherical polar coordinates:

$$\begin{aligned}
\underline{\nabla} V &= \frac{\partial V}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\underline{\hat{\phi}} \\
&= \frac{\partial V}{\partial r}\underline{\hat{r}} \quad \text{as the other partial derivatives are zero} \\
&= \frac{\partial}{\partial r}\left[\frac{Q}{4\pi\epsilon_0 r}\right]\underline{\hat{r}} \\
&= -\frac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}
\end{aligned}$$

Interpretation

So $\underline{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}$ as required.

This is a form of Coulomb's Law. A positive charge will experience a positive repulsion radially *outwards* in the field of another positive charge.

Example 26

Using spherical polar coordinates, find $\underline{\nabla} \cdot \underline{F}$ for the following vector functions.

(a) $\underline{F} = r\underline{\hat{r}}$ (b) $\underline{F} = r^2 \sin\theta \underline{\hat{r}}$ (c) $\underline{F} = r\sin\theta\, \underline{\hat{r}} + r^2 \sin\phi\, \underline{\hat{\theta}} + r\cos\theta\, \underline{\hat{\phi}}$

Solution

(a)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta \times r) + \frac{\partial}{\partial \theta}(r\sin\theta \times 0) + \frac{\partial}{\partial \phi}(r \times 0)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^3 \sin\theta) + \frac{\partial}{\partial \theta}(0) + \frac{\partial}{\partial \phi}(0)\right] = \frac{1}{r^2 \sin\theta}\left[3r^2 \sin\theta + 0 + 0\right] = 3
\end{aligned}$$

Note :- in Cartesian coordinates, the corresponding vector is $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ with $\underline{\nabla} \cdot \underline{F} = 1 + 1 + 1 = 3$ (hence consistency).

(b)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ F_r) + \frac{\partial}{\partial \theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ r^2 \sin\theta) + \frac{\partial}{\partial \theta}(r\sin\theta \times 0) + \frac{\partial}{\partial \phi}(r \times 0)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^4 \sin^2\theta) + \frac{\partial}{\partial \theta}(0) + \frac{\partial}{\partial \phi}(0)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[4r^3 \sin^2\theta + 0 + 0\right] = 4r\sin\theta
\end{aligned}$$

(c)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ F_r) + \frac{\partial}{\partial \theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\ r\sin\theta) + \frac{\partial}{\partial \theta}(r\sin\theta \times r^2\sin\phi) + \frac{\partial}{\partial \phi}(r \times\ r\cos\theta)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^3 \sin^2\theta) + \frac{\partial}{\partial \theta}(r^3\sin\theta\sin\phi) + \frac{\partial}{\partial \phi}(r^2\cos\theta)\right] \\
&= \frac{1}{r^2 \sin\theta}\left[3r^2 \sin^2\theta + r^3\cos\theta\ \sin\phi + 0\right] = 3\sin\theta + r\cot\theta\ \sin\phi
\end{aligned}$$

Example 27

Using spherical polar coordinates, find $\underline{\nabla} \times \underline{F}$ for the following vector fields $\underline{F}$.

(a) $\underline{F} = r^k \underline{\hat{r}}$, where k is a constant (b) $\underline{F} = r^2\cos\theta\,\underline{\hat{r}} + \sin\theta\,\underline{\hat{\theta}} + \sin^2\theta\,\underline{\hat{\phi}}$

Solution

(a)

$$\begin{aligned}
\underline{\nabla}\times\underline{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^k & r\times 0 & r\sin\theta\times 0 \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(0) - \frac{\partial}{\partial\phi}(0)\right)\underline{\hat{r}} + \left(\frac{\partial}{\partial\phi}(r^k) - \frac{\partial}{\partial r}(0)\right)r\underline{\hat{\theta}}\right. \\
&\quad \left. + \left(\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial\theta}(r^k)\right)r\sin\theta\,\underline{\hat{\phi}}\right] \\
&= 0\,\underline{\hat{r}} + 0\,\underline{\hat{\theta}} + 0\,\underline{\hat{\phi}} = \underline{0}
\end{aligned}$$

(b)

$$\begin{aligned}
\underline{\nabla}\times\underline{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\,\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^2\cos\theta & r\times\sin\theta & r\sin\theta\times\sin^2\theta \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(r\sin^3\theta) - \frac{\partial}{\partial\phi}(r\sin\theta)\right)\underline{\hat{r}} + \left(\frac{\partial}{\partial\phi}(r^2\cos\theta) - \frac{\partial}{\partial r}(r\sin^3\theta)\right)r\underline{\hat{\theta}}\right. \\
&\quad \left. + \left(\frac{\partial}{\partial r}(r\sin\theta) - \frac{\partial}{\partial\theta}(r^2\cos\theta)\right)r\sin\theta\,\underline{\hat{\phi}}\right] \\
&= \frac{1}{r^2\sin\theta}\left[\left(3r\sin^2\theta\cos\theta + 0\right)\underline{\hat{r}} + \left(0 - \sin^3\theta\right)r\underline{\hat{\theta}} + \left(\sin\theta + r^2\sin\theta\right)r\sin\theta\,\underline{\hat{\phi}}\right] \\
&= \frac{3\sin\theta\cos\theta}{r}\underline{\hat{r}} - \frac{\sin^2\theta}{r}\underline{\hat{\theta}} + \frac{(1+r^2)}{r}\sin\theta\,\underline{\hat{\phi}}
\end{aligned}$$

Task

Using spherical polar coordinates, find $\underline{\nabla} f$ for

(a) $f = r^4$

(b) $f = \dfrac{r}{r^2+1}$

(c) $f = r^2 \sin 2\theta \cos\phi$

Your solution

Answer

(a) $4r^3\underline{\hat{r}}$,

(b) $\dfrac{1-r^2}{(1+r^2)^2}\underline{\hat{r}}$,

(c) $\dfrac{\partial}{\partial r}(r^2\sin 2\theta\cos\phi)\underline{\hat{r}} + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(r^2\sin 2\theta\cos\phi)\underline{\hat{\phi}} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}(r^2\sin 2\theta\cos\phi)$

$$= 2r\sin 2\theta\cos\phi\ \underline{\hat{r}} + 2r\cos 2\theta\cos\phi\ \underline{\hat{\theta}} - 2r\cos\theta\sin\phi\ \underline{\hat{\phi}}$$

Exercises

1. For $\underline{F} = r\sin\theta\underline{\hat{r}} + r\cos\phi\underline{\hat{\theta}} + r\sin\phi\underline{\hat{\phi}}$, find $\underline{\nabla}\cdot\underline{F}$ and $\underline{\nabla}\times\underline{F}$.

2. For $\underline{F} = r^{-4}\cos\theta\underline{\hat{r}} + r^{-4}\sin\theta\underline{\hat{\theta}}$, find $\underline{\nabla}\cdot\underline{F}$ and $\underline{\nabla}\times\underline{F}$.

3. For $\underline{F} = r^2\cos\theta\underline{\hat{r}} + \cos\phi\underline{\hat{\theta}}$ find $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{F})$.

Answers

1. $\cos\phi(\cot\theta + \text{cosec}\theta) + 3\sin\theta$, $\quad \cot\dfrac{\theta}{2}\sin\phi\underline{\hat{r}} - 2\sin\phi\underline{\hat{\theta}} + (2\cos\phi - \cos\theta)\underline{\hat{\phi}}$

2. 0, $\quad -2r^{-5}\sin\theta\underline{\hat{\phi}}$

3. 0