Cauchy-Riemann Equations and Conformal Mapping

26.2

Introduction

In this Section we consider two important features of complex functions. The Cauchy-Riemann equations provide a necessary and sufficient condition for a function $f(z)$ to be analytic in some region of the complex plane; this allows us to find $f'(z)$ in that region by the rules of the previous Section.

A mapping between the z-plane and the w-plane is said to be conformal if the angle between two intersecting curves in the z-plane is equal to the angle between their mappings in the w-plane. Such a mapping has widespread uses in solving problems in fluid flow and electromagnetics, for example, where the given problem geometry is somewhat complicated.

Prerequisites

Before starting this Section you should ...

- understand the idea of a complex function and its derivative

Learning Outcomes

On completion you should be able to ...

- use the Cauchy-Riemann equations to obtain the derivative of complex functions
- appreciate the idea of a conformal mapping

1. The Cauchy-Riemann equations

Remembering that $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, we note that there is a very useful test to determine whether a function $w = f(z)$ is analytic at a point. This is provided by the **Cauchy-Riemann** equations. These state that $w = f(z)$ is differentiable at a point $z = z_0$ if, and only if,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at that point.}$$

When these equations hold then it can be shown that the complex derivative may be determined by using either $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x}$ or $\dfrac{df}{dz} = -\mathrm{i}\dfrac{\partial f}{\partial y}$.

(The use of 'if, and only if,' means that if the equations are valid, then the function is differentiable **and vice versa**.)

If we consider $f(z) = z^2 = x^2 - y^2 + 2\mathrm{i}xy$ then $u = x^2 - y^2$ and $v = 2xy$ so that

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x.$$

It should be clear that, for this example, the Cauchy-Riemann equations are always satisfied; therefore, the function is analytic everywhere. We find that

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2\mathrm{i}y = 2z \quad \text{or, equivalently,} \quad \frac{df}{dz} = -\mathrm{i}\frac{\partial f}{\partial y} = -\mathrm{i}(-2y + 2\mathrm{i}x) = 2z$$

This is the result we would expect to get by simply differentiating $f(z)$ as if it was a real function. **For analytic functions this will always be the case** i.e. for an analytic function $f'(z)$ can be found using the rules for differentiating real functions.

Example 3

Show that the function $f(z) = z^3$ is analytic everwhere and hence obtain its derivative.

Solution

$$w = f(z) = (x + \mathrm{i}y)^3 = x^3 - 3xy^2 + (3x^2y - y^3)\mathrm{i}$$

Hence

$$u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3.$$

Then

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial u}{\partial y} = -6xy, \qquad \frac{\partial v}{\partial x} = 6xy, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

The Cauchy-Riemann equations are identically true and $f(z)$ is analytic everywhere.

Furthermore $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x} = 3x^2 - 3y^2 + (6xy)\mathrm{i} = 3(x + \mathrm{i}y)^2 = 3z^2$ as we would expect.

We can easily find functions which are not analytic anywhere and others which are only analytic in a restricted region of the complex plane. Consider again the function $f(z) = \bar{z} = x - \mathrm{i}y$.

Here

$$u = x \quad \text{so that} \quad \frac{\partial u}{\partial x} = 1, \quad \text{and} \quad \frac{\partial u}{\partial y} = 0; \qquad\qquad v = -y \quad \text{so that} \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1.$$

The Cauchy-Riemann equations are never satisfied so that $\bar{z}$ is not differentiable anywhere and so is not analytic anywhere.

By contrast if we consider the function $f(z) = \dfrac{1}{z}$ we find that

$$u = \frac{x}{x^2 + y^2}; \quad v = \frac{y}{x^2 + y^2}.$$

As can readily be shown, the Cauchy-Riemann equations are satisfied everywhere except for $x^2 + y^2 = 0$, i.e. $x = y = 0$ (or, equivalently, $z = 0$.) At all other points $f'(z) = -\dfrac{1}{z^2}$. This function is analytic everywhere except at the single point $z = 0$.

Analyticity is a very powerful property of a function of a complex variable. Such functions tend to behave like functions of a real variable.

Example 4

Show that if $f(z) = z\bar{z}$ then $f'(z)$ exists only at $z = 0$.

