Contents 28

Differential Vector Calculus

Learning outcomes

In this Workbook you will learn about scalar and vector fields and how physical quantities can be represented by such fields. You will be able to 'differentiate' such fields i.e. to find how rapidly the scalar or vector field varies with position. Depending on whether the original function and the intended derivative are scalars or vectors, there are three such derivatives known as the 'gradient', the 'divergence' and the 'curl'. You will be able to evaluate these derivatives for given fields. In addition, you will be able to work out the derivatives while using polar coordinate systems.

Background to Vector Calculus

28.1

Introduction

Vector Calculus is the study of the various derivatives and integrals of a scalar or vector function of the variables defining position (x,y,z) and possibly also time (t). This Section considers functions of several variables and introduces scalar and vector fields.

Prerequisites

Before starting this Section you should ...

- be familiar with the concept of a function of two variables
- be familiar with the concept of partial differentiation
- be familiar with the concept of vectors

Learning Outcomes

On completion you should be able to ...

- state the properties of scalar and vector fields
- work with a vector function of a variable

1. Functions of several variables and partial derivatives

These functions were first studied in HELM 18. As a reminder:

- a function of the two independent variables x and y may be written as $f(x, y)$
- the first and second order partial derivatives are $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

Consider, for example, the function $f(x, y) = x^2 + 5xy + 3y^4 + 1$. The first and second partial derivatives are

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 2x + 5y \qquad \text{(differentiating with respect to } x \text{ keeping } y \text{ constant)} \\
\frac{\partial f}{\partial y} &= 5x + 12y^3 \qquad \text{(differentiating with respect to } y \text{ keeping } x \text{ constant)} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 5y) = 2 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x + 12y^3) = 36y^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 5y) = 5
\end{aligned}
$$

The number of independent variables is not restricted to two. For example, if u is a function of the three variables x, y and z, say $u = x^2 + y^2 + z^2$ then:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = 2, \quad \frac{\partial^2 u}{\partial z^2} = 2$$

Similarly, if u is a function of the four variables x, y, z and t say $u = xy^2z^3\mathrm{e}^t$ then

$$\frac{\partial u}{\partial x} = y^2z^3\mathrm{e}^t, \quad \frac{\partial u}{\partial t} = xy^2z^3\mathrm{e}^t, \quad \frac{\partial^2 u}{\partial z^2} = 6xy^2z\mathrm{e}^t, \text{ etc.}$$

2. Vector functions of a variable

Vectors were first studied in HELM 9. A vector is a quantity that has magnitude and direction and combines together with other vectors according to the triangle law. Examples are (i) a velocity of 60 mph West and (ii) a force of 98.1 newtons vertically downwards.
It is often convenient to express vectors in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$, which are unit vectors in the x, y and z directions respectively. Examples are $\underline{a} = 3\underline{i} + 4\underline{j}$ and $\underline{b} = 2\underline{i} - 2\underline{j} + \underline{k}$
The magnitudes of these vectors are $|\underline{a}| = \sqrt{3^2 + 4^2} = 5$ and $|\underline{b}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$ respectively. In this case $\underline{a}$ and $\underline{b}$ are constant vectors, but a vector could be a function of an independent variable such as t (which may represent time in certain applications).

Example 1

A particle is at the point A(3,0). At time $t = 0$ it starts moving at a constant speed of 2 m s^{-1} in a direction parallel to the positive y-axis. Find expressions for the position vector, $\underline{r}$, of the particle at time t, together with its velocity $\underline{v} = \frac{d\underline{r}}{dt}$ and acceleration $\underline{a} = \frac{d^2\underline{r}}{dt^2}$.

Solution

In the first second of its motion the particle moves 2 metres to B and it moves a further 2 metres in each subsequent second, to C, D, ... Because it moves parallel to the y-axis its velocity is $\underline{v} = 2\underline{j}$. As its velocity is constant its acceleration is $\underline{a} = \underline{0}$.
The position of the particle at $t = 0, 1, 2, 3$ is given in the table.

