Cauchy's Theorem

26.5

Introduction

In this Section we introduce Cauchy's theorem which allows us to simplify the calculation of certain contour integrals. A second result, known as Cauchy's integral formula, allows us to evaluate some integrals of the form $\displaystyle\oint_C \frac{f(z)}{z-z_0}\,dz$ where z_0 lies inside C.

Prerequisites

Before starting this Section you should ...

- be familiar with the basic ideas of functions of a complex variable
- be familiar with line integrals

Learning Outcomes

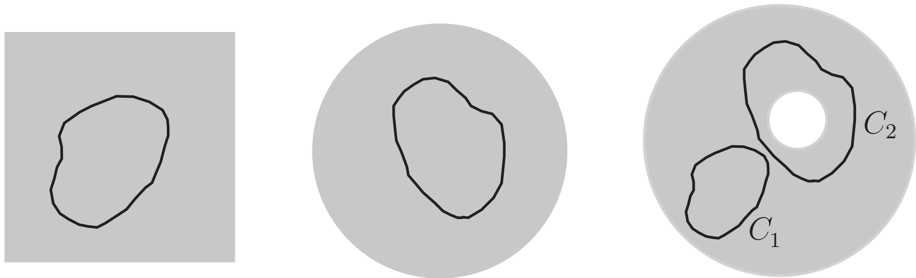
On completion you should be able to ...

- state and use Cauchy's theorem
- state and use Cauchy's integral formula

1. Cauchy's theorem

Simply-connected regions

A region is said to be simply-connected if any closed curve in that region can be shrunk to a point without any part of it leaving a region. The interior of a square or a circle are examples of simply connected regions. In Figure 11 (a) and (b) the shaded grey area is the region and a typical closed curve is shown inside the region. In Figure 11 (c) the region contains a hole (the white area inside). The shaded region between the two circles is **not** simply-connected; curve C_1 can shrink to a point but curve C_2 cannot shrink to a point without leaving the region, due to the hole inside it.



(a) (b) (c)

Figure 11

Key Point 2

Cauchy's Theorem

The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then:

$$\oint_C f(z)\,dz = 0$$

for every simple closed path C lying in the region.

This is perhaps the most important theorem in the area of complex analysis.

As a straightforward example note that $\oint_C z^2\,dz = 0$, where C is the unit circle, since z^2 is analytic everywhere (see Section 261). Indeed $\oint_C z^2\,dz = 0$ for *any* simple contour: it need not be circular.

Consider the contour shown in Figure 12 and assume $f(z)$ is analytic everywhere on and inside the contour C.

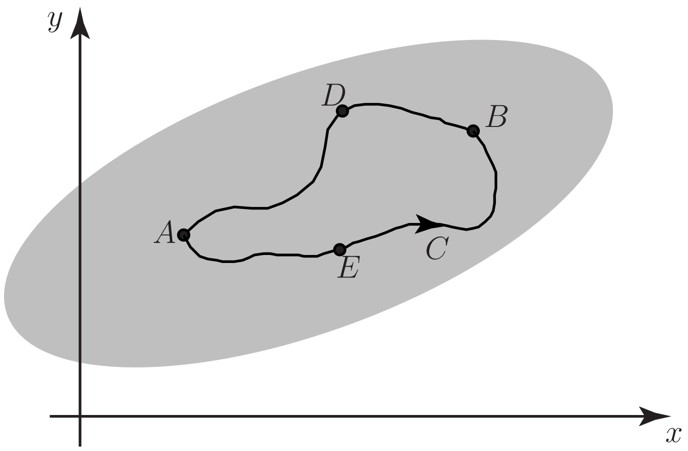


Figure 12

Then by analogy with real line integrals

$$\int_{AEB} f(z)\,dz + \int_{BDA} f(z)\,dz = \oint_C f(z)\,dz = 0 \qquad \text{by Cauchy's theorem.}$$

Therefore

$$\int_{AEB} f(z)\,dz = -\int_{BDA} f(z)\,dz = \int_{ADB} f(z)\,dz$$

(since reversing the direction of integration reverses the sign of the integral).

This implies that we may choose any path between A and B and the integral will have the same value **providing $f(z)$ is analytic in the region concerned.**

Integrals of analytic functions only depend on the positions of the points A and B, not on the path connecting them. This explains the 'coincidences' referred to previously in Section 26.4.

Task

Using 'simple' integration evaluate $\displaystyle\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\; dz$, and explain why this is valid.

Your solution

Answer

$$\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\; dz = \Big[\sin z\Big]_{\mathrm{i}}^{1+2\mathrm{i}} = \sin(1+2\mathrm{i}) - \sin\ \mathrm{i}.$$

This way of determining the integral is legitimate because $\cos z$ is analytic (everywhere).