Solution

$f(z) = x^2 + y^2$ so that $u = x^2 + y^2, \quad v = 0.$ $\qquad \dfrac{\partial u}{\partial x} = 2x, \quad \dfrac{\partial u}{\partial y} = 2y, \quad \dfrac{\partial v}{\partial x} = 0, \quad \dfrac{\partial v}{\partial y} = 0.$

Hence the Cauchy-Riemann equations are satisfied only where $x = 0$ and $y = 0$, i.e. where $z = 0$. Therefore this function is not analytic anywhere.

Analytic functions and harmonic functions

Using the Cauchy-Riemann equations in a region of the z-plane where $f(z)$ is analytic, gives

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial y^2}.$$

If these differentiations are possible then $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$

In a similar way we find that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{(Can you show this?)}$$

When $f(z)$ is analytic the functions u and v are called **conjugate harmonic functions**.

Suppose $u = u(x, y) = xy$ then it is easy to verify that u satisfies Laplace's equation (try this). We now try to find the conjugate harmonic function $v = v(x, y)$.

First, using the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x.$$

Integrating the first equation gives $v = \frac{1}{2}y^2 +$ a function of x. Integrating the second equation gives $v = -\frac{1}{2}x^2 +$ a function of y. Bearing in mind that an additive constant leaves no trace after differentiation, we pool the information above to obtain

$$v = \frac{1}{2}(y^2 - x^2) + C \qquad \text{where } C \text{ is a constant}$$

Note that $f(z) = u + \text{i}v = xy + \frac{1}{2}(y^2 - x^2)\text{i} + D$ where D is a constant (replacing Ci).

We can write $f(z) = -\frac{1}{2}\text{i}z^2 + D$ (as you can verify). This function is analytic everywhere.

Task

Given the function $u = x^2 - x - y^2$

(a) Show that u is harmonic, (b) Find the conjugate harmonic function, v.

Your solution

(a)

Answer

$$\frac{\partial u}{\partial x} = 2x - 1, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2.$$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

Your solution

(b)

Answer

Integrating $\dfrac{\partial v}{\partial y} = 2x - 1$ gives $v = 2xy - y +$ function of x.

Integrating $\dfrac{\partial v}{\partial x} = +2y$ gives $v = 2xy +$ function of y.

Ignoring the duplication, $v = 2xy - y + C$, where C is a constant.

Task

Find $f(z)$ in terms of z, where $f(z) = u + \mathrm{i}v$, where u and v are those found in the previous Task.

Your solution

Answer

$f(z) = u + iv = x^2 - x - y^2 + 2xy\mathrm{i} - \mathrm{i}y + D$, $\quad D$ constant.

Now $\quad z^2 = x^2 - y^2 + 2\mathrm{i}xy \quad$ and $\quad z = x + \mathrm{i}y \quad$ thus $\qquad f(z) = z^2 - z + D$.

Exercises

1. Find the singular point of the rational function $f(z) = \dfrac{z}{z - 2\mathrm{i}}$. Find $f'(z)$ at other points and evaluate $f'(-\mathrm{i})$.
2. Show that the function $f(z) = z^2 + z$ is analytic everywhere and hence obtain its derivative.
3. Show that the function $u = x^2 - y^2 - 2y$ is harmonic, find the conjugate harmonic function v and hence find $f(z) = u + \mathrm{i}v$ in terms of z.

Answers

1. $f(z)$ is singular at $z = 2\mathrm{i}$. Elsewhere

$$f'(z) = \frac{(z - 2\mathrm{i}).1 - z.1}{(z - 2\mathrm{i})^2} = \frac{-2\mathrm{i}}{(z - 2\mathrm{i})^2} \qquad f'(-\mathrm{i}) = \frac{-2\mathrm{i}}{(-3\mathrm{i})^2} = \frac{-2\mathrm{i}}{-9} = \frac{2}{9}\mathrm{i}$$

2. $u = x^2 + x - y^2$ and $\quad v = 2xy + y$

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

Here the Cauchy-Riemann equations are identically true and $f(z)$ is analytic everywhere.

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 1 + 2y\mathrm{i} = 2z + 1$$

Answer

3. $\dfrac{\partial^2 u}{\partial x^2} = 2, \quad \dfrac{\partial^2 u}{\partial y^2} = -2$ therefore u is harmonic.

