Applications of PDEs 25.2

Introduction

In this Section we discuss briefly some of the most important PDEs that arise in various branches of science and engineering. We shall see that some equations can be used to describe a variety of different situations.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of partial differentiation

Learning Outcomes

On completion you should be able to . . .

- recognise the heat conduction equation and the wave equation and have some knowledge of their applicability

Key Point 4 by no means exhausts the types of PDE which are important in applications. In this Section we will discuss those three PDEs in Key Point 4 in more detail and briefly discuss other PDEs over a wide range of applications. We will omit detailed derivations.

1. Wave equation

The simplest situation to give rise to the one-dimensional wave equation is the motion of a stretched string - specifically the **transverse vibrations** of a string such as the string of a musical instrument. Assume that a string is placed along the x-axis, is stretched and then fixed at ends $x = 0$ and $x = L$; it is then deflected and at some instant, which we call $t = 0$, is released and allowed to vibrate. The quantity of interest is the deflection u of the string at any point x, $0 \leq x \leq L$, and at any time $t > 0$. We write $u = u(x, t)$. Figure 2 shows a possible displacement of the string at a fixed time t.

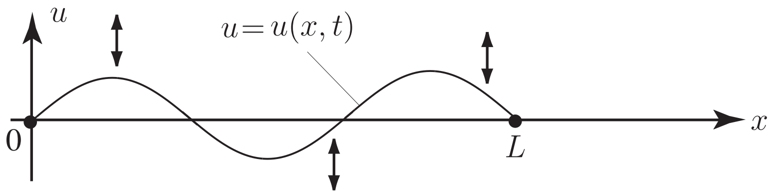

Figure 2

Subject to various assumptions $\cdots$

1. damping forces such as air resistance are negligible
2. the weight of the string is negligible
3. the tension in the string is tangential to the curve of the string at any point
4. the string performs small transverse oscillations i.e. every particle of the string moves strictly vertically and such that its deflection and slope as every point on the string is small.

$\cdots$ it can be shown, by applying Newton's law of motion to a small segment of the string, that u satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where $c^2 = \dfrac{T}{\rho}$, ρ being the mass per unit length of the string and T being the (constant) horizontal component of the tension in the string. To determine $u(x, t)$ uniquely, we must also know

1. the initial definition of the string at the time $t = 0$ at which it is released
2. the initial velocity of the string.

Thus we must be given **initial conditions**

$$u\,(x, 0) = f(x) \qquad 0 \leq x \leq L \qquad \text{(initial position)}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \qquad 0 \leq x \leq L \qquad \text{(initial velocity)}$$

where $f(x)$ and $g(x)$ are known.

These two initial conditions are in addition to the two **boundary conditions**

$$u(0,t) = u(L,t) = 0 \qquad \text{for } t \geq 0$$

which indicate that the string is fixed at each end. In Example 2 discussed in Section 25.1 we had

$$f(x) = u_0 \ \sin\left(\frac{\pi x}{\ell}\right)$$

$$g(x) = 0 \qquad \text{(string initially at rest).}$$

The PDE (1) is the (undamped) wave equation. We will discuss solutions of it for various initial conditions later. More complicated forms of the wave equation would arise if some of the assumptions were modified. For example:

(a) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} - g$ if the weight of the string was allowed for,

(b) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} - \alpha\dfrac{\partial u}{\partial t}$ if a damping force proportional to the velocity of the string (with damping constant α) was included.

Equation (1) is referred to as the one-dimensional wave equation because only one space variable, x, is present. The two-dimensional (undamped) wave equation is, in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2)$$

This arises for example when we model the transverse vibrations of a membrane. See Figures 3(a), 3(b). Here $u(x, y, t)$ is the definition of a point (x, y) on the membrane at time t. Again, a boundary condition must be specified: commonly

$$u = 0 \qquad t \geq 0$$

on the boundary of the membrane, if this is fixed (clamped). Also initial conditions must be given

$$u\,(x, y, 0) = f(x, y) \quad \text{(initial position)} \qquad \frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \quad \text{(initial velocity)}$$

For a **circular** membrane, such as a drumhead, polar coordinates defined by $x = r\cos\theta,\ y = r\sin\theta$ would be more convenient than Cartesian. In this case (2) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad 0 \leq r \leq R, \qquad 0 \leq \theta \leq 2\pi$$

for a circular membrane of radius R.

