Differential Vector Calculus

28.2

Introduction

A vector field or a scalar field can be differentiated with respect to position in three ways to produce another vector field or scalar field. This Section studies the three derivatives, that is: (i) the gradient of a scalar field (ii) the divergence of a vector field and (iii) the curl of a vector field.

Prerequisites

Before starting this Section you should . . .

- be familiar with the concept of a function of two variables
- be familiar with the concept of partial differentiation
- be familiar with scalar and vector fields

Learning Outcomes

On completion you should be able to . . .

- find the divergence, gradient or curl of a vector or scalar field

1. The gradient of a scalar field

Consider the height ϕ above sea level at various points on a hill. Some contours for such a hill are shown in Figure 14.

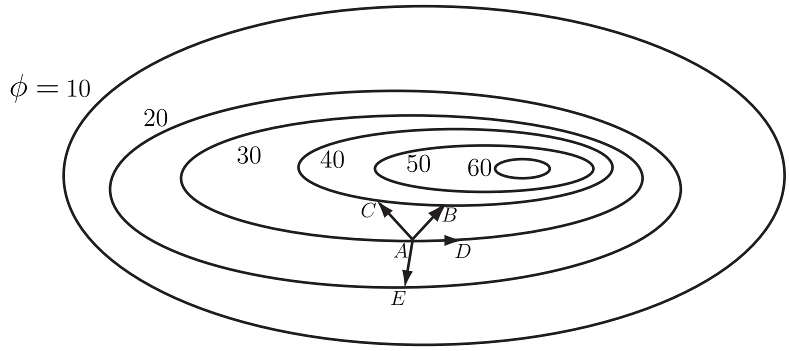


Figure 14: "Contour map" of a hill

We are interested in how ϕ changes from one point to another. Starting from A and making a displacement $\underline{d}$ the change in height (ϕ) depends on the direction of the displacement. The magnitude of each $\underline{d}$ is the same.

Displacement	Change in ϕ
AB	$40 - 30 = 10$
AC	$40 - 30 = 10$
AD	$30 - 30 = 0$
AE	$20 - 30 = -10$

The change in ϕ clearly depends on the direction of the displacement. For the paths shown ϕ increases most rapidly along AB, does not increase at all along AD (as A and D are both on the same contour and so are both at the same height) and decreases along AE.

The direction in which ϕ changes fastest is along the line of greatest slope which is orthogonal (i.e. perpendicular) to the contours. Hence, at each point of a scalar field we can define a vector field giving the magnitude and direction of the greatest rate of change of ϕ locally.

A vector field, called the gradient, written grad ϕ, can be associated with a scalar field ϕ so that at every point the direction of the vector field is orthogonal to the scalar field contour. This vector field is the direction of the maximum rate of change of ϕ.

For a second example consider a metal plate heated at one corner and cooled by an ice bag at the opposite corner. All edges and surfaces are insulated. After a while a steady state situation exists in which the temperature ϕ at any point remains the same. Some temperature contours are shown in Figure 15.

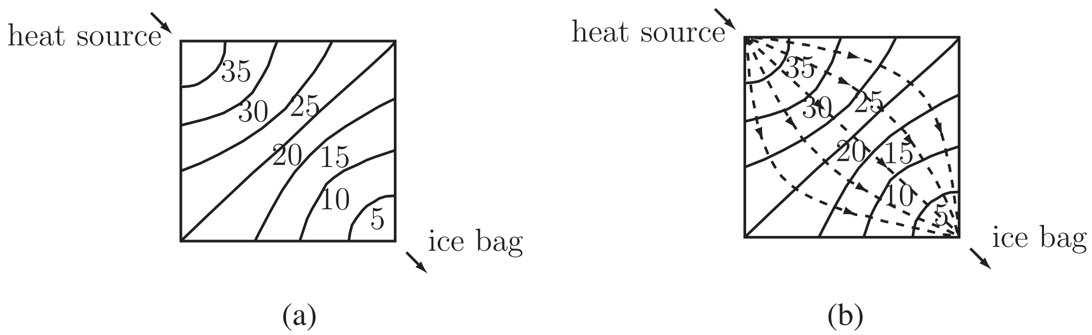


Figure 15: Temperature contours and heat flow lines for a metal plate

The direction of the heat flow is along the flow lines which are orthogonal to the contours (see the dashed lines in Figure 15(b)); this heat flow is proportional to the vector field grad ϕ.

Definition

The gradient of the scalar field $\phi = f(x, y, z)$ is $\quad$ grad $\phi = \underline{\nabla}\phi = \dfrac{\partial \phi}{\partial x}\underline{i} + \dfrac{\partial \phi}{\partial y}\underline{j} + \dfrac{\partial \phi}{\partial z}\underline{k}$

Often, instead of grad ϕ, the notation $\underline{\nabla}\phi$ is used. ($\underline{\nabla}$ is a vector differential operator called 'del' or 'nabla' defined by $\dfrac{\partial}{\partial x}\underline{i} + \dfrac{\partial}{\partial y}\underline{j} + \dfrac{\partial}{\partial z}\underline{k}$. As a vector differential operator, it retains the characteristics of a vector while also carrying out differentiation.)

