Basic Complex Integration

26.4

➡ Introduction

Complex variable techniques have been used in a wide variety of areas of engineering. This has been particularly true in areas such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity. With the rapid developments in computer technology and the consequential use of sophisticated algorithms for analysis and design in engineering there has been, in recent years, less emphasis on the use of complex variable techniques and a shift towards numerical techniques applied directly to the underlying full partial differential equations which model the situation. However it is useful to have an analytical solution, possibly for an idealized model in order to develop a better understanding of the solution and to develop confidence in numerical estimates for the solution of more sophisticated models.

The design of aerofoil sections for aircraft is an area where the theory was developed using complex variable techniques. Throughout engineering, transforms defined as complex integrals in one form or another play a major role in analysis and design. The use of complex variable techniques allows us to develop criteria for the stability of systems.

➡ **Prerequisites** Before starting this Section you should ...	• be able to carry out integration of simple real-valued functions • be familiar with the basic ideas of functions of a complex variable • be familiar with line integrals
➡ **Learning Outcomes** On completion you should be able to ...	• understand the concept of complex integrals

1. Complex integrals

If $f(z)$ is a single-valued, continuous function in some region R in the complex plane then we define the **integral** of $f(z)$ **along a path** C in R (see Figure 7) as
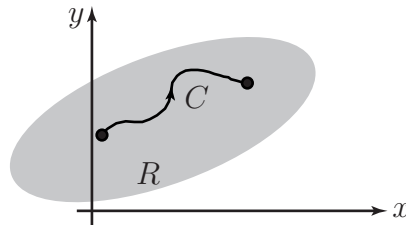
$$\int_C f(z)\, dz = \int_C (u + \mathrm{i}v)(dx + \mathrm{i}\, dy).$$

Figure 7

Here we have written $f(z)$ and dz in real and imaginary parts:

$$f(z) = u + \mathrm{i}v \quad \text{and} \quad dz = dx + \mathrm{i}\, dy.$$

Then we can separate the integral into real and imaginary parts as

$$\int_C f(z)\, dz = \int_C (u\, dx - v\, dy) + \mathrm{i} \int_C (v\, dx + u\, dy).$$

We often interpret real integrals in terms of area; now we define complex integrals in terms of line integrals over paths in the complex plane. The line integrals are evaluated as described in HELM 29.

Example 10

Obtain the complex integral:

$$\int_C z\, dz$$

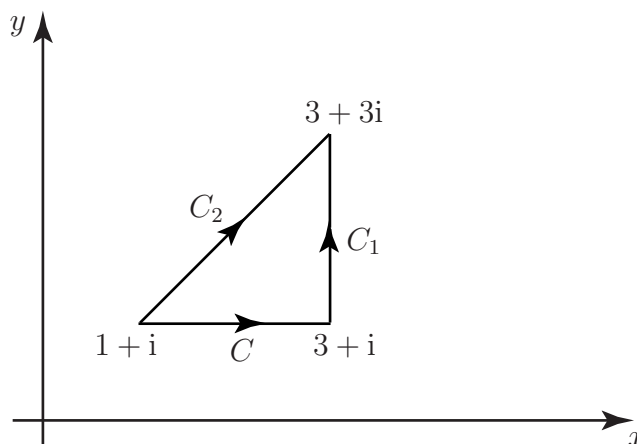where C is the straight line path from $z = 1 + \mathrm{i}$ to $z = 3 + \mathrm{i}$. See Figure 8.

Figure 8

Solution

Here, since y is constant $(y = 1)$ along the given path then $z = x + i$, implying that $u = x$ and $v = 1$. Also, as y is constant, $dy = 0$.

Therefore,

$$
\begin{aligned}
\int_C z \, dz &= \int_C (u \, dx - v \, dy) + i \int_C (v \, dx + u \, dy) \\
&= \int_1^3 x \, dx + i \int_1^3 1 \, dx \\
&= \left[\frac{x^2}{2} \right]_1^3 + i \left[x \right]_1^3 \\
&= \left(\frac{9}{2} - \frac{1}{2} \right) + i(3 - 1) = 4 + 2i.
\end{aligned}
$$

Task

Evaluate $\displaystyle\int_{C_1} z \, dz$ where C_1 is the straight line path from $z = 3 + i$ to $z = 3 + 3i$.

First obtain expressions for u, v, dx and dy by finding an appropriate expression for z along the path:

Your solution

Answer

Along the path $z = 3 + iy$, implying that $u = 3$ and $v = y$. Also $dz = 0 + idy$.