Time t	0	1	2	3
Position $\underline{r}$	$3\underline{i}$	$3\underline{i}+2\underline{j}$	$3\underline{i}+4\underline{j}$	$3\underline{i}+6\underline{j}$

In general, after t seconds, the position vector of the particle is $\underline{r} = 3\underline{i} + 2t\underline{j}$

Example 2

The position vector of a particle at time t is given by $\underline{r} = 2t\underline{i} + t^2\underline{j}$. Find its equation in Cartesian form and sketch the path followed by the particle.

Tabulating $\underline{r} = x\underline{i} + y\underline{j}$ at different times t:

Time t	0	1	2	3	4
x	0	2	4	6	8
y	0	1	4	9	16
$\underline{r}$	$\underline{0}$	$2\underline{i}+\underline{j}$	$4\underline{i}+4\underline{j}$	$6\underline{i}+9\underline{j}$	$8\underline{i}+16\underline{j}$

Solution

To find the Cartesian equation of the curve we eliminate t between $x = 2t$ and $y = t^2$. Re-arrange $x = 2t$ as $t = \frac{1}{2}x$. Then $y = t^2 = \left(\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$, which is a parabola. This is the path followed by the particle. See Figure 1.

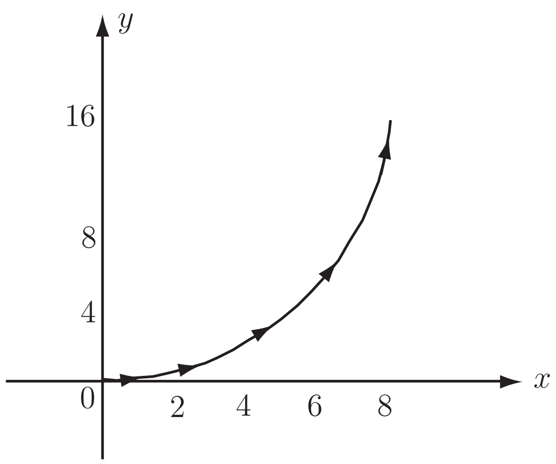


Figure 1: Path followed by a particle

In general, a three-dimensional vector function of one variable t is of the form

$$\underline{u} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}.$$

Such functions may be differentiated one or more times and the rules of differentiation are derived from those for ordinary scalar functions. In particular, if $\underline{u}$ and $\underline{v}$ are vector functions of t and if c is a constant, then:

Rule 1. $\dfrac{d}{dt}(\underline{u} + \underline{v}) = \dfrac{d\underline{u}}{dt} + \dfrac{d\underline{v}}{dt}$

Rule 2. $\dfrac{d}{dt}(c\underline{u}) = c\dfrac{d\underline{u}}{dt}$

Rule 3. $\dfrac{d}{dt}(\underline{u} \cdot \underline{v}) = \underline{u} \cdot \dfrac{d\underline{v}}{dt} + \dfrac{d\underline{u}}{dt} \cdot \underline{v}$

Rule 4. $\dfrac{d}{dt}(\underline{u} \times \underline{v}) = \underline{u} \times \dfrac{d\underline{v}}{dt} + \dfrac{d\underline{u}}{dt} \times \underline{v}$

Also, if a particle moves so that its position vector at time t is $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$ then the velocity of the particle is

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{dx(t)}{dt}\underline{i} + \frac{dy(t)}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k} = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}$$

and its acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = \frac{d^2x(t)}{dt^2}\underline{i} + \frac{d^2y(t)}{dt^2}\underline{j} + \frac{d^2z(t)}{dt^2}\underline{k} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

Example 3

Find the derivative (with respect to t) of the position vector $\underline{r} = t^2\underline{i} + 3t\underline{j} + 4\underline{k}$. Also find a unit vector tangential to the curve traced out by the position vector at the point where $t = 2$.

Solution

Differentiating $\underline{r}$ with respect to t,

$$\dot{\underline{r}} = \frac{d\underline{r}}{dt} = 2t\underline{i} + 3\underline{j}$$

so

$$\dot{\underline{r}}(2) = 4\underline{i} + 3\underline{j}$$

A unit vector in this direction, which is tangential to the curve, is

$$\frac{\dot{\underline{r}}(2)}{|\dot{\underline{r}}(2)|} = \frac{4\underline{i} + 3\underline{j}}{\sqrt{4^2 + 3^2}} = \frac{4}{5}\underline{i} + \frac{3}{5}\underline{j}$$

Example 4

For the position vectors (i) $\underline{r} = 3\underline{i} + 2t\underline{j}$ and (ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$ use the general expressions for velocity and acceleration to confirm the values of $\underline{v}$ and $\underline{a}$ found earlier in Examples 1 and 2.