The vector grad ϕ gives the magnitude and direction of the greatest rate of change of ϕ at any point, and is always orthogonal to the contours of ϕ. For example, in Figure 14, grad ϕ points in the direction of AB while the contour line is parallel to AD i.e. perpendicular to AB. Similarly, in Figure 15(b), the various intersections of the contours with the lines representing grad ϕ occur at right-angles.

For the hill considered earlier the direction and magnitude of grad ϕ are shown at various points in Figure 16. Note that the magnitude of grad ϕ is greatest (as indicated by the length of the arrow) when the hill is at its steepest (as indicated by the closeness of the contours).

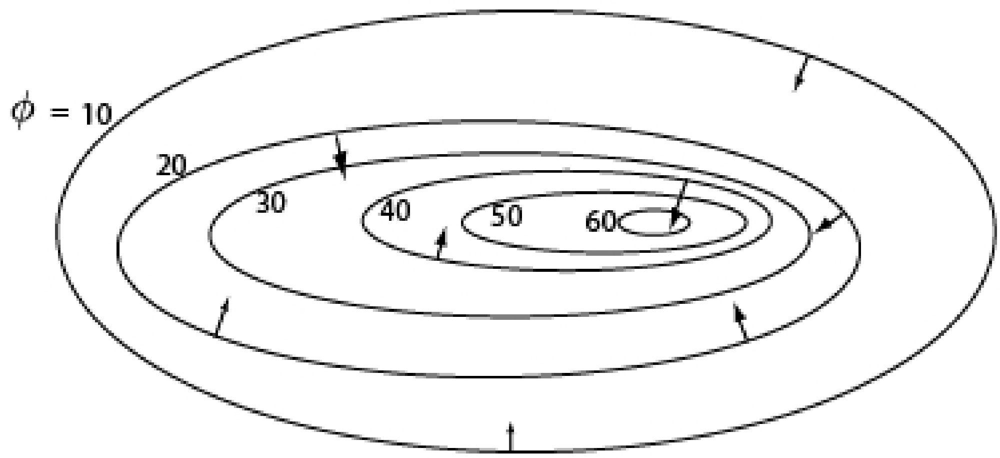


Figure 16: Grad ϕ and the steepest ascent direction for a hill

Key Point 3

ϕ is a scalar field but grad ϕ is a vector field.

Example 9

Find grad ϕ for

(a) $\phi = x^2 - 3y$ (b) $\phi = xy^2z^3$

Solution

(a) grad $\phi = \dfrac{\partial}{\partial x}(x^2-3y)\underline{i} + \dfrac{\partial}{\partial y}(x^2-3y)\underline{j} + \dfrac{\partial}{\partial z}(x^2-3y)\underline{k} = 2x\underline{i} + (-3)\underline{j} + 0\underline{k} = 2x\underline{i} - 3\underline{j}$

(b) grad $\phi = \dfrac{\partial}{\partial x}(xy^2z^3)\underline{i} + \dfrac{\partial}{\partial y}(xy^2z^3)\underline{j} + \dfrac{\partial}{\partial z}(xy^2z^3)\underline{k} = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$

Example 10

For $f = x^2 + y^2$ find grad f at the point $A(1,2)$. Show that the direction of grad f is orthogonal to the contour at this point.

Solution

$$\text{grad } f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} = 2x\underline{i} + 2y\underline{j} + 0\underline{k} = 2x\underline{i} + 2y\underline{j}$$

and at $A(1,2)$ this equals $2 \times 1\underline{i} + 2 \times 2\underline{j} = 2\underline{i} + 4\underline{j}$.

Since $f = x^2 + y^2$ then the contours are defined by $x^2 + y^2 =$ constant, so the contours are circles centred at the origin. The vector grad f at $A(1,2)$ points directly away from the origin and hence grad f and the contour are orthogonal; see Figure 17. Note that $\underline{r}(A) = \underline{i} + 2\underline{j} = \dfrac{1}{2}\,\text{grad}\, f$.

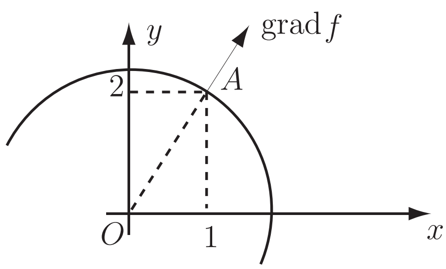


Figure 17: Grad f is perpendicular to the contour lines

The change in a function ϕ in a given direction (specified as a unit vector $\underline{a}$) is determined from the scalar product (grad ϕ) $\cdot\, \underline{a}$. This scalar quantity is called the directional derivative.
Note:

- $\underline{a}$ along a contour implies $\underline{a}$ is perpendicular to grad ϕ which implies $\underline{a} \cdot$ grad $\phi = 0$.
- $\underline{a}$ perpendicular to a contour implies $\underline{a} \cdot$ grad ϕ is a maximum.