Now find limits on y:

Your solution

Answer

The limits on y are: $y = 1$ to $y = 3$.

Now evaluate the integral:

Your solution

Answer

$$\int_{C_1} z\,dz = \int_{C_1} (u\,dx - v\,dy) + i\int_{C_1} (v\,dx + u\,dy)$$

$$= \int_1^3 -y\,dy + i\int_1^3 3\,dy$$

$$= \left[\frac{-y^2}{2}\right]_1^3 + i\left[3y\right]_1^3 = \left(-\frac{9}{2} + \frac{1}{2}\right) + i(9 - 3)$$

$$= -4 + 6i.$$

Task

Evaluate $\int_{C_2} z\,dz$ where C_2 is the straight line path from $z = 1 + i$ to $z = 3 + 3i$.

Your solution

Answer

We first need to find the equation of the line C_2 in the Argand plane.

We note that both points lie on the line $y = x$ so the complex equation of the straight line is $z = x + ix$ giving $u = x$ and $v = x$.   Also $dz = dx + idx = (1 + i)dx$.

$$\therefore \quad \int_{C_2} z\,dz = \int_{C_2} (x\,dx - x\,dx) + i\int_{C_2} (x\,dx + x\,dx).$$

$$= i\int_{C_2} (2x\,dx)$$

Next, we see that the limits on x are $x = 1$ to $x = 3$. We are now in a position to evaluate the integral:

$$\int_{C_2} z\,dz = i\int_1^3 2x\,dx = i\left[x^2\right]_1^3 = i(9 - 1) = 8i.$$

Note that this result is the sum of the integrals along C and C_1. You might have expected this.

A more intricate example now follows.

Example 11

Evaluate $\int_{C_1} z^2 \, dz$ where C_1 is that part of the unit circle going anticlockwise from the point $z = 1$ to the point $z = i$. See Figure 9.
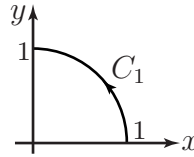
Figure 9

Solution

First, note that $z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$ and $dz = dx + i \, dy$ giving

$$\int_{C_1} z^2 \, dx = \int_{C_1} \left\{ (x^2 - y^2) \, dx - 2xy \, dy \right\} + i \int_{C_1} \left\{ 2xy \, dx + (x^2 - y^2) dy \right\}.$$

This is obtained by simply expressing the integral in real and imaginary parts. These integrals cannot be evaluated in this form since y and x are related. Instead we re-write them in terms of the single variable θ.

Note that on the unit circle: $\quad x = \cos\theta, \; y = \sin\theta \quad$ so that $\quad dx = -\sin\theta \, d\theta$ and $dy = \cos\theta \, d\theta$.

The expressions $(x^2 - y^2)$ and $2xy$ can be expressed in terms of 2θ since

$$x^2 - y^2 = \cos^2\theta - \sin^2\theta \equiv \cos 2\theta \qquad 2xy = 2\cos\theta\sin\theta \equiv \sin 2\theta.$$

Now as the point z moves from $z = 1$ to $z = i$ along the path C_1 the parameter θ changes from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$. Hence,

$$\int_{C_1} f(z) \, dz = \int_0^{\frac{\pi}{2}} \left\{ -\cos 2\theta \sin\theta \, d\theta - \sin 2\theta \cos\theta \, d\theta \right\} + i \int_0^{\frac{\pi}{2}} \left\{ -\sin 2\theta \sin\theta \, d\theta + \cos 2\theta \cos\theta \, d\theta \right\}.$$

We can simplify these daunting-looking integrals by using the trigonometric identities:

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B \qquad \text{and} \qquad \cos(A + B) \equiv \cos A \cos B - \sin A \sin B.$$

We obtain (choosing $A = 2\theta$ and $B = \theta$ in both expressions):

$$-\cos 2\theta \sin\theta - \sin 2\theta \cos\theta \equiv -(\sin\theta \cos 2\theta + \cos\theta \sin 2\theta) \equiv -\sin 3\theta.$$

Also $\quad -\sin 2\theta \sin\theta + \cos 2\theta \cos\theta \equiv \cos 3\theta.$

Now we can complete the evaluation of our integral:

$$\int_{C_1} f(z) \, dx = \int_0^{\frac{\pi}{2}} (-\sin 3\theta) d\theta + i \int_0^{\frac{\pi}{2}} \cos 3\theta \, d\theta$$

$$= \left[\frac{1}{3} \cos 3\theta \right]_0^{\frac{\pi}{2}} + i \left[\frac{1}{3} \sin 3\theta \right]_0^{\frac{\pi}{2}} = (0 - \frac{1}{3}) + i \left(-\frac{1}{3} - 0 \right) = -\frac{1}{3} - \frac{1}{3}i \equiv -\frac{1}{3}(1 + i).$$

In the last Task we integrated z^2 over a given path. We had to perform some intricate mathematics to get the value. It would be convenient if there was a simpler way to obtain the value of such complex integrals. This is explored in the following Tasks.