$\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 2x$ therefore $v = 2xy+$ function of y

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 2y+2$ therefore $v = 2xy + 2x+$ function of x

$\therefore \quad v = 2xy + 2x +$ constant

$f(z) = x^2 + 2\mathrm{i}xy - y^2 + 2x\mathrm{i} - 2y = z^2 + 2\mathrm{i}z$

2. Conformal mapping

In Section 26.1 we saw that the real and imaginary parts of an analytic function each satisfies Laplace's equation. We shall show now that the curves

$u(x, y) =$ constant and $v(x, y) =$ constant

intersect each other at right angles (i.e. are **orthogonal**). To see this we note that along the curve $u(x, y) =$ constant we have $du = 0$. Hence

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

Thus, on these curves the gradient at a general point is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}.$$

Similarly along the curve $v(x, y) =$ constant, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}.$$

The product of these gradients is

$$\frac{(\dfrac{\partial u}{\partial x})(\dfrac{\partial v}{\partial x})}{(\dfrac{\partial u}{\partial y})(\dfrac{\partial v}{\partial y})} = -\frac{(\dfrac{\partial u}{\partial x})(\dfrac{\partial u}{\partial y})}{(\dfrac{\partial u}{\partial y})(\dfrac{\partial u}{\partial x})} = -1$$

where we have made use of the Cauchy-Riemann equations. We deduce that the curves are orthogonal.

As an example of the practical application of this work consider two-dimensional electrostatics. If $u =$ constant gives the **equipotential** curves then the curves $v =$ constant are the **electric lines of force**. Figure 2 shows some curves from each set in the case of oppositely-charged particles near to each other; the dashed curves are the lines of force and the solid curves are the equipotentials.

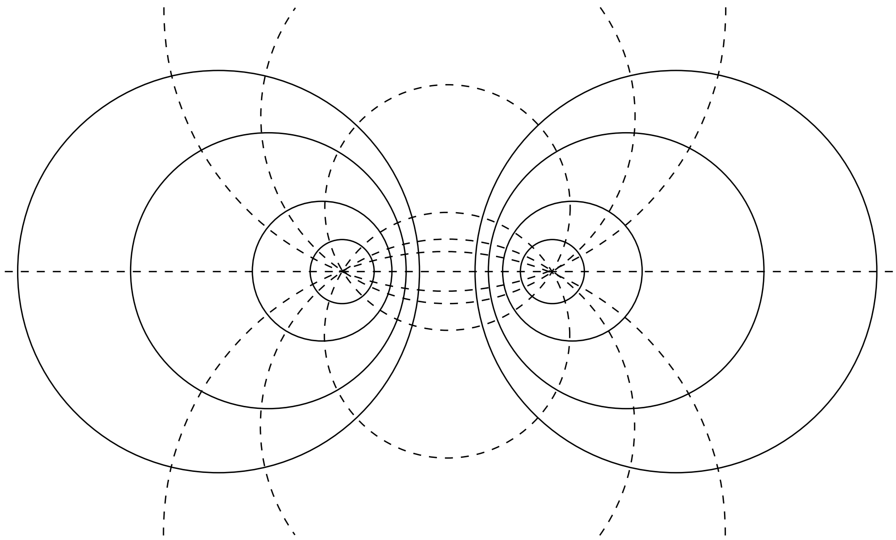

Figure 2

In ideal fluid flow the curves $v =$ constant are the **streamlines** of the flow.

In these situations the function $w = u + \text{i}v$ is the **complex potential** of the field.

Function as mapping

A function $w = f(z)$ can be regarded as a mapping, which maps a point in the z-plane to a point in the w-plane. Curves in the z-plane will be mapped into curves in the w-plane.

Consider aerodynamics where we are interested in the fluid flow in a complicated geometry (say flow past an aerofoil). We first find the flow in a simple geometry that can be mapped to the aerofoil shape (the complex plane with a circular hole works here). Most of the calculations necessary to find physical characteristics such as lift and drag on the aerofoil can be performed in the simple geometry - the resulting integrals being much easier to evaluate than in the complicated geometry.

Consider the mapping

$$w = z^2.$$

The point $z = 2 + \text{i}$ maps to $w = (2 + \text{i})^2 = 3 + 4\text{i}$. The point $z = 2 + \text{i}$ lies on the intersection of the two lines $x = 2$ and $y = 1$. To what curves do these map? To answer this question we note that a point on the line $y = 1$ can be written as $z = x + \text{i}$. Then

$$w = (x + \text{i})^2 = x^2 - 1 + 2x\text{i}$$

As usual, let $w = u + \text{i}v$, then

$$u = x^2 - 1 \qquad \text{and} \qquad v = 2x$$

Eliminating x we obtain:

$$4u = 4x^2 - 4 = v^2 - 4 \quad \text{so} \quad v^2 = 4 + 4u \text{ is the curve to which } y = 1 \text{ maps.}$$

Example 5

Onto what curve does the line $x = 2$ map?