Task

Given $\phi = x^2y^2z^2$, find

(a) grad ϕ

(b) grad ϕ at $(-1, 1, 1)$ and a unit vector in this direction.

(c) the derivative of ϕ at $(2, 1, -1)$ in the direction of

(i) $\underline{i}$ (ii) $\underline{d} = \frac{3}{5}\underline{i} + \frac{4}{5}\underline{k}$.

Your solution

Answer

(a) grad $\phi = \dfrac{\partial\phi}{\partial x}\underline{i} + \dfrac{\partial\phi}{\partial y}\underline{j} + \dfrac{\partial\phi}{\partial z}\underline{k} = 2xy^2z^2\underline{i} + 2x^2yz^2\underline{j} + 2x^2y^2z\underline{k}$

(b) At $(-1, 1, 1)$, grad $\phi = -2\underline{i} + 2\underline{j} + 2\underline{k}$
A unit vector in this direction is

$$\frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{(-2)^2 + 2^2 + 2^2}} = \frac{1}{2\sqrt{3}}(-2\underline{i} + 2\underline{j} + 2\underline{k}) = -\frac{1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{1}{\sqrt{3}}\underline{k}$$

(c) At $(2, 1, -1)$, grad $\phi = 4\underline{i} + 8\underline{j} - 8\underline{k}$

(i) To find the derivative of ϕ in the direction of $\underline{i}$ take the scalar product $(4\underline{i} + 8\underline{j} - 8\underline{k}) \cdot \underline{i} = 4 \times 1 + 0 + 0 = 4.$ So the derivative in the direction of $\underline{d}$ is 4.

(ii) To find the derivative of ϕ in the direction of $\underline{d} = \dfrac{3}{5}\underline{i} + \dfrac{4}{5}\underline{k}$ take the scalar product

$$(4\underline{i} + 8\underline{j} - 8\underline{k}) \cdot (\frac{3}{5}\underline{i} + \frac{4}{5}\underline{k}) = 4 \times \frac{3}{5} + 0 + (-8) \times \frac{4}{5} = \frac{12}{5} - \frac{32}{5} = -4.$$

So the derivative in the direction of $\underline{d}$ is -4.

Exercises

1. Find grad ϕ for the following scalar fields

   (a) $\phi = y - x$. (b) $\phi = y - x^2$, (c) $\phi = x^2 + y^2 + z^2$.

2. Find grad ϕ for each of the following two-dimensional scalar fields given that $\underline{r} = x\underline{i} + y\underline{j}$ and $r = \sqrt{x^2 + y^2}$ (you should express your answer in terms of $\underline{r}$).

   (a) $\phi = r$, (b) $\phi = \ln r$, (c) $\phi = \dfrac{1}{r}$, (d) $\phi = r^n$.

3. If $\phi = x^3y^2z$, find,

   (a) $\underline{\nabla}\phi$
   (b) a unit vector normal to the contour at the point $(1, 1, 1)$.
   (c) the rate of change of ϕ at $(1, 1, 1)$ in the direction of $\underline{i}$.
   (d) the rate of change of ϕ at $(1, 1, 1)$ in the direction of the unit vector $\underline{n} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$.

4. Find a unit vector which is normal to the sphere $x^2 + (y - 1)^2 + (z + 1)^2 = 2$ at the point $(0, 0, 0)$.

5. Find vectors normal to $\phi_1 = y - x^2$ and $\phi_2 = x + y - 2$. Hence find the angle between the curves $y = x^2$ and $y = 2 - x$ at their point of intersection in the first quadrant.

Answers

1. (a) $\dfrac{\partial}{\partial x}(y - x)\underline{i} + \dfrac{\partial}{\partial y}(y - x)\underline{j} = -\underline{i} + \underline{j}$

   (b) $-2x\underline{i} + \underline{j}$

   (c) $[\dfrac{\partial}{\partial x}(x^2 + y^2 + z^2)]\underline{i} + [\dfrac{\partial}{\partial y}(x^2 + y^2 + z^2)]\underline{j} + [\dfrac{\partial}{\partial z}(x^2 + y^2 + z^2)]\underline{k} = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

2. (a) $\dfrac{\underline{r}}{r}$, (b) $\dfrac{\underline{r}}{r^2}$, (c) $-\dfrac{\underline{r}}{r^3}$, (d) $nr^{n-2}\underline{r}$

3. (a) $3x^2y^2z\underline{i} + 2x^3yz\underline{j} + x^3y^2\underline{k}$, (b) $\dfrac{1}{\sqrt{14}}(3\underline{i} + 2\underline{j} + \underline{k})$, (c) 3, (d) $2\sqrt{3}$

