Some Special Fourier Transform Pairs

24.3

Introduction

In this final Section on Fourier transforms we shall study briefly a number of topics such as Parseval's theorem and the relationship between Fourier transform and Laplace transforms. In particular we shall obtain, intuitively rather than rigorously, various Fourier transforms of functions such as the unit step function which actually violate the basic conditions which guarantee the existence of Fourier transforms!

Prerequisites

Before starting this Section you should ...

- be aware of the definitions and simple properties of the Fourier transform and inverse Fourier transform.

Learning Outcomes

On completion you should be able to ...

- use the unit impulse function (the Dirac delta function) to obtain various Fourier transforms

1. Parseval's theorem

Recall from HELM 23.2 on Fourier series that for a periodic signal $f_T(t)$ with complex Fourier coefficients $c_n (n = 0, \pm 1, \pm 2, \ldots)$ Parseval's theorem holds:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T^2(t)dt = \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where the left-hand side is the mean square value of the function (signal) over one period.

For a non-periodic real signal $f(t)$ with Fourier transform $F(\omega)$ the corresponding result is

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

This result is particularly significant in filter theory. For reasons that we do not have space to go into, the left-hand side integral is often referred to as the **total energy** of the signal. The integrand on the right-hand side

$$\frac{1}{2\pi}|F(\omega)|^2$$

is then referred to as the **energy density** (because it is the frequency domain quantity that has to be integrated to obtain the total energy).

Task

Verify Parseval's theorem using the one-sided exponential function

$$f(t) = e^{-t}u(t).$$

Firstly evaluate the integral on the left-hand side:

Your solution

Answer

$$\int_{-\infty}^{\infty} f^2(t)dt = \int_{0}^{\infty} e^{-2t}dt = \left[\frac{e^{-2t}}{-2}\right]_0^{\infty} = \frac{1}{2}.$$

Now obtain the Fourier transform $F(\omega)$ and evaluate the right-hand side integral:

Your solution

Answer

$$F(\omega) = \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+\mathrm{i}\omega},$$

so

$$|F(\omega)|^2 = \frac{1}{(1+\mathrm{i}\omega)}.\frac{1}{(1-\mathrm{i}\omega)} = \frac{1}{1+\omega^2}.$$

Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega &= \frac{1}{\pi}\int_0^{\infty}|F(\omega)|^2 d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}d\omega = \frac{1}{\pi}\Big[\tan^{-1}\omega\Big]_0^{\infty} = \frac{1}{\pi}\times\frac{\pi}{2} = \frac{1}{2}.\end{aligned}$$

Since both integrals give the same value, Parseval's theorem is verified for this case.

2. Existence of Fourier transforms

Formally, sufficient conditions for the Fourier transform of a function $f(t)$ to exist are

(a) $\int_{-\infty}^{\infty}|f(t)|^2 dt$ is finite

(b) $f(t)$ has a finite number of maxima and minima in any finite interval

(c) $f(t)$ has a finite number of discontinuities.

Like the equivalent conditions for the existence of Fourier series these conditions are known as **Dirichlet conditions**.

If the above conditions hold then $f(t)$ has a unique Fourier transform. However certain functions, such as the unit step function, which violate one or more of the Dirichlet conditions still have Fourier transforms in a more generalized sense as we shall see shortly.

3. Fourier transform and Laplace transforms

Suppose $f(t) = 0$ for $t < 0$. Then the Fourier transform of $f(t)$ becomes

$$\mathcal{F}\{f(t)\} = \int_0^{\infty} f(t)e^{-\mathrm{i}\omega t}dt. \qquad (1)$$

As you may recall from earlier units, the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt. \qquad (2)$$

Comparison of (1) and (2) suggests that for such one-sided functions, the Fourier transform of $f(t)$ can be obtained by simply replacing s by $\mathrm{i}\omega$ in the Laplace transform.
An obvious example where this can be done is the function

$$f(t) = e^{-\alpha t}u(t).$$

In this case $\mathcal{L}\{f(t)\} = \dfrac{1}{\alpha + s} = F(s)$ and, as we have seen earlier,

$$\mathcal{F}\{f(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} = F(\mathrm{i}\omega).$$

