Changing Coordinates 27.4

Introduction

We have seen how changing the variable of integration of a single integral or changing the coordinate system for multiple integrals can make integrals easier to evaluate. In this Section we introduce the Jacobian. The Jacobian gives a general method for transforming the coordinates of any multiple integral.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the various techniques of integration
- be familiar with the concept of a function of several variables
- be able to evaluate the determinant of a matrix

Learning Outcomes

On completion you should be able to ...

- decide which coordinate transformation simplifies an integral
- determine the Jacobian for a coordinate transformation
- evaluate multiple integrals using a transformation of coordinates

1. Changing variables in multiple integrals

When the method of substitution is used to solve an integral of the form $\displaystyle\int_a^b f(x)\,dx$ three parts of the integral are changed, the limits, the function and the infinitesimal dx. So if the substitution is of the form $x = x(u)$ the u limits, c and d, are found by solving $a = x(c)$ and $b = x(d)$ and the function is expressed in terms of u as $f(x(u))$.

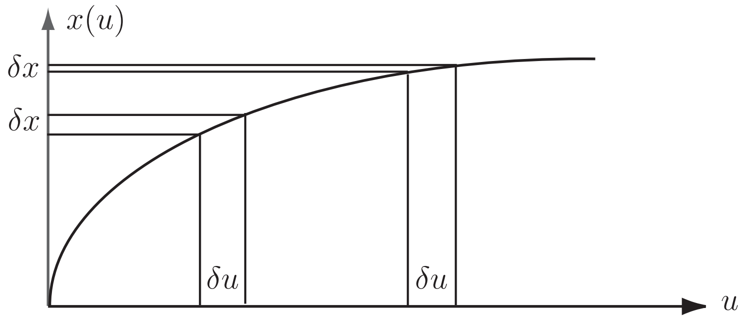


Figure 28

Figure 28 shows why the dx needs to be changed. While the δu is the same length for all u, the δx change as u changes. The rate at which they change is precisely $\dfrac{d}{du}x(u)$. This gives the relation

$$\delta x = \frac{dx}{du}\delta u$$

Hence the transformed integral can be written as

$$\int_a^b f(x)\,dx = \int_c^d f\left(x\left(u\right)\right)\frac{dx}{du}du$$

Here the $\dfrac{dx}{du}$ is playing the part of the Jacobian that we will define.

Another change of coordinates that you have seen is the transformations from cartesian coordinates (x, y) to polar coordinates (r, θ).
Recall that a double integral in polar coordinates is expressed as

$$\int\!\!\int f(x, y)\,dxdy = \int\!\!\int g(r, \theta)\,rdrd\theta$$

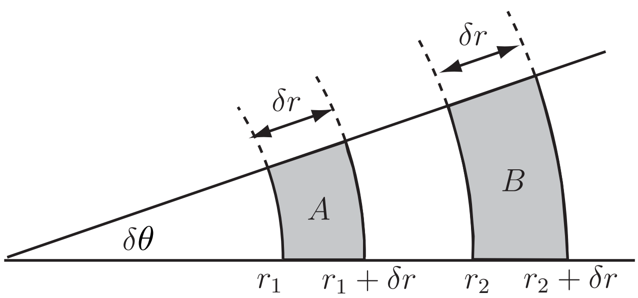


Figure 29

We can see from Figure 29 that the area elements change in size as r increases. The circumference of a circle of radius r is $2\pi r$, so the length of an arc spanned by an angle θ is $2\pi r\dfrac{\theta}{2\pi} = r\theta$. Hence the area elements in polar coordinates are approximated by rectangles of width δr and length $r\delta\theta$. Thus under the transformation from cartesian to polar coordinates we have the relation

$$\delta x \delta y \to r\delta r \delta\theta$$

that is, $r\delta r\delta\theta$ plays the same role as $\delta x\delta y$. This is why the r term appears in the integrand. Here r is playing the part of the Jacobian.