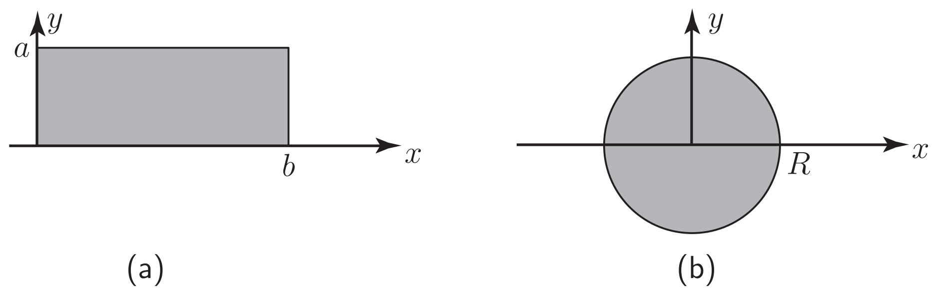

Figure 3

2. Heat conduction equation

Consider a long thin bar, or wire, of constant cross-section and of homogeneous material oriented along the x-axis (see Figure 4).

Figure 4

Imagine that the bar is thermally insulated laterally and is sufficiently thin that heat flows (by conduction) only in the x-direction. Then the temperature u at any point in the bar depends only on the x-coordinate of the point and the time t. By applying the principle of conservation of energy it can be shown that $u(x, t)$ satisfies the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{c} 0 \leq x \leq L \\ t > 0 \end{array} \qquad (3)$$

where k is a positive constant. In fact k, sometimes called the **thermal diffusivity** of the bar, is given by

$$k = \frac{\kappa}{s\rho}$$

where $\kappa =$ thermal conductivity of the material of the bar

$s =$ specific heat capacity of the material of the bar

$\rho =$ density of the material of the bar.

The PDE (3) is called the one-dimensional heat conduction equation (or, in other contexts where it arises, the **diffusion equation**).

Task

What is the obvious difference between the wave equation (1) and the heat conduction equation (3)?

Your solution

Answer

Both equations involve second derivatives in the space variable x but whereas the wave equation has a *second* derivative in the time variable t the heat conduction equation has only a *first* derivative in t. This means that the solutions of (3) are quite different in form from those of (1) and we shall study them separately later.

The fact that (3) is first order in t means that only one initial condition at $t = 0$ is needed, together with two boundary conditions, to obtain a unique solution. The usual initial condition specifies the initial temperature distribution in the bar

$$u(x, 0) = f(x)$$

where $f(x)$ is known. Various types of boundary conditions at $x = 0$ and $x = L$ are possible. For example:

(a) $u(0, t) = T_1$ and $u(L, T) = T_2$ (ends of the bar are at constant temperatures T_1 and T_2).

(b) $\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = 0$ which are **insulation** conditions since they tell us that there is no heat flow through the ends of the bar.

As you would expect, there are two-dimensional and three-dimensional forms of the heat conduction equation. The two dimensional version of (3) is

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (4)$$

where $u(x, y, t)$ is the temperature in a flat plate. The plate is assumed to be thin and insulated on its top and bottom surface so that no heat flow occurs other than in the Oxy plane. Boundary conditions and an initial condition are needed to give unique solutions of (4). For example if the plate is rectangular as in Figure 5:

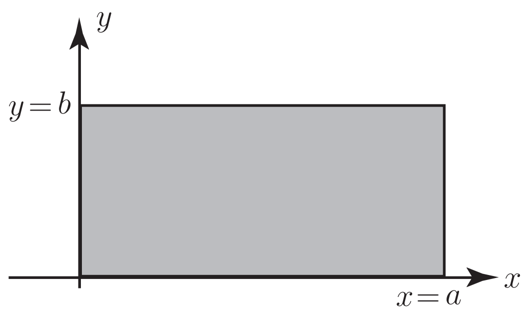

Figure 5

typical boundary conditions might be

$u(x, 0) = T_1 \qquad 0 \leq x \leq a$ (bottom side at fixed temperature)

$\dfrac{\partial u}{\partial x}(a, y) = 0 \qquad 0 \leq y \leq b$ (right-hand side insulated)

$u(x, b) = T_2 \qquad 0 \leq x \leq a$ (top side at fixed temperature)

$u(0, y) = 0 \qquad 0 \leq y \leq b$ (left hand side at zero fixed temperature).