Solution

(i) $\underline{r} = 3\underline{i} + 2t\underline{j}$. Then

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{d}{dt}(3\underline{i} + 2t\underline{j}) = \frac{d(3)}{dt}\underline{i} + \frac{d(2t)}{dt}\underline{j} = 0\underline{i} + 2\underline{j} = 2\underline{j}$$

and

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{\underline{r}} = \frac{d}{dt}(2\underline{j}) = \frac{d(2)}{dt}\underline{j} = 0\underline{j} = \underline{0}$$

which agree with those found earlier.

(ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$. Then

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{d}{dt}(2t\underline{i} + t^2\underline{j}) = \frac{d(2t)}{dt}\underline{i} + \frac{d(t^2)}{dt}\underline{j} = 2\underline{i} + 2t\underline{j}$$

and

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{\underline{r}} = \frac{d}{dt}(2\underline{i} + 2t\underline{j}) = \frac{d(2)}{dt}\underline{i} + \frac{d(2t)}{dt}\underline{j} = 0\underline{i} + 2\underline{j} = 2\underline{j}$$

which agree with those found earlier.

Example 5

A particle of mass $m = 1$ kg has position vector $\underline{r}$. The torque (moment of force) $\underline{H}$ relative to the origin acting on the particle as a result of a force $\underline{F}$ is defined as $\underline{H} = \underline{r} \times \underline{F}$, where, by Newton's second law, $\underline{F} = m\ddot{\underline{r}}$. The angular momentum (moment of momentum) $\underline{L}$ of the particle is defined as $\underline{L} = \underline{r} \times m\dot{\underline{r}}$. Find $\underline{L}$ and $\underline{H}$ for the particle where (i) $\underline{r} = 3\underline{i} + 2t\underline{j}$ and (ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$, and show that in each case the torque law $\underline{H} = \dot{\underline{L}}$ is satisfied.

Solution

(i) Here $\underline{r} = 3\underline{i} + 2t\underline{j}$ so $\dot{\underline{r}} = 2\underline{j}$ and $\underline{a} = \underline{0}$. Then

$$\underline{L} = \underline{r} \times m\dot{\underline{r}} = (3\underline{i} + 2t\underline{j}) \times 2\underline{j} = 6\underline{k} \quad \text{so} \quad \dot{\underline{L}} = \frac{d}{dt}(6)\underline{k} = \underline{0}$$

and

$$\underline{H} = \underline{r} \times \underline{F} = \underline{r} \times m\ddot{\underline{r}} = (3\underline{i} + 2t\underline{j}) \times \underline{0} = \underline{0} \quad \text{giving} \quad \underline{H} = \dot{\underline{L}} \quad \text{as required.}$$

Solution (contd.)

(ii) Here $\underline{r} = 2t\underline{i} + t^2\underline{j}$ so $\dot{\underline{r}} = 2\underline{i} + 2t\underline{j}$ and $\underline{a} = 2\underline{j}$. Then

$$\underline{L} = \underline{r} \times m\dot{\underline{r}} = (2t\underline{i} + t^2\underline{j}) \times (2\underline{i} + 2t\underline{j}) = (4t^2 - 2t^2)\underline{k} = 2t^2\underline{k} \quad \text{so} \quad \dot{\underline{L}} = 4t\underline{k}$$

and

$$\underline{H} = \underline{r} \times \underline{F} = \underline{r} \times m\ddot{\underline{r}} = (2t\underline{i} + t^2\underline{j}) \times 2\underline{j} = 4t\underline{k} \quad \text{giving} \quad \underline{H} = \dot{\underline{L}} \text{ as required.}$$

Task

A particle moves so that its position vector is $\underline{r} = 12t\underline{i} + (19t - 5t^2)\underline{j}$.

(a) Find $\dfrac{d\underline{r}}{dt}$ and $\dfrac{d^2\underline{r}}{dt^2}$.

(b) When is the $\underline{j}$-component of $\dfrac{d\underline{r}}{dt}$ equal to zero?