2. The Jacobian

Given an integral of the form $\displaystyle\int\!\!\int_A f(x,y)\,dxdy$

Assume we have a change of variables of the form $x = x(u,v)$ and $y = y(u,v)$ then the Jacobian of the transformation is defined as

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Key Point 10

Jacobian in Two Variables

For given transformations $x = x(u,v)$ and $y = y(u,v)$ the Jacobian is

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Notice the pattern occurring in the x, y, u and v. Across a row of the determinant the numerators are the same and down a column the denominators are the same.

Notation

Different textbooks use different notation for the Jacobian. The following are equivalent.

$$J(u,v) = J(x,y;u,v) = J\left(\frac{x,y}{u,v}\right) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

The Jacobian correctly describes how area elements change under such a transformation. The required relationship is

$$dxdy \to |J(u,v)|\ dudv$$

that is, $|J(u,v)|\ dudv$ plays the role of $dxdy$.

Key Point 11

Jacobian for Transforming Areas

When transforming area elements employing the Jacobian it is the **modulus** of the Jacobian that must be used.

Example 24

Find the area of the circle of radius R.

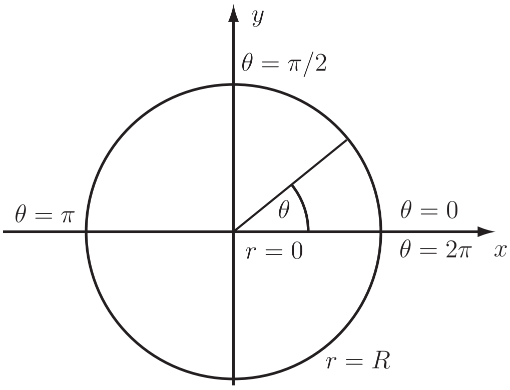


Figure 30

Solution

Let A be the region bounded by a circle of radius R centred at the origin. Then the area of this region is $\int_A dA$. We will calculate this area by changing to polar coordinates, so consider the usual transformation $x = r\cos\theta, y = r\sin\theta$ from cartesian to polar coordinates. First we require all the partial derivatives

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Thus

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \times r\cos\theta - (-r\sin\theta)\times\sin\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Solution (contd.)

This confirms the previous result for polar coordinates, $dxdy \to rdrd\theta$. The limits on r are $r = 0$ (centre) to $r = R$ (edge). The limits on θ are $\theta = 0$ to $\theta = 2\pi$, i.e. starting to the right and going once round anticlockwise. The required area is

$$\int_A dA = \int_0^{2\pi}\int_0^R |J(r,\theta)|\,drd\theta = \int_0^{2\pi}\int_0^R r\,drd\theta = 2\pi\frac{R^2}{2} = \pi R^2$$

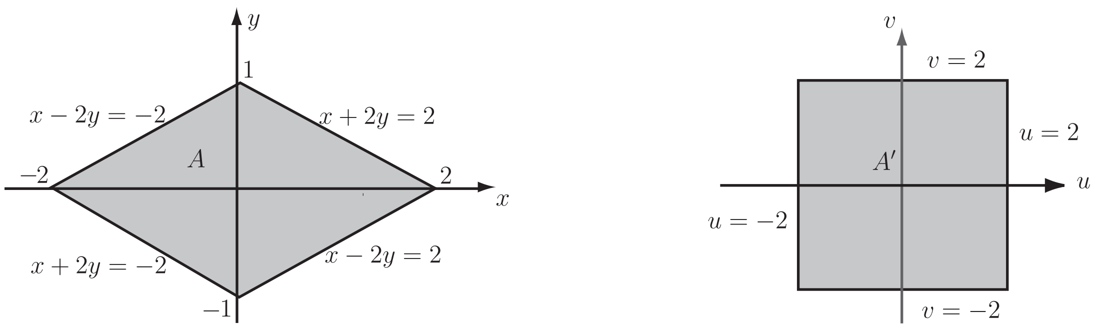
Note that here $r > 0$ so $|J(r,\theta)| = J(r,\theta) = r$.