However, care must be taken with such substitutions. We must be sure that the conditions for the existence of the Fourier transform are met. Thus, for the unit step function,

$$\mathcal{L}\{u(t)\} = \frac{1}{s},$$

whereas, $\mathcal{F}\{u(t)\} \neq \dfrac{1}{\mathrm{i}\omega}$. (We shall see that $\mathcal{F}\{u(t)\}$ does actually exist but is not equal to $\dfrac{1}{\mathrm{i}\omega}$.)

We should also point out that some of the properties we have discussed for Fourier transforms are similar to those of the Laplace transforms e.g. the time-shift properties:

Fourier: $\mathcal{F}\{f(t-t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$ Laplace: $\mathcal{L}\{f(t-t_0)\} = e^{-st_0}F(s)$.

4. Some special Fourier transform pairs

As mentioned in the previous subsection it is possible to obtain Fourier transforms for some important functions that violate the Dirichlet conditions. To discuss this situation we must introduce the **unit impulse function**, also known as the **Dirac delta function**. We shall study this topic in an inituitive, rather than rigorous, fashion.

Recall that a symmetrical rectangular pulse

$$p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$$

has a Fourier transform

$$P_a(\omega) = \frac{2}{\omega}\sin \omega a.$$

If we consider a pulse whose height is $\dfrac{1}{2a}$ rather than 1 (so that the pulse encloses unit area), then we have, by the linearity property of Fourier transforms,

$$\mathcal{F}\left\{\frac{1}{2a}p_a(t)\right\} = \frac{\sin \omega a}{\omega a}.$$

As the value of a becomes smaller, the rectangular pulse becomes narrower and taller but still has unit area.

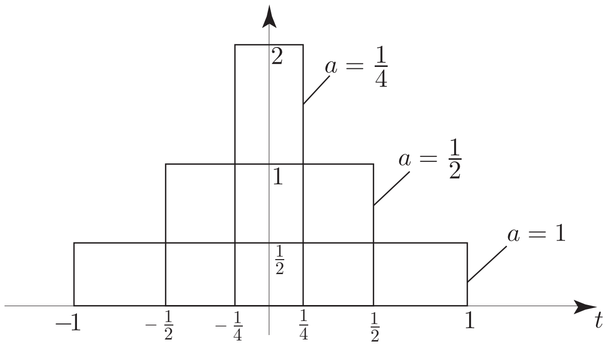


Figure 7

We define the unit impulse function $\delta(t)$ as

$$\delta(t) = \lim_{a\to 0} \frac{1}{2a} p_a(t)$$

and show it graphically as follows:

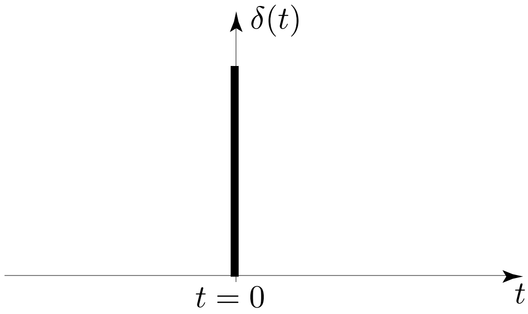


Figure 8

Then,

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= \mathcal{F}\left\{\lim_{a\to 0}\frac{1}{2a}p_a(t)\right\} = \lim_{a\to 0}\mathcal{F}\left\{\frac{1}{2a}p_a(t)\right\}\\ &= \lim_{a\to 0}\frac{\sin\omega a}{\omega a}\\ &= 1.\end{aligned}$$

Here we have assumed that interchanging the order of taking the Fourier transform with the limit operation is valid.