(c) Find a unit vector normal to its trajectory when $t = 1$.

Your solution

Answer

(a) $\dfrac{d\underline{r}}{dt} = 12\underline{i} + (19 - 10t)\underline{j}, \qquad \dfrac{d^2\underline{r}}{dt^2} = -10\underline{j}.$

(b) The $\underline{j}$-component of $\dfrac{d\underline{r}}{dt}$, (also written $\dot{\underline{r}}$) is zero when $t = 1.9$.

(c) When $t = 1$ $\dot{\underline{r}} = 12\underline{i} + 9\underline{j}$. A vector perpendicular to this is $\dot{\underline{r}} = 9\underline{i} - 12\underline{j}$. Its magnitude is $\sqrt{81 + 144} = 15$. So a unit vector in this direction is $\frac{9}{15}\underline{i} - \frac{12}{15}\underline{j} = \frac{3}{5}\underline{i} - \frac{4}{5}\underline{j}$. The unit vector $-\dfrac{3}{5}\underline{i} + \dfrac{4}{5}\underline{j}$ is also a solution.

Task

A particle moving at a constant speed around a circle moves so that

$\underline{r} = \cos(\pi t)\underline{i} + \sin(\pi t)\underline{j}$

(a) Find $\dfrac{d\underline{r}}{dt}$ and $\dfrac{d^2\underline{r}}{dt^2}$.

(b) Find $\underline{r} \cdot \dfrac{d\underline{r}}{dt}$ and $\underline{r} \times \dfrac{d^2\underline{r}}{dt^2}$.

Your solution

Answer

(a) $\dfrac{d\underline{r}}{dt} = -\pi \sin \pi t\underline{i} + \pi \cos \pi t\underline{j}, \quad \dfrac{d^2\underline{r}}{dt^2} = -\pi^2 \cos \pi t\underline{i} - \pi^2 \sin \pi t\underline{j} = -\pi^2\underline{r},$

(b) $\underline{r}.\dfrac{d\underline{r}}{dt} = -\pi \cos \pi t \sin \pi t + \pi \cos \pi t \sin \pi t = 0 \;\Rightarrow\; \dfrac{d\underline{r}}{dt}$ is perpendicular to $\underline{r}$

$$\underline{r} \times \frac{d^2\underline{r}}{dt^2} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \pi t & \sin \pi t & 0 \\ -\pi^2 \cos \pi t & -\pi^2 \sin \pi t & 0 \end{vmatrix} = \underline{0} \;\Rightarrow\; \frac{d^2\underline{r}}{dt^2} \text{ is parallel to } \underline{r}.$$

Task

If $\underline{r} = \sin(2t)\underline{i} + \cos(2t)\underline{j} + t^2\underline{k}$ and $(1+t^2)\,|\ddot{\underline{r}}|^2 = c\,|\dot{\underline{r}}|^2$, find the value of c.

Your solution

Answer

$\dot{\underline{r}} = 2\cos(2t)\underline{i} - 2\sin(2t)\underline{j} + 2t\underline{k}, \quad \ddot{\underline{r}} = -4\sin(2t)\underline{i} - 4\cos(2t)\underline{j} + 2\underline{k}$

$|\ddot{\underline{r}}|^2 = 16\sin^2(2t) + 16\cos^2(2t) + 4 = 20 \quad |\dot{\underline{r}}|^2 = 4\cos^2(2t) + 4\sin^2(2t) + 4t^2 = 4(1+t^2)$

$\therefore \quad 20(1+t^2) = 4c(1+t^2)$ so that $c = 5$.

3. Scalar fields

A **scalar field** is a distribution of scalar values over a region of space (which may be 1D, 2D or 3D) so that a scalar value is associated with each point of space. Examples of scalar fields follow.

1.

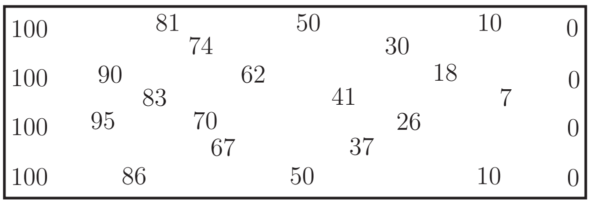


Figure 2: Temperature in a plate, one side held at 100°C the other at 0°C

2.