Example 25

The diamond shaped region A in Figure 31(a) is bounded by the lines $x + 2y = 2$, $x - 2y = 2$, $x + 2y = -2$ and $x - 2y = -2$. We wish to evaluate the integral

$$I = \iint_A (3x + 6y)^2\,dA$$

over this region. Since the region A is neither vertically nor horizontally simple, evaluating I without changing coordinates would require separating the region into two simple triangular regions. So we use a change of coordinates to transform A to a square region in Figure 31(b) and evaluate I.



(a) (b)

Figure 31

Solution

By considering the equations of the boundary lines of region A it is easy to see that the change of coordinates

$$du = x + 2y \qquad (1) \qquad\qquad v = x - 2y \qquad (2)$$

will transform the boundary lines to $u = 2$, $u = -2$, $v = 2$ and $v = -2$. These values of u and v are the new limits of integration. The region A will be transformed to the square region A' shown above.

We require the inverse transformations so that we can substitute for x and y in terms of u and v. By adding (1) and (2) we obtain $u + v = 2x$ and by subtracting (1) and (2) we obtain $u - v = 4y$, thus the required change of coordinates is

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{4}(u-v)$$

Substituting for x and y in the integrand $(3x + 6y)^2$ of I gives

$$\left(\tfrac{3}{2}(u+v) + \tfrac{6}{4}(u-v)\right)^2 = 9u^2$$

We have the new limits of integration and the new form of the integrand, we now require the Jacobian. The required partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial x}{\partial v} = \frac{1}{2} \qquad \frac{\partial y}{\partial u} = \frac{1}{4} \qquad \frac{\partial y}{\partial v} = -\frac{1}{4}$$

Then the Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = -\frac{1}{4}$$

Then $dA' = |J(u,v)|dA = \frac{1}{4}dA$. Using the new limits, integrand and the Jacobian, the integral can be written

$$I = \int_{-2}^{2}\int_{-2}^{2} \frac{9}{4}u^2\, dudv.$$

You should evaluate this integral and check that $I = 48$.

Task This Task concerns using a transformation to evaluate $\iint (x^2 + y^2)\, dxdy$.

(a) Given the transformations $u = x + y$, $v = x - y$ express x and y in terms of u and v to find the inverse transformations:

Your solution

Answer

$$u = x + y \qquad (1)$$

$$v = x - y \qquad (2)$$

Add equations (1) and (2) $\quad u + v = 2x$

Subtract equation (2) from equation (1) $\quad u - v = 2y$

So $\quad x = \frac{1}{2}(u + v) \qquad\qquad y = \frac{1}{2}(u - v)$

(b) Find the Jacobian $J(u, v)$ for the transformation in part (a):

Your solution

Answer

Evaluating the partial derivatives, $\dfrac{\partial x}{\partial u} = \dfrac{1}{2}$, $\dfrac{\partial x}{\partial v} = \dfrac{1}{2}$, $\dfrac{\partial y}{\partial u} = \dfrac{1}{2}$ and $\dfrac{\partial y}{\partial v} = -\dfrac{1}{2}$ so the Jacobian

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

(c) Express the integral $I = \iint (x^2 + y^2)\ dxdy$ in terms of u and v, using the transformations introduced in (a) and the Jacobian found in (b):

Your solution

Answer

On letting $x = \dfrac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$ and $dxdy = |J|\ dudv = \frac{1}{2}\ dudv$, the integral $\iint (x^2+y^2)\ dxdy$ becomes

$$\begin{aligned} I &= \iint \left(\frac{1}{4}(u+v)^2 + \frac{1}{4}(u-v)^2\right) \times \frac{1}{2}\, dudv \\ &= \iint \frac{1}{2}\left(u^2+v^2\right) \times \frac{1}{2}\, dudv \\ &= \iint \frac{1}{4}\left(u^2+v^2\right)\ dudv \end{aligned}$$