Now consider a shifted unit impulse $\delta(t - t_0)$:

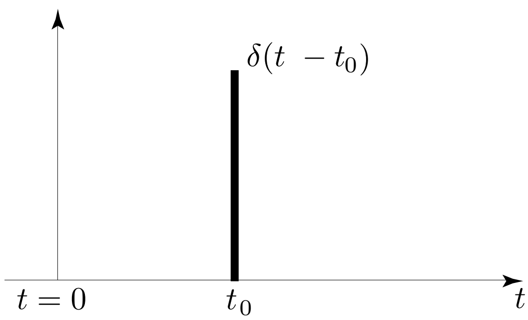


Figure 9

We have, by the time shift property

$$\mathcal{F}\{\delta(t - t_0)\} = e^{-\mathrm{i}\omega t_0}(1) = e^{-\mathrm{i}\omega t_0}.$$

These results are summarized in the following Key Point:

Key Point 4

The Fourier transform of a Unit Impulse

$$\mathcal{F}\{\delta(t-t_0)\} = e^{-\mathrm{i}\omega t_0}.$$

If $\quad t_0 = 0 \qquad$ then $\qquad \mathcal{F}\{\delta(t)\} = 1.$

Task

Apply the duality property to the result

$$\mathcal{F}\{\delta(t)\} = 1.$$

(From the way we have introduced the unit impluse function it must clearly be treated as an **even** function.)

Your solution

Answer

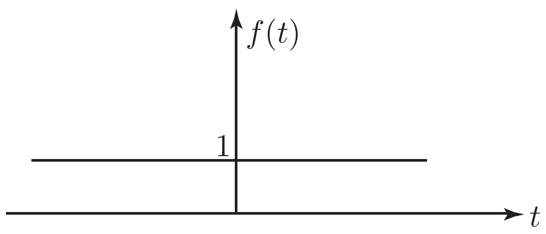
We have $\mathcal{F}\{\delta(t)\} = 1$. Therefore by the duality property

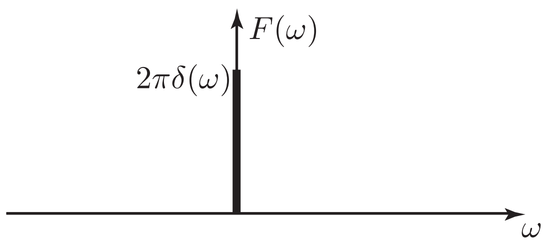
$$\mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega).$$

We see that the signal

$$f(t) = 1, \qquad -\infty < t < \infty$$

which is infinitely wide, has Fourier transform $F(\omega) = 2\pi\delta(\omega)$ which is infinitesimally narrow. This reciprocal effect is characteristic of Fourier transforms.





This result is intuitively plausible since a constant signal would be expected to have a frequency representation which had only a component at zero frequency ($\omega = 0$).

Task

Use the result $\mathcal{F}\{1\} = 2\pi\delta(\omega)$ and the frequency shift property to obtain

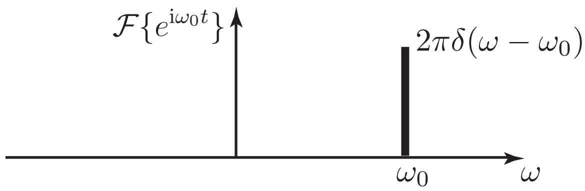
$$\mathcal{F}\{e^{\mathrm{i}\omega_0 t}\}.$$

Your solution

Answer

$\mathcal{F}\{e^{\mathrm{i}\omega_0 t}\} = \mathcal{F}\{e^{\mathrm{i}\omega_0 t}f(t)\}$ where $f(t) = 1, \qquad -\infty < t < \infty.$

But $\mathcal{F}\{f(t)\} = 2\pi\delta(\omega)$, therefore, by the frequency shift property $\mathcal{F}\{e^{\mathrm{i}\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$.



Task

Obtain the Fourier transform of a pure cosine wave

$$f(t) = \cos\omega_0 t \qquad -\infty < t < \infty$$

by writing $f(t)$ in terms of complex exponentials and using the result of the previous Task.

