Contents 25

Partial Differential Equations

Learning outcomes

By studying this Workbook you will learn to recognise the two-dimensional Laplace's equation and the one-dimensinal diffusion and wave equations.

You will learn how to verify solutions of these equations and how to find solutions by using the separation of variable method and by using Fourier series.

Partial Differential Equations

25.1

Introduction

A **partial differential equation** (PDE) is a differential equation involving partial derivatives of one dependent variable with respect to two or more independent variables. The independent variables may be space variables only or one or more space variables and time. Mathematical modelling of many situations involving natural phenomena leads to PDEs.

The subject of PDEs is a very large one. We shall discuss only a few special PDEs which model a wide range of applied problems.

Prerequisites

Before starting this Section you should ...

- be able to carry out partial differentiation
- be able to solve constant coefficient ordinary differential equations

Learning Outcomes

On completion you should be able to ...

- verify solutions of given partial differential equations arising in engineering and science

1. Introduction

You have already studied ordinary differential equations (ODEs) and have learnt how to obtain the solution of certain types. Since a knowledge of the solution of certain ODEs (i.e. those with constant coefficients) will be required in solving partial differential equations (PDEs), we will begin this unit reminding you of some important results.

Key Point 1

The first order ODE

$$\frac{dy}{dx} = ky$$

has general solution

$$y = Ae^{kx}$$

Here k is a constant which can be positive or negative and A is an arbitrary constant.

In Key Point 1 the quantity A in the general solution is a constant. To obtain the value of A we have to know the value of y at some value of x, perhaps $x = 0$. In other words, we need to know an **initial condition**.

Task

Find y as a function of x if

$$\frac{dy}{dx} = -2y$$

and the initial condition is $y(0) = 3$.

Your solution

Answer

From Key Point 1 with $k = -2$ we have the general solution

$$y = Ae^{-2x}$$

Putting $x = 0$ and $y = 3$ into this we obtain $3 = Ae^0$ i.e. $A = 3$ so the solution to the given initial value problem is

$$y = 3e^{-2x}$$

We shall also need to be familiar with solutions to second order, homogeneous, constant coefficient ODEs, summarised in Key Point 2.

Key Point 2

A second order ODE of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

where a, b, c are constants, has an auxiliary equation

$$\mathrm{a}m^2 + bm + c = 0 \qquad (2)$$

obtained by inserting the trial solution $y = e^{mx}$ in (1).
The general solution of (1) then depends on the solutions (or roots) of the quadratic equation (2).

(a) If (2) has real, distinct roots $m = m_1$ and $m = m_2$ then

$$y = Ae^{m_1x} + Be^{m_2x}$$

(b) If (2) has a repeated root $m = m_1$ then

$$y = (A + Bx)e^{m_1x}$$

(c) If (2) has complex roots (which will be a conjugate pair) $m = \alpha \pm j\beta$ then

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Note that in each of these cases (a) to (c) the general solution is a linear combination of two particular solutions:

For (a) they are e^{m_1x} and e^{m_2x}.

For (b) they are e^{m_1x} and $x\mathrm{e}^{m_1x}$.

For (c) they are $\mathrm{e}^{\alpha x}\cos\beta x$ and $\mathrm{e}^{\alpha x}\sin\beta x$.

Task

Use Key Point 2 to find the general solution of $\dfrac{d^2y}{dx^2} - 4y = 0$.

First write down the auxiliary equation:

Your solution

Answer

$m^2 - 4 = 0$

Now find the roots of the auxiliary equation:

Your solution

Answer

$m = \pm 2$

Finally give the general solution to the ODE:

Your solution

Answer

$y = Ae^{2x} + Be^{-2x}$ (Since the roots of the auxiliary equation are real and distinct.)

Task

Find the general solution of $\dfrac{d^2y}{dx^2} + 9y = 0$

First write down the auxiliary equation:

Your solution

Answer

$m^2 + 9 = 0$

Now Find the roots of this auxiliary equation:

Your solution

Answer

$m = \pm 3\text{i}$

Finally give the general solution to the ODE:

Your solution

Answer
$y = A\cos 3x + B\sin 3x$

(Since the roots of the auxiliary equation are complex conjugates with real part $\alpha = 0$ and imaginary part $\beta = 3$.)