Task

Evaluate $\left[\dfrac{1}{3}z^3\right]_1^i$

Your solution

Answer

We obtain $-\dfrac{1}{3}(1+i)$ again, which is the same result as from the previous Task.

It would seem that, by carrying out an analogue of real integration (simply integrating the function and substituting in the limits) we can obtain the answer much more easily. Is this coincidence?

If you return to the first Task of this Section you will note:

$$
\begin{aligned}
\left[\frac{1}{2}z^2\right]_{1+i}^{3+3i} &= \frac{1}{2}\left\{(3+3i)^2 - (1+i)^2\right\} \\
&= \frac{1}{2}\left\{9 + 18i - 9 - 1 - 2i + 1\right\} \\
&= \frac{1}{2}(16i) = 8i,
\end{aligned}
$$

the result we obtained earlier.

We shall investigate these 'coincidences' in Section 26.5.

As a variation on this example, suppose that the path C_1 is the entire circumference of the unit circle travelled in an anti-clockwise direction. The limits are $\theta = 0$ and $\theta = 2\pi$. Hence

$$
\begin{aligned}
\int_{C_1} f(z)\,dz &= \int_0^{2\pi} (-\sin 3\theta)d\theta + i\int_0^{2\pi} \cos 3\theta\,d\theta \\
&= \left[\frac{1}{3}\cos 3\theta\right]_0^{2\pi} + i\left[\frac{1}{3}\sin 3\theta\right]_0^{2\pi} \\
&= \left(\frac{1}{3} - \frac{1}{3}\right) + i(0 - 0) = 0.
\end{aligned}
$$

Is there an underlying reason for this result? (We shall see in Section 26.5.)

Another technique for evaluating integrals taken around the unit circle is shown in the next example, in which we need to evaluate

$$\oint_C \frac{1}{z}dz \qquad \text{where } C \text{ is the unit circle.}$$

Note the use of $\oint$ since we have a closed path; we could have used this notation earlier.

Task

Evaluate $\oint_C \dfrac{1}{z} dz$ where C is the unit circle.

First show that a point z on the unit circle can be written $z = e^{i\theta}$ and hence find dz in terms of θ:

Your solution

Answer

On the unit circle a point (x, y) is such that $x = \cos\theta, \quad y = \sin\theta \quad$ and hence $\quad z = \cos\theta + i\sin\theta$ which, using De Moivre's theorem, can be seen to be $\quad z = e^{i\theta}$.

Then $\quad \dfrac{dz}{d\theta} = ie^{i\theta} \quad$ so that $\quad dz = ie^{i\theta} d\theta$.

Now evaluate the integral $\oint_C \dfrac{1}{z} dz$.

Your solution

Answer

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

We now quote one of the most important results in complex integration which incorporates the last result.

Key Point 1

If n is an integer and C is the circle centre $z = z_0$ and radius r, that is, it has equation $\quad |z - z_0| = r$ then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 0, & n \neq 1; \\ 2\pi i, & n = 1. \end{cases}$$

Note that the result is independent of the value of r.

Engineering Example 1

Two-dimensional fluid flow

Introduction

Functions of a complex variable find a very elegant application in the mathematical treatment of two-dimensional fluid flow.

Problem in words

Find the forces and moments due to fluid flowing past a cylinder.

Mathematical statement of the problem

Figure 10 shows a cross section of a cylinder (not necessarily circular), whose boundary is C, placed in a steady non-viscous flow of an ideal fluid; the flow takes place in planes parallel to the xy plane. The cylinder is out of the plane of the paper. The flow of the fluid exerts forces and turning moments upon the cylinder. Let X, Y be the components, in the x and y directions respectively, of the force on the cylinder and let M be the anticlockwise moment (on the cylinder) about the orgin.
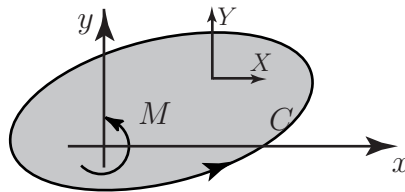

Figure 10

Blasius' theorem (which we shall not prove) states that

$$X - \mathrm{i}Y = \frac{1}{2}\mathrm{i}\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz \qquad \text{and} \qquad M = \mathrm{Re}\left\{ -\frac{1}{2}\rho \oint_C z\left(\frac{dw}{dz}\right)^2 dz \right\}$$

where Re denotes the real part, ρ is the (constant) density of the fluid and $w = u + iv$ is the complex potential (see Section 261) for the flow. Both ρ and ω are presumed known.