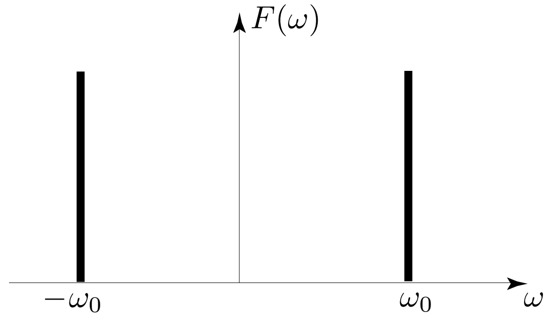
Your solution

Answer

We have $f(t) = \cos\omega_0 t = \frac{1}{2}\left\{e^{\mathrm{i}\omega_0 t} + e^{-\mathrm{i}\omega_0 t}\right\}$

so

$$\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathcal{F}\{e^{\mathrm{i}\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-\mathrm{i}\omega_0 t}\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



Note that because $\displaystyle\int_{-\infty}^{\infty} |\cos\omega_0 t|\,dt$ diverges, one of the Dirichlet conditions is violated. Nevertheless, as we can see via the use of the unit impulse functions, the Fourier transform of $\cos\omega_0 t$ exists.

By similar reasoning we can readily show

$$\mathcal{F}\{\sin\omega_0 t\} = \frac{\pi}{\mathrm{i}}\delta(\omega-\omega_0) - \frac{\pi}{\mathrm{i}}\delta(\omega+\omega_0).$$

Note that the usual results for Fourier transforms of even and odd functions still hold.

5. Fourier transform of the unit step function

We have already pointed out that although

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

we cannot simply replace s by $\mathrm{i}\omega$ to obtain the Fourier transform of the unit step.

We proceed via the Fourier transform of the **signum function** sgn(t) which is defined as

$$\text{sgn } t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

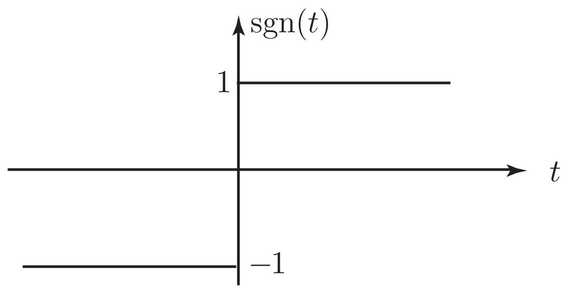


Figure 10

We obtain $\mathcal{F}\{\text{sgn}(t)\}$ as follows.

Consider the odd two-sided exponential function $f_\alpha(t)$ defined as

$$f_\alpha(t) = \begin{cases} e^{-\alpha t} & t > 0 \\ -e^{\alpha t} & t < 0 \end{cases},$$

where $\alpha > 0$:

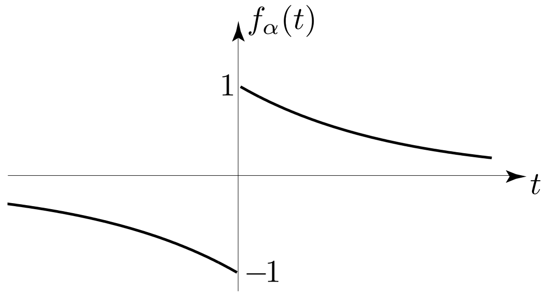


Figure 11

By slightly adapting our earlier calculation for the even two-sided exponential function we find

$$\begin{aligned} \mathcal{F}\{f_\alpha(t)\} &= -\frac{1}{(\alpha - \mathrm{i}\omega)} + \frac{1}{(\alpha + \mathrm{i}\omega)} \\ &= \frac{-(\alpha + \mathrm{i}\omega) + (\alpha - \mathrm{i}\omega)}{\alpha^2 + \omega^2} \\ &= -\frac{2\mathrm{i}\omega}{\alpha^2 + \omega^2}. \end{aligned}$$