(d) Find the limits on u and v for the rectangle with vertices $(x,y) = (0,0),\ (2,2),\ (-1,5),\ (-3,3)$:

Your solution

Answer

For $(0,0)$, $u = 0$ and $v = 0$
For $(2,2)$, $u = 4$ and $v = 0$
For $(-1,5)$, $u = 4$ and $v = -6$
For $(-3,3)$, $u = 0$ and $v = -6$
Thus, the limits on u are $u = 0$ to $u = 4$ while the limits on v are $v = -6$ to $v = 0$.

(e) Finally evaluate I:

Your solution

Answer
The integral is

$$\begin{aligned} I &= \int_{v=-6}^{0}\int_{u=0}^{4} \frac{1}{4}\left(u^2+v^2\right)\ dudv \\ &= \frac{1}{4}\int_{v=-6}^{0}\left[\frac{1}{3}u^3+uv^2\right]_{u=0}^{4}\ dudv = \int_{v=-6}^{0}\left[\frac{16}{3}+v^2\right]\ dv \\ &= \left[\frac{16}{3}v+\frac{1}{3}v^3\right]_{-6}^{0} = 0-\left[\frac{16}{3}\times(-6)+\frac{1}{3}\times(-216)\right] = 104 \end{aligned}$$

3. The Jacobian in 3 dimensions

When changing the coordinate system of a triple integral

$$I = \iiint_V f(x,y,z)\ dV$$

we need to extend the above definition of the Jacobian to 3 dimensions.

Key Point 12

Jacobian in Three Variables

For given transformations $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ the Jacobian is

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

The same pattern persists as in the 2-dimensional case (see Key Point 10). Across a row of the determinant the numerators are the same and down a column the denominators are the same.

The volume element $dV = dxdydz$ becomes $dV = |J(u,v,w)|\, dudvdw$. As before the limits and integrand must also be transformed.

Example 26

Use spherical coordinates to find the volume of a sphere of radius R.

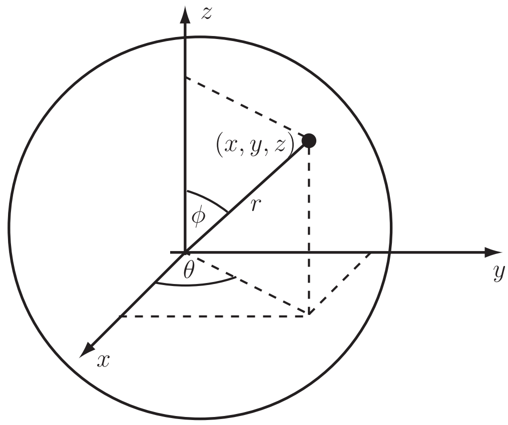


Figure 32

Solution

The change of coordinates from Cartesian to spherical polar coordinates is given by the transformation equations

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi$$

We now need the nine partial derivatives

$$\frac{\partial x}{\partial r} = \cos\theta\sin\phi \quad \frac{\partial x}{\partial \theta} = -r\sin\theta\sin\phi \quad \frac{\partial x}{\partial \phi} = r\cos\theta\cos\phi$$
$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$
$$\frac{\partial z}{\partial r} = \cos\phi \quad \frac{\partial z}{\partial \theta} = 0 \quad \frac{\partial z}{\partial \phi} = r\sin\phi$$

Hence we have

$$J(r,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$J(r,\theta,\phi) = \cos\phi \begin{vmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + 0 - r\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi \end{vmatrix}$$

Check that this gives $J(r,\theta,\phi) = -r^2\sin\phi$. Notice that $J(r,\theta,\phi) \leq 0$ for $0 \leq \phi \leq \pi$, so $|J(r,\theta,\phi)| = r^2\sin\phi$. The limits are found as follows. The variable ϕ is related to 'latitude' with $\phi = 0$ representing the 'North Pole' with $\phi = \pi/2$ representing the equator and $\phi = \pi$ representing the 'South Pole'.