Solution

A point on the line is $z = 2 + y\text{i}$. Then

$$w = (2 + y\text{i})^2 = 4 - y^2 + 4y\text{i}$$

Hence $u = 4 - y^2$ and $v = 4y$ so that, eliminating y we obtain

$$16u = 64 - v^2 \quad \text{or} \quad v^2 = 64 - 16u$$

In Figure 3(a) we sketch the lines $x = 2$ and $y = 1$ and in Figure 3(b) we sketch the curves into which they map. Note these curves intersect at the point $(3, 4)$.

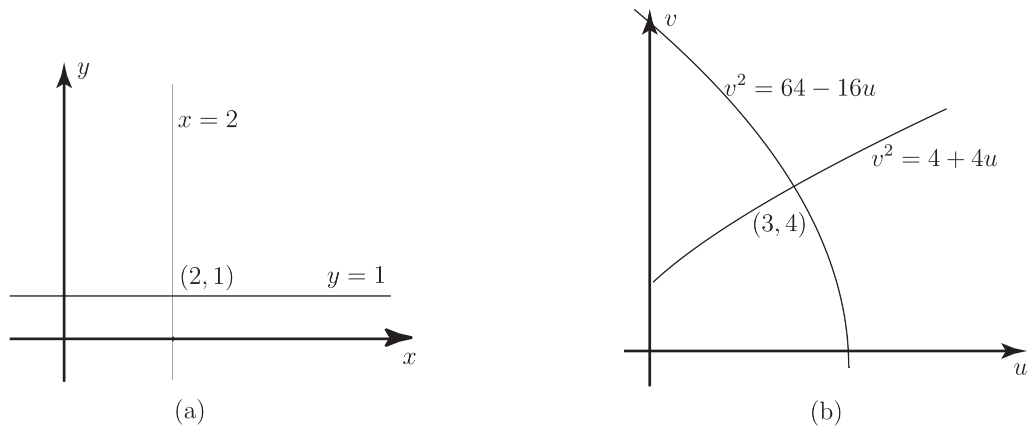

Figure 3

The angle between the original lines in (a) is clearly 90^0; what is the angle between the curves in (b) at the point of intersection?

The curve $v^2 = 4 + 4u$ has a gradient $\dfrac{dv}{du}$. Differentiating the equation implicitly we obtain

$$2v\frac{dv}{du} = 4 \quad \text{or} \quad \frac{dv}{du} = \frac{2}{v}$$

At the point $(3, 4)$ $\dfrac{dv}{du} = \dfrac{1}{2}$.

Task

Find $\dfrac{dv}{du}$ for the curve $v^2 = 64 - 16u$ and evaluate it at the point $(3, 4)$.

Your solution

Answer

$2v\dfrac{dv}{du} = -16 \qquad \therefore \quad \dfrac{dv}{du} = -\dfrac{8}{v}$. At $v = 4$ we obtain $\dfrac{dv}{du} = -2$.

Note that the product of the gradients at $(3, 4)$ is -1 and therefore the angle between the curves at their point of intersection is also 90^0. Since the angle between the lines and the angle between the curves is the same we say the **angle is preserved**.

In general, if two curves in the z-plane intersect at a point z_0, and their image curves under the mapping $w = f(z)$ intersect at $w_0 = f(z_0)$ and the angle between the two original curves at z_0 equals the angle between the image curves at w_0 we say that the mapping is **conformal** at z_0.

An analytic function is conformal everywhere except where $f'(z) = 0$.

Task

At which points is $w = e^z$ not conformal?

Your solution

Answer

$f'(z) = e^z$. Since this is never zero the mapping is conformal everywhere.

Inversion

The mapping $\quad w = f(z) = \dfrac{1}{z} \quad$ is called an **inversion**. It maps the interior of the unit circle in the z-plane to the exterior of the unit circle in the w-plane, and vice-versa. Note that

$$w = u + \mathrm{i}v = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\mathrm{i} \quad \text{and similarly} \quad z = x + \mathrm{i}y = \frac{u}{u^2+v^2} - \frac{v}{u^2+v^2}\mathrm{i}$$

so that

$$u = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v = -\frac{y}{x^2+y^2}.$$

A line through the origin in the z-plane will be mapped into a line through the origin in the w-plane. To see this, consider the line $y = mx$, for m constant. Then

$$u = \frac{x}{x^2 + m^2x^2} \qquad \text{and} \qquad v = -\frac{mx}{x^2 + m^2x^2}$$

so that $v = -mu$, which is a line through the origin in the w-plane.