4. The vector field $\underline{\nabla}\phi$ where $\phi = x^2 + (y - 1)^2 + (z + 1)^2$ is $2x\underline{i} + 2(y - 1)\underline{j} + 2(z + 1)\underline{k}$
   The value that this vector field takes at the point $(0, 0, 0)$ is $-2\underline{j} + 2\underline{k}$ which is a vector normal to the sphere.
   Dividing this vector by its magnitude forms a unit vector: $\dfrac{1}{\sqrt{2}}(-\underline{j} + \underline{k})$

5. $108°$ or $72°$ (intersect at $(1, 1)$) [At intersection, grad $\phi_1 = -2\underline{i} + \underline{j}$ and grad $\phi_2 = \underline{i} + \underline{j}$.]

2. The divergence of a vector field

Consider the vector field $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$.

In 3D cartesian coordinates the **divergence** of $\underline{F}$ is defined to be

$$\text{div } \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that $\underline{F}$ is a vector field but div $\underline{F}$ is a scalar.
In terms of the differential operator $\underline{\nabla}$, div $\underline{F} = \underline{\nabla} \cdot \underline{F}$ since

$$\underline{\nabla} \cdot \underline{F} = (\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}) \cdot (F_1\underline{i} + F_2\underline{j} + F_3\underline{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Physical Significance of the Divergence

The meaning of the divergence is most easily understood by considering the behaviour of a fluid and hence is relevant to engineering topics such as thermodynamics. The divergence (of the vector field representing velocity) at a point in a fluid (liquid or gas) is a measure of the rate per unit volume at which the fluid is flowing away from the point. A negative divergence is a convergence indicating a flow towards the point. Physically positive divergence means that either the fluid is expanding or that fluid is being supplied by a source external to the field. Conversely convergence means a contraction or the presence of a sink through which fluid is removed from the field. The lines of flow diverge from a source and converge to a sink.

If there is no gain or loss of fluid anywhere then div $\underline{v} = 0$ which is the equation of continuity for an incompressible fluid.

The divergence also enters engineering topics such as electromagnetism. A magnetic field ($\underline{B}$) has the property $\underline{\nabla} \cdot \underline{B} = 0$, that is there are no isolated sources or sinks of magnetic field (no magnetic monopoles).

Key Point 4

$\underline{F}$ is a vector field but div $\underline{F}$ is a scalar field.

Example 11

Find the divergence of the following vector fields.

(a) $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$

(b) $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

(c) $\underline{v} = -x\underline{i} + y\underline{j} + 2\underline{k}$

Solution

(a) $\text{div } \underline{F} = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(y^2) + \dfrac{\partial}{\partial z}(z^2) = 2x + 2y + 2z$

(b) $\text{div } \underline{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(c) $\text{div } \underline{v} = \dfrac{\partial}{\partial x}(-x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(2) = -1 + 1 + 0 = 0$

Example 12

Find the value of a for which $\underline{v} = (2x^2y + z^2)\underline{i} + (xy^2 - x^2z)\underline{j} + (axyz - 2x^2y^2)\underline{k}$ is the vector field of an incompressible fluid.

Solution

$\underline{v}$ is incompressible if div $\underline{v} = 0$.

$$\text{div } \underline{v} = \frac{\partial}{\partial x}(2x^2y + z^2) + \frac{\partial}{\partial y}(xy^2 - x^2z) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 4xy + 2xy + axy$$

which is zero if $a = -6$.

Task

Find the divergence of the following vector field, in general terms and at the point $(1, 0, 3)$.

$$\underline{F}_1 = x^3\underline{i} + y^3\underline{j} + z^3\underline{k}$$

Your solution

Answer

(a) $3x^2 + 3y^2 + 3z^2$, 30

Task

Find the divergence of $\underline{F}_2 = x^2y\underline{i} - 2xy^2\underline{j}$, in general terms and at $(1, 0, 3)$.

Your solution

Answer

$-2xy$, 0,

Task

Find the divergence of $\underline{F}_3 = x^2z\underline{i} - 2y^3z^3\underline{j} + xyz^2\underline{k}$, in general terms and at the point $(1, 0, 3)$.

Your solution

Answer

$2xz - 6y^2z^3 + 2xyz, \;\; 6$

3. The curl of a vector field

The curl of the vector field given by $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ is defined as the vector field

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \underline{k}$$

Physical significance of curl

The divergence of a vector field represents the outflow rate from a point; however the curl of a vector field represents the rotation at a point.

Consider the flow of water down a river (Figure 18). The surface velocity $\underline{v}$ of the water is revealed by watching a light floating object such as a leaf. You will notice two types of motion. First the leaf floats down the river following the streamlines of $\underline{v}$, but it may also rotate. This rotation may be quite fast near the bank, but slow or zero in midstream. Rotation occurs when the velocity, and hence the drag, is greater on one side of the leaf than the other.