Solution (contd.)

The variable θ is related to 'longitude' with values of 0 to 2π covering every point for each value of ϕ. Thus limits on ϕ are 0 to π and limits on θ are 0 to 2π. The limits on r are $r = 0$ (centre) to $r = R$ (surface).

To find the volume of the sphere we then integrate the volume element $dV = r^2 \sin\phi \; drd\theta d\phi$ between these limits.

$$\begin{aligned}\text{Volume} = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\phi \; drd\theta d\phi &= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}R^3 \sin\phi \; d\theta d\phi \\ &= \int_0^{\pi} \frac{2\pi}{3}R^3 \sin\phi \; d\phi = \frac{4}{3}\pi R^3\end{aligned}$$

Example 27

Find the volume integral of the function $f(x, y, z) = x - y$ over the parallelepiped with the vertices of the base at

$$(x, y, z) = (0, 0, 0), (2, 0, 0), (3, 1, 0) \text{ and } (1, 1, 0)$$

and the vertices of the upper face at

$$(x, y, z) = (0, 1, 2),\ (2, 1, 2),\ (3, 2, 2) \text{ and } (1, 2, 2).$$

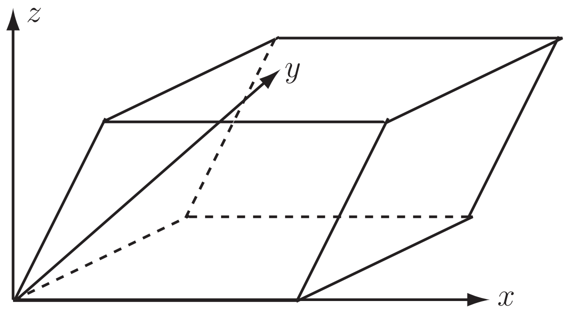


Figure 33

Solution

This will be a difficult integral to derive limits for in terms of x, y and z. However, it can be noted that the base is described by $z = 0$ while the upper face is described by $z = 2$. Similarly, the front face is described by $2y - z = 0$ with the back face being described by $2y - z = 2$. Finally the left face satisfies $2x - 2y + z = 0$ while the right face satisfies $2x - 2y + z = 4$.

The above suggests a change of variable with the new variables satisfying $u = 2x-2y+z$, $v = 2y-z$ and $w = z$ and the limits on u being 0 to 4, the limits on v being 0 to 2 and the limits on w being 0 to 2.

Inverting the relationship between u, v, w and x, y and z, gives

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(v+w) \qquad z = w$$

The Jacobian is given by

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 \\[2ex] 0 & \dfrac{1}{2} & \dfrac{1}{2} \\[2ex] 0 & 0 & 1 \end{vmatrix} = \frac{1}{4}$$

Note that the function $f(x,y,z) = x - y$ equals $\frac{1}{2}(u+v) - \frac{1}{2}(v+w) = \frac{1}{2}(u-w)$. Thus the integral is

$$\begin{aligned}
\int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{2}(u-w)\frac{1}{4}\,dudvdw &= \int_{w=0}^{2}\int_{v=0}^{2}\int_{u=0}^{4} \frac{1}{8}(u-w)\,dudvdw \\
&= \int_{w=0}^{2}\int_{v=0}^{2} \left[\frac{1}{16}u^2 - \frac{1}{8}uw\right]_0^4 dvdw \\
&= \int_{w=0}^{2}\int_{v=0}^{2} \left(1 - \frac{1}{2}w\right) dvdw \\
&= \int_{w=0}^{2} \left[v - \frac{vw}{2}\right]_0^2 dw \\
&= \int_{w=0}^{2} (2-w)\,dw \\
&= \left[2w - \frac{1}{2}w^2\right]_0^2 \\
&= 4 - \frac{4}{2} - 0 \\
&= 2
\end{aligned}$$