The two Tasks above can be generalised as in Key Point 3.

Key Point 3

(1) The general solution to: $\dfrac{d^2y}{dx^2} - n^2y = 0$ is

$$y = Ae^{nx} + Be^{-nx}$$

or, equivalently using hyperbolic functions,

$$y = C\cosh nx + D\sinh nx$$

(2) The general solution to: $\dfrac{d^2y}{dx^2} + n^2y = 0$ is

$$y = A\cos nx + B\sin nx$$

Those of you who are familiar with elementary dynamics will recognise the second differential equation in Key Point 3 as modelling **simple harmonic motion**.

2. Partial differential equations

In all the above examples we had a function y of a single variable x, y being the solution of an **ordinary** differential equation.

In engineering and science ODEs arise as models for systems where there is **one** independent variable (often x) and **one** dependent variable (often y). Obvious examples are lumped electrical circuits where the current i is a function only of time t (and not of position in the circuit) and lumped mechanical systems (such as the simple harmonic oscillator referred to above) where the displacement of a moving particle depends only on t.

However, in problems where one variable, say u, depends on **more than one** independent variable, say both x and t, then any derivatives of u will be **partial derivatives** such as $\dfrac{\partial u}{\partial x}$ or $\dfrac{\partial^2 u}{\partial t^2}$ and any differential equation arising will be known as a **partial differential equation.** In particular, one-dimensional (1-D) time-dependent problems where u depends on a position coordinate x and the time t and two-dimensional (2-D) time-independent problems where u is a function of the two position coordinates x and y both give rise to PDEs involving **two** independent variables. This is the case we shall concentrate on. A two-dimensional time-dependent problem would involve 3 independent variables x, y, t as would a three-dimensional time-independent problem where x, y, z would be the independent variables.

Example 1

Show that $u = \sin x \cosh y$ satisfies the PDE $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

This PDE is known as **Laplace's equation in two dimensions** and it arises in many applications e.g. electrostatics, fluid flow, heat conduction.

Solution

$u = \sin x \cosh y \qquad \Rightarrow \qquad \dfrac{\partial u}{\partial x} = \cos x \cosh y$ and $\dfrac{\partial u}{\partial y} = \sin x \sinh y$

Differentiating again gives $\dfrac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$ and $\dfrac{\partial^2 u}{\partial y^2} = \sin x \cosh y$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0$$

so the given function $u(x, y)$ is indeed a solution of the PDE.

Task

Show that $u = e^{-2\pi^2 t} \sin \pi x$ is a solution of the PDE $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}\dfrac{\partial u}{\partial t}$

First find $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$:

Your solution

Answer

$\dfrac{\partial u}{\partial t} = -2\pi^2 e^{-2\pi^2 t} \sin \pi x \qquad \dfrac{\partial u}{\partial x} = \pi e^{-2\pi^2 t} \cos \pi x$

Now find $\dfrac{\partial^2 u}{\partial x^2}$ and complete the Task:

Your solution

Answer

$\dfrac{\partial^2 u}{\partial x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x$, and we see that $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}\dfrac{\partial u}{\partial t}$ as required.

The PDE in the above Task has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k is a positive constant. This equation is referred to as the **one-dimensional heat conduction equation** (or sometimes as the **diffusion equation**). In a heat conduction context the dependent variable u represents the temperature $u(x,t)$.

The third important PDE involving two independent variables is known as the **one-dimensional wave equation**. This has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

(Note that both partial derivatives in the wave equation are second-order in contrast to the heat conduction equation where the time derivative is first order.)

Example 2

(a) Verify that $u(x,t) = u_0 \sin\left(\frac{\pi x}{\ell}\right)\ \cos\left(\frac{\pi c t}{\ell}\right)$ (where u_0, ℓ and c are constants) satisfies the one-dimensional wave equation.

(b) Verify the boundary conditions i.e. $u(0,t) = u(\ell,t) = 0$.

(c) Verify the initial conditions i.e. $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = u_0 \sin\left(\frac{\pi x}{\ell}\right)$.