Task

Consider the line $ax + by + c = 0$ where $c \neq 0$. This represents a line in the z-plane which does **not** pass through the origin. To what type of curve does it map in the w-plane?

Your solution

Answer

The mapped curve is

$$\frac{au}{u^2 + v^2} - \frac{bv}{u^2 + v^2} + c = 0$$

Hence $au - bv + c(u^2 + v^2) = 0$. Dividing by c we obtain the equation:

$$u^2 + v^2 + \frac{a}{c}u - \frac{b}{c}v = 0$$

which is the equation of a circle in the w-plane which passes through the origin.

Similarly, it can be shown that a circle in the z-plane passing through the origin maps to a line in the w-plane which does not pass through the origin. Also a circle in the z-plane which does not pass through the origin maps to a circle in the w-plane which does pass through the origin. The inversion mapping is an example of the **bilinear transformation**:

$$w = f(z) = \frac{az + b}{cz + d} \qquad \text{where we demand that} \quad ad - bc \neq 0$$

(If $ad - bc = 0$ the mapping reduces to $f(z) =$ constant.)

Task

Find the set of bilinear transformations $w = f(z) = \dfrac{az + b}{cz + d}$ which map $z = 2$ to $w = 1$.

Your solution

Answer

$1 = \dfrac{2a+b}{2c+d}$. Hence $2a + b = 2c + d$.

Any values of a, b, c, d satisfying this equation will do provided $ad - bc \neq 0$.

Task

Find the bilinear transformations for which $z = -1$ is mapped to $w = 3$.

Your solution

Answer

$3 = \dfrac{-a+b}{-c+d}$. Hence $-a + b = -3c + 3d$.

Example 6

Find the bilinear transformation which maps

(a) $z = 2$ to $w = 1$, **and**

(b) $z = -1$ to $w = 3$, **and**

(c) $z = 0$ to $w = -5$

Solution

We have the answers to (a) and (b) from the previous two Tasks:

$$\begin{aligned} 2a + b &= 2c + d \\ -a + b &= -3c + 3d \end{aligned}$$

If $z = 0$ is mapped to $w = -5$ then $-5 = \dfrac{b}{d}$ so that $b = -5d$. Substituting this last relation into the first two obtained we obtain

$$\begin{aligned} 2a - 2c - 6d &= 0 \\ -a + 3c - 8d &= 0 \end{aligned}$$

Solving these two in terms of d we find $2c = 11d$ and $2a = 17d$. Hence the transformation is:

$w = \dfrac{17z - 10}{11z + 2}$ (note that the d's cancel in the numerator and denominator).

Some other mappings are shown in Figure 4.