Task

Find the Jacobian for the following transformation:

$$x = 2u + 3v - w, \; y = v - 5w, \; z = u + 4w$$

Your solution

Answer

Evaluating the partial derivatives,

$$\frac{\partial x}{\partial u} = 2, \qquad \frac{\partial x}{\partial v} = 3, \qquad \frac{\partial x}{\partial w} = -1,$$
$$\frac{\partial y}{\partial u} = 0, \qquad \frac{\partial y}{\partial v} = 1, \qquad \frac{\partial y}{\partial w} = -5,$$
$$\frac{\partial z}{\partial u} = 1, \qquad \frac{\partial z}{\partial v} = 0, \qquad \frac{\partial z}{\partial w} = 4$$

so the Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & -5 \\ 1 & 0 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -5 \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 1 & -5 \end{vmatrix} = 2 \times 4 + 1 \times (-14) = -6$$

where expansion of the determinant has taken place down the first column.

Engineering Example 3

Volume of liquid in an ellipsoidal tank

Introduction

An ellipsoidal tank (elliptical when viewed from along x-, y- or z-axes) has a volume of liquid poured into it. It is useful to know in advance how deep the liquid will be. In order to make this calculation, it is necessary to perform a multiple integration and calculate a Jacobian.

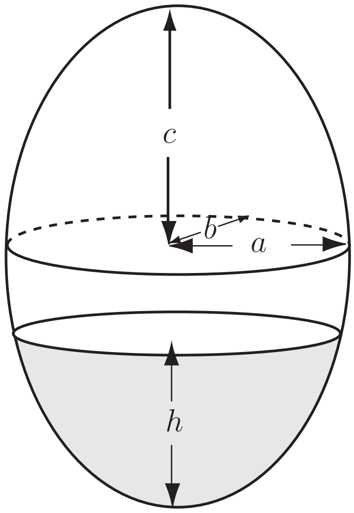


Figure 34

Problem in words

The metal tank is in the form of an ellipsoid, with semi-axes a, b and c. A volume V of liquid is poured into the tank ($V < \frac{4}{3}\pi abc$, the volume of the ellipsoid) and the problem is to calculate the depth, h, of the liquid.

Mathematical statement of problem

The shaded area is expressed as the triple integral

$$V = \int_{z=0}^{h}\int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} dxdydz$$

where limits of integration

$$x_1 = -a\sqrt{1 - \frac{y^2}{b^2} - \frac{(z-c)^2}{c^2}} \quad \text{and} \quad x_2 = +a\sqrt{1 - \frac{y^2}{b^2} - \frac{(z-c)^2}{c^2}}$$

which come from rearranging the equation of the ellipsoid $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{(z-c)^2}{c^2} = 1\right)$ and limits

$$y_1 = -\frac{b}{c}\sqrt{c^2 - (z-c)^2} \quad \text{and} \quad y_2 = +\frac{b}{c}\sqrt{c^2 - (z-c)^2}$$

from the equation of an ellipse in the y-z plane $\left(\dfrac{y^2}{b^2} + \dfrac{(z-c)^2}{c^2} = 1\right)$.

Mathematical analysis

To calculate V, use the substitutions

$$\begin{aligned} x &= a\tau\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \\ y &= b\tau\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}} \\ z &= z \end{aligned}$$

now expressing the triple integral as

$$V=\int_{z=0}^{h}\int_{\phi=\phi_1}^{\phi_2}\int_{\tau=\tau_1}^{\tau_2} J\ d\tau d\phi dz$$

where J is the Jacobian of the transformation calculated from

$$J=\begin{vmatrix} \dfrac{\partial x}{\partial \tau} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \tau} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \tau} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix}$$

and reduces to

$$\begin{aligned} J &= \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \phi}-\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \tau} \qquad \text{since } \frac{\partial z}{\partial \tau}=\frac{\partial z}{\partial \phi}=0 \\ &= \left\{a\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\ b\tau\cos\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\right\} \\ &\quad - \left\{-a\tau\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\ b\sin\phi\left(1-\frac{(z-c)^2}{c^2}\right)^{\frac{1}{2}}\right\} \\ &= ab\tau\left(\cos^2\phi+\sin^2\phi\right)\left(1-\frac{(z-c)^2}{c^2}\right) \\ &= ab\tau\left(1-\frac{(z-c)^2}{c^2}\right) \end{aligned}$$