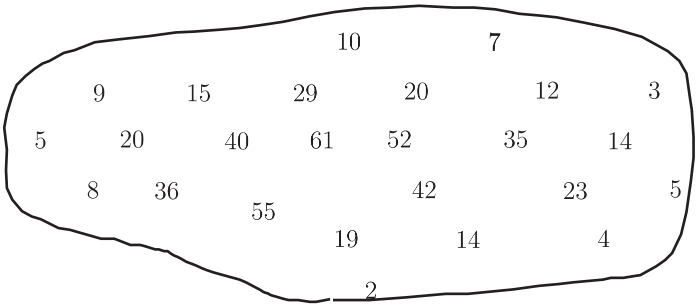


Figure 3: Height of land above sea level

3. The mean annual rainfall at different locations in Britain.

4. The light intensity near a 100 watt light bulb.

To define a **scalar field** we need to:

- Describe the region of space where it is found (this is the **domain**)
- Give a rule to show how the value of the scalar is related to every point in the domain.

Consider the scalar field defined by $\phi(x, y) = x + y$ over the rectangle $0 \leq x \leq 4$, $0 \leq y \leq 2$. We can calculate, and plot, values of ϕ at different (x, y) points. For example $\phi(0, 2) = 0 + 2 = 2$, $\phi(4, 1) = 4 + 1 = 5$ and so on.

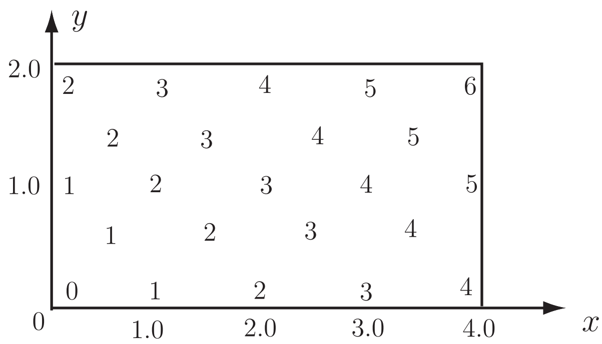


Figure 4: The scalar field $\phi(x, y) = x + y$

Contours

A contour on a map is a curve joining points that are the same height above sea level. These contours give far more information about the shape of the land than selected spot heights.

For example, the contours near the top of a hill might look like those shown in Figure 5 where the numbers are the values of the heights above sea level.

In general for a scalar field $\phi(x, y, z)$, contour curves are the family of curves given by $\phi = c$, for different values of the constant c.

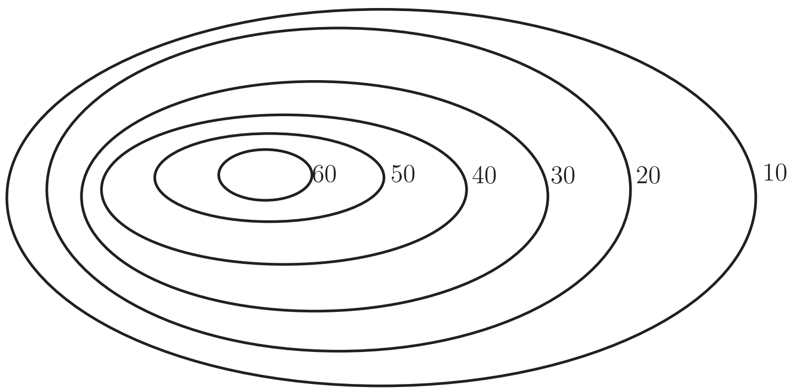


Figure 5: Contour lines

Example 6

Describe contour curves for the following scalar fields and sketch typical contours for (a) and (b).

(a) $\phi(x,y) = x + y$

(b) $\phi(x,y) = 9 - x^2 - y^2$

(c) $\phi(x,y) = \dfrac{1}{x^2 + y^2 + z^2}$

Solution

(a) The contour curves for $\phi(x,y) = x + y$ are $x + y = c$ or $y = -x + c$. These are straight lines of gradient -1. See Figure 6(a).

(b) For $\phi(x,y) = 9 - x^2 - y^2$, the contour curves are $9 - x^2 - y^2 = c$, or $x^2 + y^2 = 9 - c$. See Figure 6(b). These are circles, centered at the origin, radius $\sqrt{9 - c}$.