We now investigate what occurs when the closed path of integration does not necessarily lie within a simply-connected region. Consider the situation described in Figure 13.

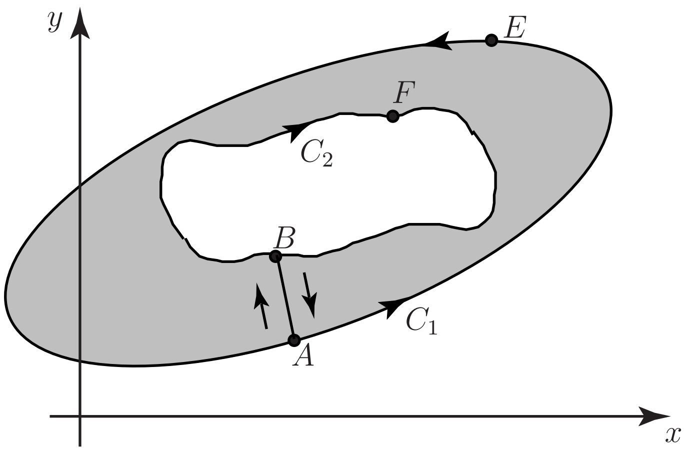


Figure 13

Let $f(z)$ be analytic in the region bounded by the closed curves C_1 and C_2. The region is cut by the line segment joining A and B.

Consider now the closed curve $AEABFBA$ travelling in the direction indicated by the arrows. No line can cross the cut AB and be regarded as remaining in the region. Because of the cut the shaded region is **simply connected**. Cauchy's theorem therefore applies (see Key Point 2).

Therefore

$$\oint_{AEABFBA} f(z)\,dz = 0 \quad \text{since } f(z) \text{ is analytic within and on the curve } AEABFBA.$$

Note that

$$\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz, \qquad \text{being a simple change of direction.}$$

Also, we can divide the closed curve into smaller sections:

$$\begin{aligned}\oint_{AEABFBA} f(z)\,dz &= \int_{AEA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BFB} f(z)\,dz + \int_{BA} f(z)\,dz \\ &= \int_{AEA} f(z)\,dz + \int_{BFB} f(z)\,dz = 0.\end{aligned}$$

i.e.

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0$$

(since we assume that closed paths are travelled anticlockwise).

Therefore $\displaystyle\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$

This allows us to evaluate $\displaystyle\oint_{C_1} f(z)\,dz$ by replacing C_1 by any curve C_2 such that the region between them contains no singularities (see Section 261) of $f(z)$. Often we choose a circle for C_2.

Example 12

Determine $\displaystyle\oint_C \frac{6}{z(z-3)}\,dz$ where C is the curve $|z-3| = 5$ shown in Figure 14.

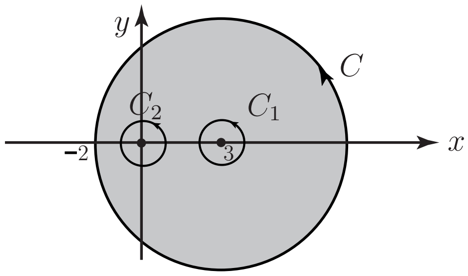


Figure 14

Solution

We observe that $f(z) = \dfrac{6}{z(z-3)}$ is analytic everywhere except at $z = 0$ and $z = 3$.

Let C_1 be the circle of unit radius centred at $z = 3$ and C_2 be the unit circle centered at the origin. By analogy with the previous example we state that

$$\oint_C \frac{6}{z(z-3)}\,dz = \oint_{C_1} \frac{6}{z(z-3)}\,dz + \oint_{C_2} \frac{6}{z(z-3)}\,dz.$$

(To show this you would need two cuts: from C to C_1 and from C to C_2.)

The remaining parts of this problem are presented as two Tasks.

Task

Expand $\dfrac{6}{z(z-3)}$ into partial functions.

Your solution

Answer

Let $\dfrac{6}{z(z-3)} \equiv \dfrac{A}{z} + \dfrac{B}{z-3} \equiv \dfrac{A(z-3)+Bz}{z(z-3)}$. Then $A(z-3)+Bz \equiv 6$.

If $z = 0$ $\quad A(-3) = 6 \quad \therefore \quad A = -2.$ $\quad$ If $z = 3 \quad B \times 3 = 6 \quad \therefore \quad B = 2.$

$$\therefore \quad \frac{6}{z(z-3)} \equiv -\frac{2}{z} + \frac{2}{z-3}.$$

Thus:

$$\oint_C \frac{6}{z(z-3)}\,dz = \oint_{C_1} \frac{2}{z-3}\,dz - \oint_{C_1} \frac{2}{z}\,dz + \oint_{C_2} \frac{2}{z-3}\,dz - \oint_{C_2} \frac{2}{z}\,dz = I_1 - I_2 + I_3 - I_4.$$

Task

Find the values of I_1, I_2, I_3, I_4, using Key Point 1 (page 35):

(a) Find the value of I_1:

Your solution

Answer

Using Key Point 1 we find that $I_1 = 2 \times 2\pi\mathrm{i} = 4\pi\mathrm{i}$.