To determine limits of integration for ϕ, note that the substitutions above are similar to a cylindrical polar co-ordinate system, and so ϕ goes from 0 to 2π. For τ, setting $\tau=0 \Rightarrow x=0$ and $y=0$, i.e. the z-axis.

Setting $\tau=1$ gives

$$\frac{x^2}{a^2}=\cos^2\phi\left(1-\frac{(z-c)^2}{c^2}\right) \qquad (1)$$

and

$$\frac{y^2}{b^2}=\sin^2\phi\left(1-\frac{(z-c)^2}{c^2}\right) \qquad (2)$$

Summing both sides of Equations (1) and (2) gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (\cos^2\phi + \sin^2\phi)\left(1 - \frac{(z-c)^2}{c^2}\right)$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1$$

which is the equation of the ellipsoid, i.e. the outer edge of the volume. Therefore the range of τ should be 0 to 1. Now

$$\begin{aligned}
V &= ab\int_{z=0}^{h}\left(1 - \frac{(z-c)^2}{c^2}\right)\int_{\phi=0}^{2\pi}\int_{\tau=0}^{1}\tau\ d\tau d\phi dz \\
&= \frac{ab}{c^2}\int_{z=0}^{h}(2zc - z^2)\int_{\phi=0}^{2\pi}\left[\frac{\tau^2}{2}\right]_{\tau=0}^{1}\ d\phi dz \\
&= \frac{ab}{2c^2}\int_{z=0}^{h}(2zc - z^2)\left[\phi\right]_{\phi=0}^{2\pi}\ dz \\
&= \frac{\pi ab}{c^2}\left[cz^2 - \frac{z^3}{3}\right]_{z=0}^{h} \\
&= \frac{\pi ab}{c^2}\left(ch^2 - \frac{h^3}{3}\right)
\end{aligned}$$

Interpretation

Suppose the tank has actual dimensions of $a = 2$ m, $b = 0.5$ m and $c = 3$ m and a volume of 7 m^3 is to be poured into it. (The total volume of the tank is $4\pi\ m^3 \approx$ 12.57 m^3). Then, from above

$$V = \frac{\pi ab}{c^2}\left(ch^2 - \frac{h^3}{3}\right)$$

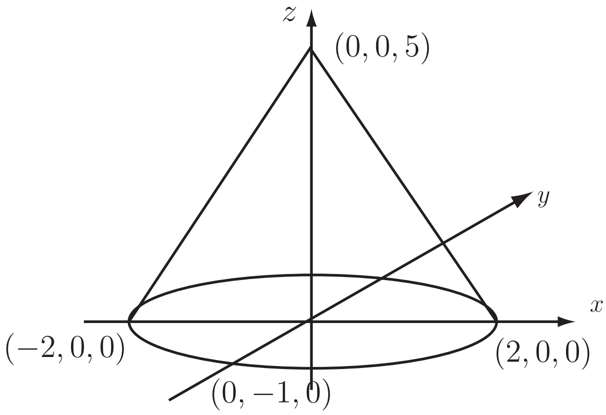
which becomes

$$7 = \frac{\pi}{9}\left(3h^2 - \frac{h^3}{3}\right)$$

with solution $h =$ 3.23 m (2 d.p.), compared to the maximum height of the ellipsoid of 6 m.