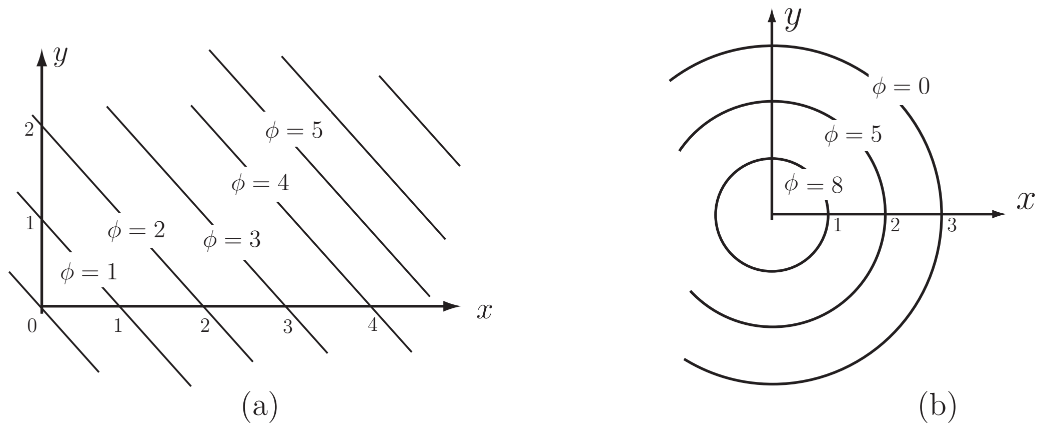


Figure 6: Contours for (a) $x + y$ (b) $9 - x^2 - y^2$

(c) For the three-dimensional scalar field $\phi(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2}$ the contour surfaces are $\dfrac{1}{x^2 + y^2 + z^2} = c$ or $x^2 + y^2 + z^2 = \dfrac{1}{c}$. These are spheres, centered at the origin and of radius $\dfrac{1}{\sqrt{c}}$.

Describe the contours for the following scalar fields

(a) $\phi = y - x$ (b) $\phi = x^2 + y^2$ (c) $\phi = y - x^2$

Your solution

Answer

(a) Straight lines of gradient 1, (b) Circles; centred at origin, (c) Parabolas $y = x^2 + c$.

Key Point 1

A scalar field F (in three-dimensional space) returns a real value for the function F for every point (x, y, z) in the domain of the field.

4. Vector fields

A vector field is a distribution of vectors over a region of space such that a vector is associated with each point of the region. Examples are:

1. The velocity of water flowing in a river (Figure 7).

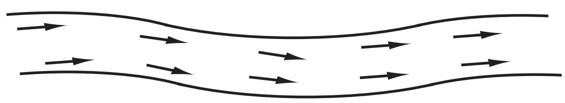


Figure 7: Velocity of water in a river

2. The gravitational pull of the Earth (Figure 8). At every point there is a gravitational pull towards the centre of the Earth.

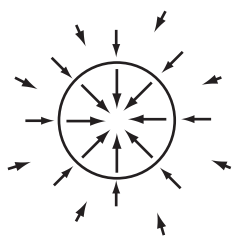


Figure 8: Gravitational pull of the Earth

Note: the length of the vector is used to indicate its magnitude (i.e. greater near the centre of the Earth.)

3. The flow of heat in a metal plate insulated on its sides (Figure 9). Heat flows from the hot portion on the left to the cool portion on the right.

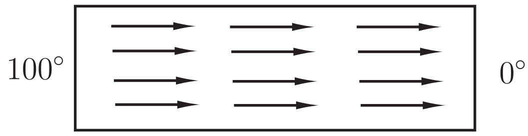


Figure 9: Flow of heat in a metal plate

To define a **vector field** we need to :

- Describe the region of space where the vectors are found (the domain)
- Give a rule for associating a vector with each point of the domain.

Note that in the case of the heat flowing in a plate, the temperature can be described by a scalar field while the flow of heat is described by a vector field.
Consider the flow of water in different situations.

(a) In a pond where the water is motionless everywhere, the velocity at all points is zero. That is, $\underline{v}(x, y, z) = \underline{0}$, or for brevity, $\underline{v} = \underline{0}$.