(d) Give a physical interpretation of this problem.

Solution

(a) By straightforward partial differentiation of the given function $u(x,t)$:

$$\frac{\partial u}{\partial x} = u_0\frac{\pi}{\ell}\cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right) \qquad \frac{\partial^2 u}{\partial x^2} = -u_0\left(\frac{\pi}{\ell}\right)^2\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)$$

$$\frac{\partial u}{\partial t} = -u_0\left(\frac{\pi c}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{\pi c t}{\ell}\right) \qquad \frac{\partial^2 u}{\partial t^2} = -u_0\left(\frac{\pi c}{\ell}\right)^2\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)$$

We see that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ which completes the verification.

(b) Putting $x = 0$, and leaving t arbitrary, in the given solution for $u(x,t)$ gives

$$u(x,0) = u_0 \sin 0\ \cos\left(\frac{\pi c t}{\ell}\right) = 0 \text{ for all } t$$

Similarly putting $x = \ell,\ t$ arbitrary: $\quad u(\ell,0) = u_0 \sin\pi\ \cos\left(\frac{\pi c t}{\ell}\right) = 0$ for all t

Solution

(c) Evaluating $\dfrac{\partial u}{\partial t}$ firstly for general x and t

$$\frac{\partial u}{\partial t} = -u_0\left(\frac{\pi c}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{\pi ct}{\ell}\right)$$

Now putting $t = 0$ leaving x arbitrary

$$\frac{\partial u}{\partial t}(x,0) = -u_0\left(\frac{\pi c}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right)\sin 0 = 0.$$

Also, putting $t = 0$ in the expression for $u(x,t)$ gives

$$u(x,0) = u_0\sin\left(\frac{\pi x}{\ell}\right)\cos 0 = u_0\sin\left(\frac{\pi x}{\ell}\right).$$

(d) Mathematically we have now proved that the given function $u(x,t)$ satisfies the 1-D wave equation specified in (a), the two **boundary conditions** specified in (b) and the two **initial conditions** specified in (c).

One possible physical interpretation of this problem is that $u(x,t)$ represents the displacement of a string stretched between two points at $x = 0$ and $x = \ell$. Clearly the position of any point P on the vibrating string will depend upon its distance x from one end and on the time t.

The boundary conditions (b) represent the fact that the string is fixed at these end-points.

The initial condition $u(x,0) = u_0\sin\left(\dfrac{\pi x}{\ell}\right)$ represents the displacement of the string at $t = 0$.

The initial condition $\dfrac{\partial u}{\partial t}(x,0) = 0$ tells us that the string is at rest at $t = 0$.

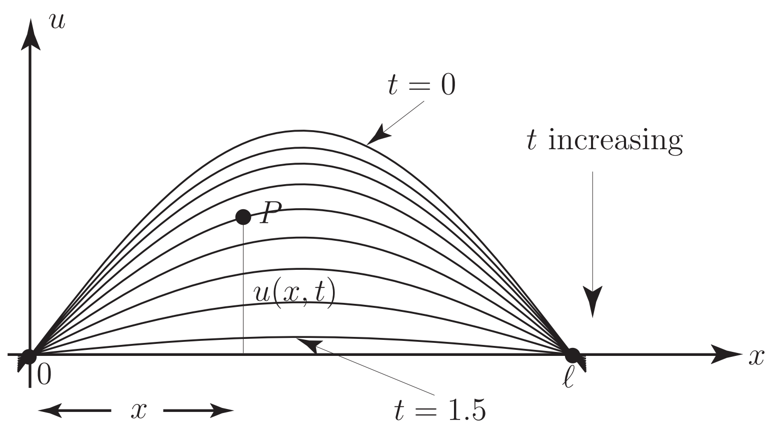

Figure 1

Note that it can be proved formally that if T is the tension in the string and if ρ is the mass per unit length of the string then u does, under certain conditions, satisfy the 1-D wave equation with $c^2 = \dfrac{T}{\rho}$.

Key Point 4

The three PDEs of greatest general interest involving two independent variables are:

(a) The two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(b) The one-dimensional heat conduction equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

(c) The one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$