Mathematical analysis

We shall find X, Y and M if the cylinder has a circular cross section and the boundary is specified by $|z| = a$. Let the flow be a uniform stream with speed U.

Now, using a standard result, the complex potential describing this situation is:

$$w = U\left(z + \frac{a^2}{z}\right) \qquad \text{so that} \qquad \frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) \text{ and } \left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right).$$

Using Key Point 1 with $z_0 = 0$:

$$X - \mathrm{i}Y = \frac{1}{2}\mathrm{i}\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz = \frac{1}{2}\mathrm{i}\rho U^2 \oint \left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right) dz = 0 \quad \text{so} \quad X = Y = 0.$$

Also, $z\left(\dfrac{dw}{dz}\right)^2 = U^2\left(z - \dfrac{2a^2}{z} + \dfrac{a^4}{z^3}\right)$. The only term to contribute to M is $\dfrac{-2a^2U^2}{z}$.

Again using Key Point 1, this leads to $-4\pi a^2 U^2 \mathrm{i}$ and this has zero real part. Hence $M = 0$, also.

Interpretation

The implication is that no net force or moment acts on the cylinder. This is not so in practice. The discrepancy arises from neglecting the viscosity of the fluid.

Exercises

1. Obtain the integral $\displaystyle\int_C z\,dz$ along the straight-line paths

   (a) from $z = 2 + 2\mathrm{i}$ to $z = 5 + 2\mathrm{i}$

   (b) from $z = 5 + 2\mathrm{i}$ to $z = 5 + 5\mathrm{i}$

   (c) from $z = 2 + 2\mathrm{i}$ to $z = 5 + 5\mathrm{i}$

2. Find $\displaystyle\int_C (z^2 + z)\,dz$ where C is the part of the unit circle going anti-clockwise from the point $z = 1$ to the point $z = \mathrm{i}$.

3. Find $\displaystyle\oint_C f(z)\,dz$ where C is the circle $|z - z_0| = r$ for the cases

   (a) $f(z) = \dfrac{1}{z^2}$. $z_0 = 1$

   (b) $f(z) = \dfrac{1}{(z-1)^2}$, $z_0 = 1$

   (c) $f(z) = \dfrac{1}{z - 1 - \mathrm{i}}$, $z_0 = 1 + \mathrm{i}$

Answers

1. (a) Here y is constant along the given path $z = x + 2i$ so that $u = x$ and $v = 2$. Also $dy = 0$. Thus

$$\int_C z\, dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) = \int_2^5 x\,dx + i\int_2^5 2dx$$

$$= \left[\frac{x^2}{2}\right]_2^5 + i\left[2x\right]_2^5 = (\frac{25}{2} - \frac{4}{2}) + i(10 - 4) = \frac{21}{2} + 6i.$$

(b) Here $dx = 0$, $v = y$, $u = 5$. Thus

$$\int_C z\, dz = \int_2^5 (-y)dy + i\int_2^5 5dy$$

$$= \left[-\frac{y^2}{2}\right]_2^5 + i\left[5y\right]_2^5 = (-\frac{25}{2} + \frac{4}{2}) + i(25 - 10) = -\frac{21}{2} + 15i.$$

(c) $z = x + ix$, $u = x$, $v = x$, $dz = (1 + i)dx$, so

$$\int_C z\, dz = \int_C (x\,dx - x\,dx) + i\int_C (x\,dx + x\,dx) = i\int_C 2x\,dx = 2i\left[\frac{x^2}{2}\right]_2^5 = 21i.$$

Note that the result in (c) is the sum of the results in (a) and (b).

2. $\displaystyle\int_C (z^2 + z)\, dz = \left[\frac{z^3}{3} + \frac{z^2}{2}\right]_1^i = (\frac{1}{3}i^3 + \frac{i^2}{2}) - (\frac{1}{3} + \frac{1}{2}) = -\frac{4}{3} - \frac{1}{3}i.$

3. Using Key Point 1 we have (a) 0, (b) 0, (c) $2\pi i$.

Note that in all cases the result is independent of r.