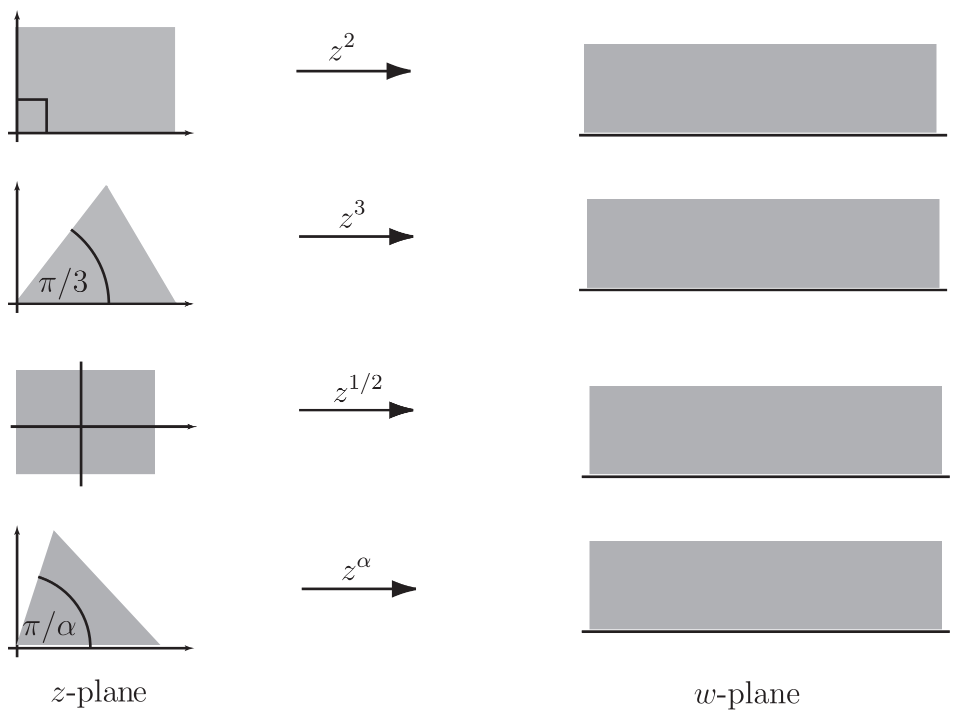

Figure 4

As an engineering application we consider the Joukowski transformation

$$w = z - \frac{\ell^2}{z} \qquad \text{where } \ell \text{ is a constant.}$$

It is used to map circles which contain $z = 1$ as an interior point and which pass through $z = -1$ into shapes resembling aerofoils. Figure 5 shows an example:

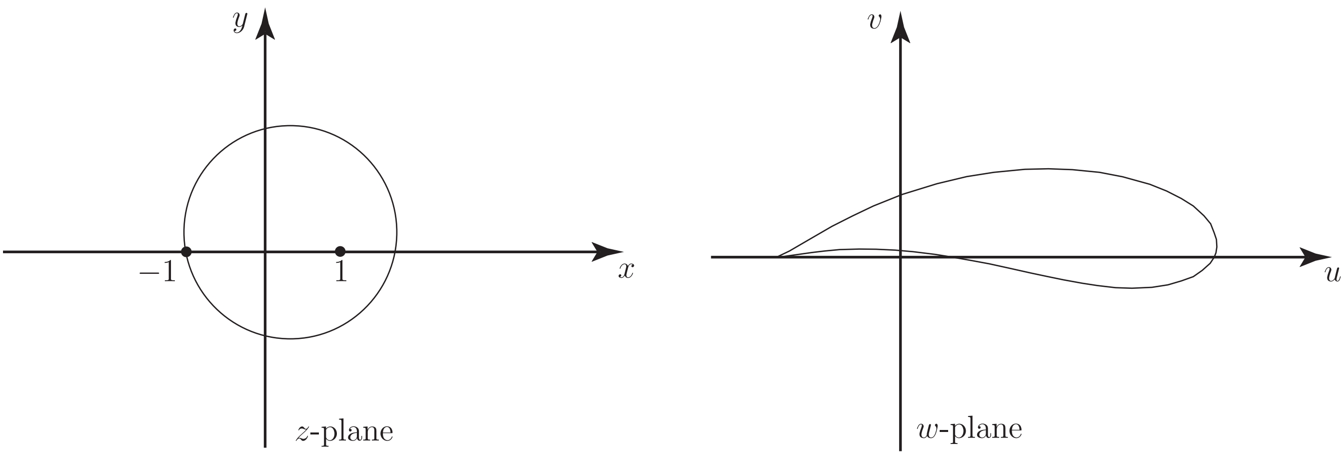

Figure 5

This creates a cusp at which the associated fluid velocity can be infinite. This can be avoided by adjusting the fluid flow in the z-plane. Eventually, this can be used to find the lift generated by such an aerofoil in terms of physical characteristics such as aerofoil shape and air density and speed.

Exercise

Find a bilinear transformation $w = \dfrac{az+b}{cz+d}$ which maps

(a) $z = 0$ into $w = \mathrm{i}$

(b) $z = -1$ into $w = 0$

(c) $z = -\mathrm{i}$ into $w = 1$

Answer

(a) $z = 0,\ w = \mathrm{i}$ gives $\mathrm{i} = \dfrac{b}{d}$ so that $b = d\mathrm{i}$

(b) $z = -1,\ w = 0$ gives $0 = \dfrac{-a+b}{-c+d}$ so $-a+b = 0$ so $a = b$.

(c) $z = -\mathrm{i},\ w = 1$ gives $1 = \dfrac{-a\mathrm{i}+b}{-c\mathrm{i}+d}$ so that $-c\mathrm{i}+d = -a\mathrm{i}+b = d+d\mathrm{i}$ (using (a) and (b))

We conclude from (c) that $-c = d$. We also know that $a = b = d\mathrm{i}$.

Hence $w = \dfrac{d\mathrm{i}z + d\mathrm{i}}{-dz+d} = \dfrac{\mathrm{i}z+\mathrm{i}}{-z+1}$