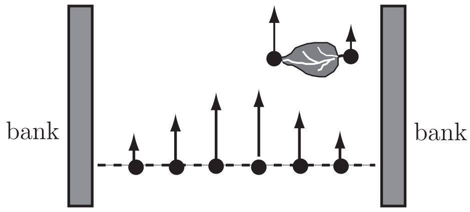


Figure 18: Rotation of a leaf in a stream

Note that for a two-dimensional vector field, such as $\underline{v}$ described here, curl $\underline{v}$ is perpendicular to the motion, and this is the direction of the axis about which the leaf rotates. The magnitude of curl $\underline{v}$ is related to the speed of rotation.

For motion in three dimensions a particle will tend to rotate about the axis that points in the direction of curl $\underline{v}$, with its magnitude measuring the speed of rotation.

If, at any point P, curl $\underline{v} = \underline{0}$ then there is no rotation at P and $\underline{v}$ is said to be **irrotational at P**. If curl $\underline{v} = \underline{0}$ at all points of the domain of $\underline{v}$ then the vector field is an **irrotational vector field**.

Key Point 5

Note that $\underline{F}$ is a vector field and that curl $\underline{F}$ is also a vector field.

Example 13

Find curl $\underline{v}$ for the following two-dimensional vector fields

(a) $\underline{v} = x\underline{i} + 2\underline{j}$ (b) $\underline{v} = -y\underline{i} + x\underline{j}$

If $\underline{v}$ represents the surface velocity of the flow of water, describe the motion of a floating leaf.

Solution

$$(a) \qquad \underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2)\right)\underline{i} + \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(x)\right)\underline{k} = \underline{0}$$

A floating leaf will travel along the streamlines without rotating.

Solution (contd.)

(b)

$$\begin{aligned}
\underline{\nabla} \times \underline{v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right)\underline{i} + \left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\underline{k} \\
&= 0\underline{i} + 0\underline{j} + 2\underline{k} = 2\underline{k}
\end{aligned}$$

A floating leaf will travel along the streamlines (anti-clockwise around the origin) and will rotate anticlockwise (as seen from above).
An analogy of the right-hand screw rule is that a positive (anti-clockwise) rotation in the xy plane represents a positive z-component of the curl. Similar results apply for the other components.

Example 14

(a) Find the curl of $\underline{u} = x^2\underline{i} + y^2\underline{j}$. When is $\underline{u}$ irrotational?

(b) Given $\underline{F} = (xy - xz)\underline{i} + 3x^2\underline{j} + yz\underline{k}$, find curl $\underline{F}$ at the origin $(0, 0, 0)$ and at the point $P = (1, 2, 3)$.

Solution

(a)

$$\begin{aligned}
\text{curl } \underline{u} &= \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2)\right)\underline{i} + \left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right)\underline{k} \\
&= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}
\end{aligned}$$

curl $\underline{u} = \underline{0}$ so $\underline{u}$ is irrotational everywhere.

Solution (contd.)

(b)

$$\begin{aligned}
\text{curl } \underline{F} &= \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy - xz & 3x^2 & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(3x^2) \right) \underline{i} + \left(\frac{\partial}{\partial z}(xy - xz) - \frac{\partial}{\partial x}(yz) \right) \underline{j} \\
&\quad + \left(\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(xy - xz) \right) \underline{k} \\
&= z\underline{i} - x\underline{j} + 5x\underline{k}
\end{aligned}$$

At the point $(0, 0, 0)$, curl $\underline{F} = \underline{0}$. At the point $(1, 2, 3)$, curl $\underline{F} = 3\underline{i} - \underline{j} + 5\underline{k}$.

Engineering Example 1

Current associated with a magnetic field

Introduction

In a magnetic field $\underline{B}$, an associated current is given by:

$$\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$$

Problem in words

Given the magnetic field $\underline{B} = B_0 x \underline{k}$ find the associated current $\underline{I}$.

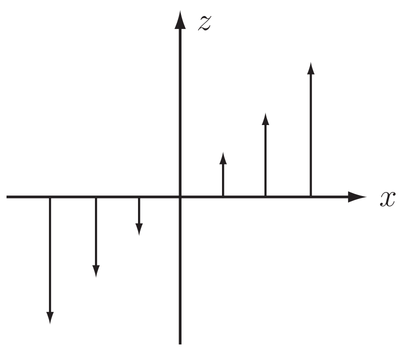


Figure 19: Magnetic field profile

Mathematical statement of problem

We need to evaluate the curl of $\underline{B}$.

Mathematical analysis

$$
\begin{aligned}
\underline{\nabla} \times \underline{B} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & B_0 x \end{vmatrix} \\
&= 0\underline{i} - B_0\underline{j} + 0\underline{k} \\
&= -B_0\underline{j}
\end{aligned}
$$

and so $\underline{I} = -\dfrac{B_0}{\mu_0}\underline{j}$.