An initial condition would have the form $u(x, y, 0) = f(x, y)$, where f is a given function.

3. Transmission line equations

In a long electrical cable or a telephone wire both the current and voltage depend upon position along the wire as well as the time (see Figure 6).

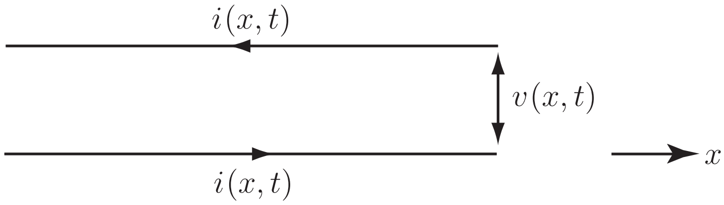

Figure 6

It is possible to show, using basic laws of electrical circuit theory, that the electrical current $i(x,t)$ satisfies the PDE

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC+GL)\frac{\partial i}{\partial t} + RGi \qquad (5)$$

where the constants R, L, C and G are, for unit length of cable, respectively the resistance, inductance, capacitance and leakage conductance. The voltage $v(x,t)$ also satisfies (5). Special cases of (5) arise in particular situations. For a submarine cable G is negligible and frequencies are low so inductive effects can also be neglected. In this case (5) becomes

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \qquad (6)$$

which is called the **submarine equation** or **telegraph equation**. For high frequency alternating currents, again with negligible leakage, (5) can be approximated by

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \qquad (7)$$

which is called the **high frequency line equation**.

Task

What PDEs, already discussed, have the same form as equations (6) or (7)?

Your solution

Answer

(6) has the same form as the one-dimensional heat conduction equation.

(7) has the same form as the one-dimensional wave equation.

4. Laplace's equation

If you look back at the two-dimensional heat conduction equation (4):

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

it is clear that if the heat flow is steady, i.e. time independent, then $\dfrac{\partial u}{\partial t} = 0$ so the temperature $u(x, y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (8)$$

(8) is the **two-dimensional Laplace equation**. Both this, and its three-dimensional counterpart

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (9)$$

arise in a wide variety of applications, quite apart from steady state heat conduction theory. Since time does not arise in (8) or (9) it is evident that Laplace's equation is always a model for **equilibrium** situations. In any problem involving Laplace's equation we are interested in solving it in a specific region R for given boundary conditions. Since conditions may involve

(a) u specified on the boundary curve C (two dimensions) or boundary surface S (three dimensions) of the region R. Such boundary conditions are called **Dirichlet conditions**.

(b) The derivative of u normal to the boundary, written $\dfrac{\partial u}{\partial n}$, specified on C or S. These are referred to as **Neumann boundary conditions**.

(c) A mixture of (a) and (b).

Some areas in which Laplace's equation arises are

(a) electrostatics (u being the electrostatic potential in a charge free region)

(b) gravitation (u being the gravitational potential in free space)

(c) steady state flow of inviscid fluids

(d) steady state heat conduction (as already discussed)

5. Other important PDEs in science and engineering

1. **Poisson's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{(two-dimensional form)}$$

where $f(x, y)$ is a given function. This equation arises in electrostatics, elasticity theory and elsewhere.

2. **Helmholtz's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \qquad \text{(two dimensional form)}$$

which arises in wave theory.

3. **Schrödinger's equation**

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) = E\psi$$

which arises in quantum mechanics. (h is Planck's constant)

4. **Transverse vibrations equation**

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

for a homogeneous rod, where $u(x, t)$ is the displacement at time t of the cross section through x.

All the PDEs we have discussed are **second order** (because the highest order derivatives that arise are second order) apart from the last example which is **fourth order**.