(b) Consider a straight river with steady flow downstream (see Figure 10). The surface velocity $\underline{v}$ can be seen by watching the motion of a light floating object, such as a leaf. The leaf will float downstream parallel to the bank so $\underline{v}$ will be a multiple of $\underline{j}$. However, the speed is usually smallest near the bank and fastest in the middle of the river. In this simple model, the velocity $\underline{v}$ is assumed to be independent of the depth z. That is, $\underline{v}$ varies, in the $\underline{i}$, or x, direction so that $\underline{v}$ will be of the form $\underline{v} = f(x)\underline{j}$.

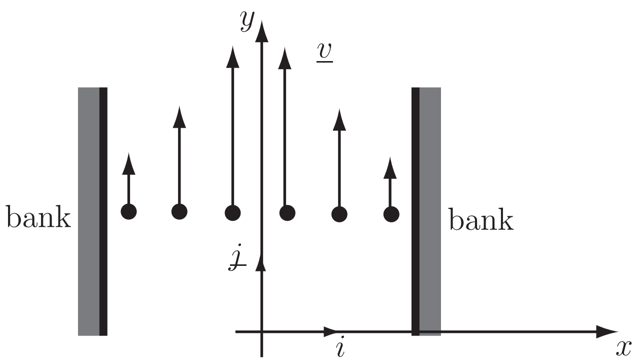


Figure 10: Flow in a straight river

(c) In a more realistic model $\underline{v}$ would vary as we move downstream and would be different at different depths due to, for example, rocks or bends. The velocity at any point could also depend on when the observation was made (for example the speed would be higher shortly after heavy rain) and so in general the velocity would be a function of the four variables x, y, z and t, and be of the form $\underline{v} = f_1(x, y, z, t)\underline{i} + f_2(x, y, z, t)\underline{j} + f_3(x, y, z, t)\underline{k}$, for suitable functions f_1, f_2 and f_3.

Example 7

Sketch sample vectors at the points $(3, 2)$, $(-2, 2)$, $(-3, -1)$, $(1, -4)$ for the two-dimensional vector field defined by $\underline{v} = x\underline{i} + 2\underline{j}$.

Solution

At $(3, 2)$, $\underline{v} = 3\underline{i} + 2\underline{j}$
At $(-2, 2)$, $\underline{v} = -2\underline{i} + 2\underline{j}$
At $(-3, -1)$, $\underline{v} = -3\underline{i} + 2\underline{j}$
At $(1, -4)$, $\underline{v} = \underline{i} + 2\underline{j}$
Plotting these vectors $\underline{v}$ gives the arrows in Figure 11.

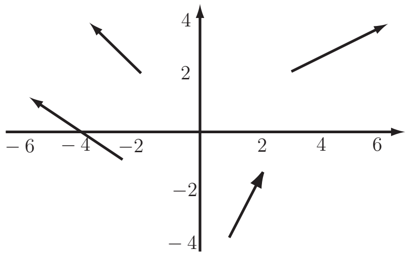


Figure 11: Sample vectors for the vector field $\underline{v} = x\underline{i} + 2\underline{j}$

It is possible to construct curves which start from and are in the same direction as any one vector and are guided by the direction of successive vectors. Starting at different points gives a set of non-intersecting lines called, depending on the context, vector field lines, lines of flow, streamlines or lines of force.

For example, consider the vector field $\underline{F} = -y\underline{i} + x\underline{j}$; $\underline{F}$ can be calculated at various points in the xy plane. Some of the individual vectors can be seen in Figure 12(a) while Figure 12(b) shows them converted seamlessly to field lines. For this function $\underline{F}$ the field lines are circles centered at the origin.