Interpretation

The current is perpendicular to the field and to the direction of variation of the field.

Task

Find the curl of the following two-dimensional vector field (a) in general terms and (b) at the point $(1, 2)$.

$\underline{F}_2 = y^2\underline{i} + xy\underline{j}$

Your solution

Answer

(a) $\underline{\nabla} \times F_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + (y - 2y)\underline{k} = -y\underline{k}$

(b) $-2\underline{k}$

Exercises

1. Find the curl of each of the following two-dimensional vector fields. Give each in general terms and also at the point $(1, 2)$.

   (a) $\underline{F}_1 = 2x\underline{i} + 2y\underline{j}$

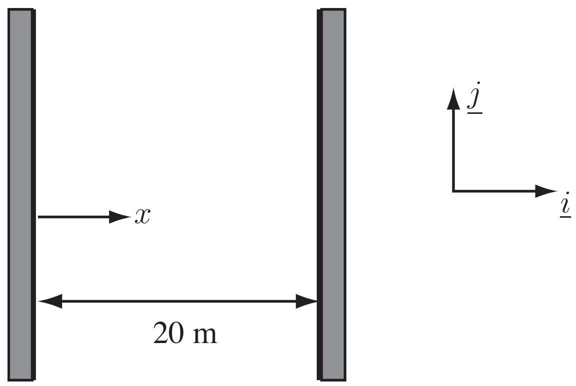
   (b) $\underline{F}_3 = x^2y^3\underline{i} - x^3y^2\underline{j}$

2. Find the curl of each of the following three-dimensional vector fields. Give each in general terms and also at the point $(2, 1, 3)$.

   (a) $\underline{F}_1 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$

   (b) $\underline{F}_2 = (xy + z^2)\underline{i} + x^2\underline{j} + (xz - 2)\underline{k}$

3. The surface water velocity on a straight uniform river 20 metres wide is modelled by the vector $\underline{v} = \frac{1}{50}x(20 - x)\underline{j}$ where x is the distance from the west bank (see diagram).



   (a) Find the velocity $\underline{v}$ at each bank and at midstream.

   (b) Find $\underline{\nabla} \times \underline{v}$ at each bank and at midstream.

4. The velocity field on the surface of an emptying bathroom sink can be modelled by two functions, the first describing the swirling vortex of radius a near the plughole and the second describing the more gently rotating fluid outside the vortex region. These functions are

$$\underline{u}(x, y) = w(-y\underline{i} + x\underline{j}), \qquad \left(\sqrt{x^2 + y^2} \leq a\right)$$

$$\underline{v}(x, y) = \frac{wa^2(-y\underline{i} + x\underline{j})}{x^2 + y^2} \qquad \left(\sqrt{x^2 + y^2} \geq a\right)$$

   Find (a) curl $\underline{u}$ and (b)curl $\underline{v}$.

Answers

1. (a) $\underline{0}$; $\underline{0}$ (b) $-6x^2y^2\underline{k}$, $-24\underline{k}$

2. (a) $\underline{0}$; $\underline{0}$ (b) $z\underline{j} + x\underline{k}$, $3\underline{j} + 2\underline{k}$

3. (a) $\underline{0}$; $\underline{0}$; $2\underline{j}$, (b) $+0.4\underline{k}$; $-0.4\underline{k}$; $\underline{0}$

4. (a) $2w\underline{k}$; (b) $\underline{0}$

4. The Laplacian

The Laplacian of a function ϕ is written as $\nabla^2\phi$ and is defined as: Laplacian $\phi =$ div grad ϕ, that is

$$\begin{aligned}\nabla^2\phi &= \underline{\nabla}\cdot\underline{\nabla}\phi\\ &= \underline{\nabla}\cdot\left(\frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}\right)\\ &= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\end{aligned}$$

The equation $\nabla^2\phi = 0$, that is $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2} = 0$ is known as Laplace's equation and has applications in many branches of engineering including Heat Flow, Electrical and Magnetic Fields and Fluid Mechanics.

Example 15

Find the Laplacian of $u = x^2y^2z + 2xz$.

Solution

$\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2} = 2y^2z + 2x^2z + 0 = 2(x^2+y^2)z$

5. Examples involving grad, div, curl and the Laplacian

The vector differential operators can be combined in several ways as the following examples show.