Exercises

1. The function $f = x^2 + y^2$ is to be integrated over an elliptical cone with base being the ellipse, $x^2/4 + y^2 = 1$, $z = 0$ and apex (point) at $(0,0,5)$. The integral can be made simpler by means of the change of variables $x = 2(1 - \dfrac{w}{5})\tau\cos\theta$, $y = (1 - \dfrac{w}{5})\tau\sin\theta$, $z = w$.



   (a) Find the limits on the variables τ, θ and w.

   (b) Find the Jacobian $J(\tau, \theta, w)$ for this transformation.

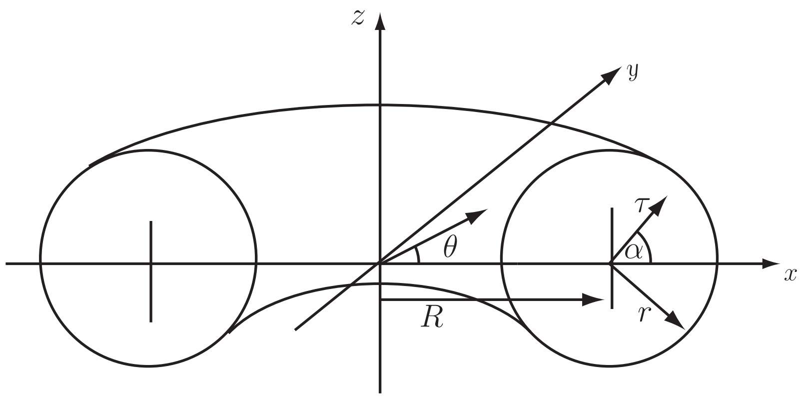
   (c) Express the integral $\displaystyle\iiint (x^2 + y^2)\, dxdydz$ in terms of τ, θ and w.

   (d) Evaluate this integral. [Hint:- it may be worth noting that $\cos^2\theta \equiv \frac{1}{2}(1 + \cos 2\theta)$].

   Note: This integral has relevance in topics such as moments of inertia.

2. Using cylindrical polar coordinates, integrate the function $f = z\sqrt{x^2 + y^2}$ over the volume between the surfaces $z = 0$ and $z = 1 + x^2 + y^2$ for $0 \leq x^2 + y^2 \leq 1$.

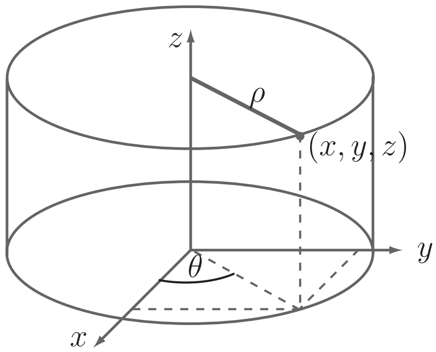
3. A torus (doughnut) has major radius R and minor radius r. Using the transformation $x = (R + \tau\cos\alpha)\cos\theta$, $y = (R + \tau\cos\alpha)\sin\theta$, $z = \tau\sin\alpha$, find the volume of the torus. [Hints:- limits on α and θ are 0 to 2π, limits on τ are 0 to r. Show that Jacobian is $\tau(R + \tau\cos\alpha)$].



4. Find the Jacobian for the following transformations.

   (a) $x = u^2 + vw$, $y = 2v + u^2w$, $z = uvw$

   (b) Cylindrical polar coordinates. $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$



Answers

1. (a) $\tau : 0$ to 1, $\theta : 0$ to 2π, $w : 0$ to 5

   (b) $2(1 - \dfrac{w}{5})^2\tau$

   (c) $2\displaystyle\int_{\tau=0}^{1}\int_{\theta=0}^{2\pi}\int_{w=0}^{5}(1 - \frac{w}{5})^4\tau^3(4\cos^2\theta + \sin^2\theta)\, dwd\theta d\tau$

   (d) $\dfrac{5}{2}\pi$

2. $\dfrac{92}{105}\pi$

3. $2\pi^2Rr^2$

4. (a) $4u^2v - 2u^4w + u^2vw^2 - 2v^2w$, (b) ρ