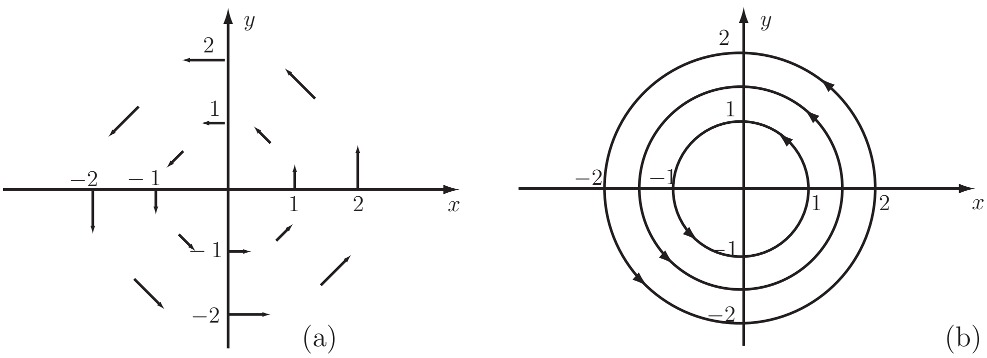


Figure 12: (a) Vectors at various points (b) Converted to field lines

Example 8

The Earth is affected by the gravitational force field of the Sun. This vector field is such that each vector $\underline{F}$ is directed towards the Sun and has magnitude proportional to $\dfrac{1}{r^2}$, where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the Sun to the Earth. Derive an equation for $\underline{F}$ and sketch some field lines.

Solution

The field has magnitude proportional to $r^{-2} = (x^2 + y^2 + z^2)^{-1}$ and points directly towards the Sun (the origin) i.e. parallel to a unit vector pointing towards the origin. At the point given by $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, a unit vector pointing towards the origin is $\dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{|-x\underline{i} - y\underline{j} - z\underline{k}|} = \dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{\sqrt{x^2 + y^2 + z^2}}$.
Multiplying the unit vector by the required magnitude $r^{-2} = (x^2 + y^2 + z^2)^{-1}$ (and by a constant of proportionality c) gives $\underline{F} = c\dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{(x^2 + y^2 + z^2)^{3/2}}$. Figure 13 shows some field lines for $\underline{F}$.

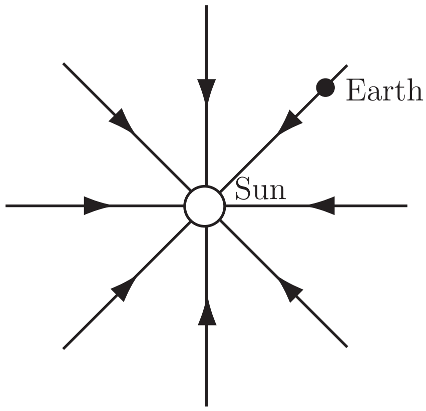


Figure 13: Gravitational field of the Sun

Key Point 2

A vector field $\underline{F}(x, y, z)$ (in three-dimensional coordinates) returns a vector $\underline{F}_0 = \underline{F}(x_0, y_0, z_0)$ for every point (x_0, y_0, z_0) in the domain of the field.

Exercises

1. Which of the following are scalar fields and which are vector fields?

   (a) $F = x^2 - yz$

   (b) $G = \dfrac{2x - z}{\sqrt{x^2 + y^2 + z^2 + 1}}$

   (c) $\underline{f} = x\underline{i} + y\underline{j} + z\underline{k}$

   (d) $H = \dfrac{y-1}{z^2+1}x + \dfrac{z-1}{x^2+1}y + \dfrac{x-1}{y^2+1}z$

   (e) $\underline{g} = (y+z)\underline{i}$

2. Draw vector diagrams for the vector fields

   (a) $\underline{f} = \underline{i} + 2\underline{j}$

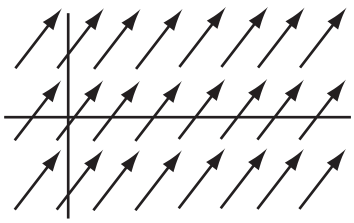
   (b) $\underline{g} = \underline{i} + y^2\underline{j}$

Answers

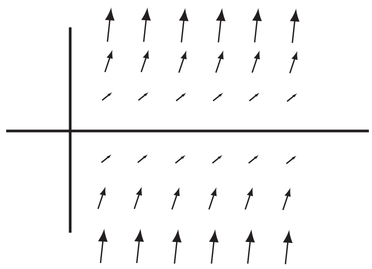
1. (a), (b) and (d) are scalar fields as the quantities defined are scalars.

   (c) and (e) are vector fields as the quantities defined are vectors.

2.



(a) The vectors point in the same direction everywhere



(b) As $|y|$ increases, the y-component increases