Example 16

If $\underline{A} = 2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k}$, $\underline{B} = x^2\underline{i} + yz\underline{j} - xy\underline{k}$ and $\phi = 2x^2yz^3$, find

(a) $(\underline{A}\cdot\underline{\nabla})\phi$ (b) $\underline{A}\cdot\underline{\nabla}\phi$ (c) $\underline{B}\times\underline{\nabla}\phi$ (d) $\nabla^2\phi$

Solution

(a)

$$\begin{aligned}(\underline{A}\cdot\underline{\nabla})\phi &= \left[(2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k})\cdot(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k})\right]\phi \\ &= \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]2x^2yz^3 \\ &= 2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial z}(2x^2yz^3) \\ &= 2yz(4xyz^3) - x^2y(2x^2z^3) + xz^2(6x^2yz^2) \\ &= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4\end{aligned}$$

(b)

$$\begin{aligned}\underline{\nabla}\phi &= \frac{\partial}{\partial x}(2x^2yz^3)\underline{i} + \frac{\partial}{\partial y}(2x^2yz^3)\underline{j} + \frac{\partial}{\partial z}(2x^2yz^3)\underline{k} \\ &= 4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}\end{aligned}$$

So
$$\begin{aligned}\underline{A}\cdot\underline{\nabla}\phi &= \left(2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k}\right)\cdot(4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}) \\ &= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4\end{aligned}$$

(c) $\underline{\nabla}\phi = 4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}$ so

$$\begin{aligned}\underline{B}\times\underline{\nabla}\phi &= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k} \\ x^2 & yz & -xy \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2\end{vmatrix} \\ &= \underline{i}(6x^2y^2z^3 + 2x^3yz^3) + \underline{j}(-4x^2y^2z^3 - 6x^4yz^2) + \underline{k}(2x^4z^3 - 4xy^2z^4)\end{aligned}$$

(d) $\nabla^2\phi = \dfrac{\partial^2}{\partial x^2}(2x^2yz^3) + \dfrac{\partial^2}{\partial y^2}(2x^2yz^3) + \dfrac{\partial^2}{\partial z^2}(2x^2yz^3) = 4yz^3 + 0 + 12x^2yz$

Example 17

For each of the expressions below determine whether the quantity can be formed and, if so, whether it is a scalar or a vector.

(a) grad(div $\underline{A}$)

(b) grad(grad ϕ)

(c) curl(div $\underline{F}$)

(d) div [curl ($\underline{A}\times$grad ϕ)]

Solution

(a) $\underline{A}$ is a vector and div$\underline{A}$ can be calculated and is a scalar. Hence, grad(div $\underline{A}$) can be formed and is a vector.

(b) ϕ is a scalar so grad ϕ can be formed and is a vector. As grad ϕ is a vector, it is not possible to take grad(grad ϕ).

(c) $\underline{F}$ is a vector and hence div $\underline{F}$ is a scalar. It is not possible to take the curl of a scalar so curl(div $\underline{F}$) does not exist.

(d) ϕ is a scalar so grad ϕ exists and is a vector. $\underline{A}\times$grad ϕ exists and is also a vector as is curl $\underline{A}\times$grad ϕ. The divergence can be taken of this last vector to give div [curl ($\underline{A}\times$grad ϕ)] which is a scalar.

6. Identities involving grad, div and curl

There are numerous identities involving the vector derivatives; a selection are given in Table 1.

Table 1

1	div($\phi\underline{A}$) $=$ grad $\phi\cdot\underline{A}+\phi$ div $\underline{A}$	or	$\underline{\nabla}\cdot(\phi\underline{A})=(\underline{\nabla}\phi)\cdot\underline{A}+\phi(\underline{\nabla}\cdot\underline{A})$
2	curl($\phi\underline{A}$) $=$ grad $\phi\times\underline{A}+\phi$ curl $\underline{A}$	or	$\underline{\nabla}\times(\phi\underline{A})=(\underline{\nabla}\phi)\times\underline{A}+\phi(\underline{\nabla}\times\underline{A})$
3	div ($\underline{A}\times\underline{B}$) $=\underline{B}\cdot$ curl $\underline{A}-\underline{A}\cdot$ curl $\underline{B}$	or	$\underline{\nabla}\cdot(\underline{A}\times\underline{B})=\underline{B}\cdot(\underline{\nabla}\times\underline{A})-\underline{A}\cdot(\underline{\nabla}\times\underline{B})$
4	curl ($\underline{A}\times\underline{B}$) $=$ ($\underline{B}\cdot$ grad) $\underline{A}-$ ($\underline{A}\cdot$ grad) $\underline{B}$ $+\underline{A}$ div $\underline{B}-\underline{B}$ div $\underline{A}$	or	$\underline{\nabla}\times(\underline{A}\times\underline{B})=(\underline{B}\cdot\underline{\nabla})\underline{A}-(\underline{A}\cdot\underline{\nabla})\underline{B}$ $+\underline{A}\ \underline{\nabla}\cdot\underline{B}-\underline{B}\ \underline{\nabla}\cdot\underline{A}$
5	grad ($\underline{A}\cdot\underline{B}$) $=$ ($\underline{B}\cdot$ grad) $\underline{A}+$ ($\underline{A}\cdot$ grad) $\underline{B}$ $+\underline{A}\times$ curl $\underline{B}+\underline{B}\times$ curl $\underline{A}$	or	$\underline{\nabla}(\underline{A}\cdot\underline{B})=(\underline{B}\cdot\underline{\nabla})\underline{A}+(\underline{A}\cdot\underline{\nabla})\underline{B}$ $+\underline{A}\times(\underline{\nabla}\times\underline{B})+\underline{B}\times(\underline{\nabla}\times\underline{A})$
6	curl (grad ϕ) $=\underline{0}$	or	$\underline{\nabla}\times(\underline{\nabla}\phi)=\underline{0}$
7	div (curl $\underline{A}$) $=0$	or	$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{A})=0$

Example 18

Show for any vector field $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$, that div curl $\underline{A} = \underline{0}$.