(b) Find the value of I_2:

Your solution

Answer

The function $\dfrac{1}{z}$ is analytic inside and on C_1 so that $I_2 = 0$.

(c) Find the value of I_3:

Your solution

Answer

The function $\dfrac{1}{z-3}$ is analytic inside and on C_2 so $I_3 = 0$.

(d) Find the value of I_4:

Your solution

Answer

$I_4 = 4\pi\mathrm{i}$ again using Key Point 1.

(e) Finally, calculate $I = I_1 - I_2 + I_3 - I_4$:

Your solution

Answer
$\displaystyle\oint_C \frac{6\,dz}{z(z-3)} = 4\pi\mathsf{i} - 0 + 0 - 4\pi\mathsf{i} = 0.$

Exercises

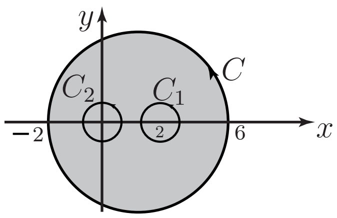
1. Evaluate $\displaystyle\int_{1+\mathsf{i}}^{2+3\mathsf{i}} \sin z\,dz.$

2. Determine $\displaystyle\oint_C \frac{4}{z(z-2)}\,dz$ where C is the contour $|z-2| = 4$.

Answers

1. $\displaystyle\int_{1+\mathsf{i}}^{2+3\mathsf{i}} \sin z\,dz = \Big[-\cos z \Big]_{1+\mathsf{i}}^{2+3\mathsf{i}} = \cos(1+\mathsf{i}) - \cos(2+3\mathsf{i})$ since $\sin z$ is analytic everywhere.

2.



$f(z) = \dfrac{4}{z(z-2)}$ is analytic everywhere except at $z = 0$ and $z = 2$.

Call $\displaystyle I = \oint_C \frac{4}{z(z-2)}\,dz = \oint_{C_1} \frac{4}{z(z-2)}\,dz + \oint_{C_2} \frac{4}{z(z-2)}\,dz.$

Now $\displaystyle\frac{4}{z(z-2)} \equiv -\frac{2}{z} + \frac{2}{z-2}$ so that

$$\begin{aligned} I &= \oint_{C_1} \frac{2}{z-2}\,dz - \oint_{C_1} \frac{2}{z}\,dz + \oint_{C_2} \frac{2}{z-2}\,dz - \oint_{C_2} \frac{2}{z}\,dz \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

I_2 and I_3 are zero because of analyticity.

$I_1 = 2 \times 2\pi\mathsf{i} = 4\pi\mathsf{i}$, by Key Point 1 and $I_4 = -4\pi\mathsf{i}$ likewise.

Hence $I = 4\pi\mathsf{i} + 0 + 0 - 4\pi\mathsf{i} = 0.$

2. Cauchy's integral formula

This is a generalization of the result in Key Point 2:

Key Point 3

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi\mathrm{i}\ f(z_0).$$

Example 13

Evaluate $\displaystyle\oint_C \frac{z}{z^2+1}\,dz$ where C is the path shown in Figure 15:

$C_1 : \quad |z-\mathrm{i}| = \frac{1}{2}$

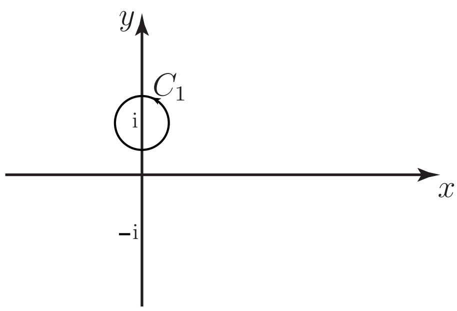


Figure 15

Solution

We note that $z^2+1 \equiv (z+\mathrm{i})(z-\mathrm{i})$.

Let $\displaystyle\frac{z}{z^2+1} = \frac{z}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{z/(z+\mathrm{i})}{z-\mathrm{i}}$.

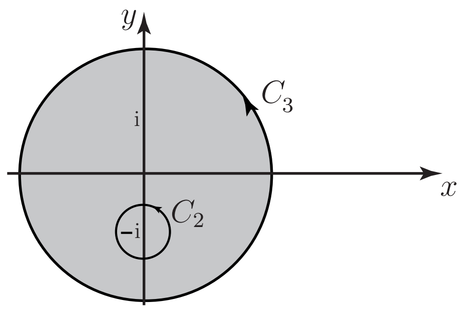
The numerator $z/(z+\mathrm{i})$ is analytic inside and on the path C_1 so putting $z_0 = \mathrm{i}$ in the Cauchy integral formula (Key Point 3)

$$\oint_{C_1} \frac{z}{z^2+1}\,dz = 2\pi\mathrm{i}\left[\frac{\mathrm{i}}{\mathrm{i}+\mathrm{i}}\right] = 2\pi\mathrm{i}.\frac{1}{2} = \pi\mathrm{i}.$$

Task

Evaluate $\oint_C \frac{z}{z^2+1}\,dz$ where C is the path (refer to the diagram)

(a) $C_2: \quad |z+\mathsf{i}| = \frac{1}{2}$ (b) $C_3: \quad |z| = 2.$



(a) Use the Cauchy integral formula to find an expression for $\oint_{C_2} \frac{z}{z^2+1}\,dz$:

Your solution

Answer

$\frac{z}{z^2+1} = \frac{z/(z-\mathsf{i})}{z+\mathsf{i}}$. The numerator is analytic inside and on the path C_2 so putting $z_0 = -\mathsf{i}$ in the Cauchy integral formula gives

$$\oint_{C_2} \frac{z}{z^2+1}\,dz = 2\pi\mathsf{i}\left[\frac{-\mathsf{i}}{-2\mathsf{i}}\right] = \pi\mathsf{i}.$$

(b) Now find $\oint_{C_3} \frac{z}{z^2+1}\,dz$:

Your solution

Answer

By analogy with the previous part,

$$\oint_{C_3} \frac{z}{z^2+1}\,dz = \oint_{C_1} \frac{z}{z^2+1}\,dz + \oint_{C_2} \frac{z}{z^2+1}\,dz = \pi\mathsf{i} + \pi\mathsf{i} = 2\pi\mathsf{i}.$$

The derivative of an analytic function

If $f(z)$ is analytic in a simply-connected region then at any interior point of the region, z_0 say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathsf{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

where C is any simple closed curve, in the region, which encloses z_0.
Note the case $n = 1$:

$$f'(z_0) = \frac{1}{2\pi \mathsf{i}} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz.$$

Example 14

Evaluate the contour integral

$$\oint_C \frac{z^3}{(z-1)^2}\, dz$$

where C is a contour which encloses the point $z = 1$.

Solution

Since $f(z) = \dfrac{z^3}{(z-1)^2}$ has a pole of order 2 at $z = 1$ then $\displaystyle\oint_C f(z)\, dz = \oint_{C'} \frac{z^3}{(z-1)^2}\, dz$

where C' is a circle centered at $z = 1$.

If $g(z) = z^3$ then $\displaystyle\oint_C f(z)\, dz = \oint_{C'} \frac{g(z)}{(z-1)^2}\, dz$

Since $g(z)$ is analytic within and on the circle C' we use Cauchy's integral formula for derivatives to show that

$$\oint_C \frac{z^3}{(z-1)^2}\, dz = 2\pi \mathsf{i} \times \frac{1}{1!}\left[g'(z)\right]_{z=1} = 2\pi \mathsf{i}\left[3z^2\right]_{z=1} = 6\pi \mathsf{i}.$$

Exercise

Evaluate $\oint_C \frac{z}{z^2+9}\,dz$ where C is the path:

(a) $C_1: \quad |z-3\text{i}|=1$ (b) $C_2: \quad |z+3\text{i}|=1$ (c) $C_3: \quad |z|=6.$

Answers

(a) We will use the fact that $\frac{z}{z^2+9} = \frac{z}{(z+3\text{i})(z-3\text{i})} = \frac{z/(z+3\text{i})}{z-3\text{i}}$

The numerator $\frac{z}{z+3\text{i}}$ is analytic inside and on the path C_1 so putting $z_0 = 3\text{i}$ in Cauchy's integral formula

$$\oint_{C_1} \frac{z}{z^2+9}\,dz = 2\pi\text{i}\left[\frac{3\text{i}}{3\text{i}+3\text{i}}\right] = 2\pi\text{i}\times\frac{1}{2} = \pi\text{i}.$$

(b) Here $\frac{z/(z-3\text{i})}{z+3\text{i}}$

The numerator is analytic inside and on the path C_2 so putting $z=-3\text{i}$ in Cauchy's integral formula:

$$\oint_{C_2} \frac{z}{z^2+9}\,dz = 2\pi\text{i}\left[\frac{-3\text{i}}{-3\text{i}-3\text{i}}\right] = \pi\text{i}.$$

(c) The integral is the sum of the two previous integrals and has value $2\pi\text{i}$.