Solution

$$\begin{aligned}
\text{div curl } \underline{A} &= \text{div} \begin{vmatrix} \underline{i} & \underline{j} & \underline{j} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\
&= \text{div} \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \underline{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \underline{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \underline{k} \right] \\
&= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\
&= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial z \partial x} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} \\
&= 0
\end{aligned}$$

N.B. This assumes $\dfrac{\partial^2 A_3}{\partial x \partial y} = \dfrac{\partial^2 A_3}{\partial y \partial x}$ etc.

Example 19

Verify identity 1 for the vector $\underline{A} = 2xy\underline{i} - 3z\underline{k}$ and the function $\phi = xy^2$.

Solution

$\phi\underline{A} = 2x^2y^3\underline{i} - 3xy^2z\underline{k}$ so

$\underline{\nabla} \cdot \phi\underline{A} = \underline{\nabla} \cdot \left(2x^2y^3\underline{i} - 3xy^2z\underline{k}\right) = \dfrac{\partial}{\partial x}(2x^2y^3) + \dfrac{\partial}{\partial z}(-3xy^2z) = 4xy^3 - 3xy^2$

So LHS $= 4xy^3 - 3xy^2$.

$\underline{\nabla}\phi = \dfrac{\partial}{\partial x}(xy^2)\underline{i} + \dfrac{\partial}{\partial y}(xy^2)\underline{j} + \dfrac{\partial}{\partial z}(xy^2)\underline{k} = y^2\underline{i} + 2xy\underline{j}$ so

$(\underline{\nabla}\phi) \cdot \underline{A} = (y^2\underline{i} + 2xy\underline{j}) \cdot (2xy\underline{i} - 3z\underline{k}) = 2xy^3$

$\underline{\nabla} \cdot \underline{A} = \underline{\nabla} \cdot (2xy\underline{i} - 3z\underline{k}) = 2y - 3$ so $\phi\underline{\nabla} \cdot \underline{A} = 2xy^3 - 3xy^2$ giving

$(\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A}) = 2xy^3 + (2xy^3 - 3xy^2) = 4xy^3 - 3xy^2$

So RHS $= 4xy^3 - 3xy^2 =$ LHS.

So $\underline{\nabla} \cdot (\phi\underline{A}) = (\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A})$ in this case.

Task

If $\underline{F} = x^2y\underline{i} - 2xz\underline{j} + 2yz\underline{k}$, find

(a) $\underline{\nabla} \cdot \underline{F}$

(b) $\underline{\nabla} \times \underline{F}$

(c) $\underline{\nabla}(\underline{\nabla} \cdot \underline{F})$

(d) $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F})$

(e) $\underline{\nabla} \times (\underline{\nabla} \times \underline{F})$

Your solution

Answer

(a) $2xy + 2y$,

(b) $(2x + 2z)\underline{i} - (x^2 + 2z)\underline{k}$,

(c) $2y\underline{i} + (2 + 2x)\underline{j}$ (using answer to (a)),

(d) 0 (using answer to (b)),

(e) $(2 + 2x)\underline{j}$ (using answer to (b))

Task

If $\phi = 2xz - y^2z$, find

(a) $\underline{\nabla}\phi$

(b) $\nabla^2\phi = \underline{\nabla} \cdot (\underline{\nabla}\phi)$

(c) $\underline{\nabla} \times (\underline{\nabla}\phi)$

Your solution

Answer

(a) $2z\underline{i} - 2yz\underline{j} + (2x - y^2)\underline{k}$, (b) $-2z$, (c) $\underline{0}$ where (b) and (c) use the answer to (a).

Exercise

Which of the following combinations of grad, div and curl can be formed? If a quantity can be formed, state whether it is a scalar or a vector.

(a) div (grad ϕ)

(b) div (div $\underline{A}$)

(c) curl (curl $\underline{F}$)

(d) div (curl $\underline{F}$)

(e) curl (grad ϕ)

(f) curl (div $\underline{A}$)

(g) div ($\underline{A} \cdot \underline{B}$)

(h) grad ($\phi_1\phi_2$)

(i) curl (div ($\underline{A}\times$ grad ϕ))

Answers

(a), (d) are scalars;

(c), (e), (h) are vectors;

(b), (f), (g) and (i) are not defined.