The parameter α controls how rapidly the exponential function varies:

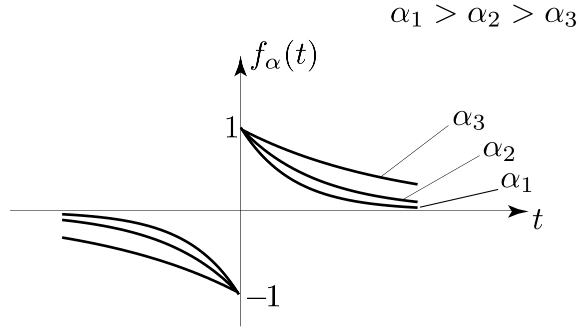


Figure 12

As we let $\alpha \to 0$ the exponential function resembles more and more closely the signum function. This suggests that

$$\begin{aligned} \mathcal{F}\{\mathrm{sgn}(t)\} &= \lim_{\alpha \to 0} \mathcal{F}\{f_\alpha(t)\} \\ &= \lim_{\alpha \to 0} \left(-\frac{2\mathrm{i}\omega}{\alpha^2 + \omega^2} \right) = -\frac{2\mathrm{i}}{\omega} = \frac{2}{\mathrm{i}\omega}. \end{aligned}$$

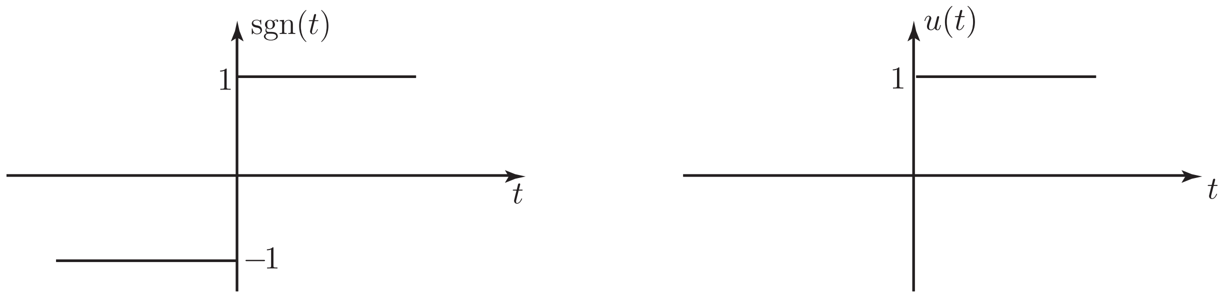
Task

Write the unit step function in terms of the signum function and hence obtain $\mathcal{F}\{u(t)\}$.

First express $u(t)$ in terms of sgn(t):

Your solution

Answer
From the graphs



the step function can be obtained by adding 1 to the signum function for all t and then dividing the resulting function by 2 i.e.

$$u(t) = \frac{1}{2}(1 + \mathsf{sgn}(t)).$$

Now, using the linearity property of Fourier transforms and previously obtained Fourier transforms, find $\mathcal{F}\{u(t)\}$:

Your solution

Answer

We have, using linearity,

$$\mathcal{F}\{u(t)\} \quad = \quad \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathsf{sgn}(t)\} = \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{\mathrm{i}\omega} = \pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}$$

Thus, the Fourier transform of the unit step function contains the additional impulse term $\pi\delta(\omega)$ as well as the odd term $\dfrac{1}{\mathrm{i}\omega}$.

Exercises

1. Use Parserval's theorem and the Fourier transform of a 'two-sided' exponential function to show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{(a^2+\omega^2)^2} = \frac{\pi}{2|a|^3}$$

2. Using $\mathcal{F}\{\mathsf{sgn}(t)\} = \dfrac{2}{\mathrm{i}\omega}$ find the Fourier transforms of (a) $f_1(t) = \dfrac{1}{t}$ (b) $f_2(t) = |t|$

   Hence obtain the transforms of (c) $f_3(t) = -\dfrac{1}{t^2}$ (d) $f_4(t) = \dfrac{2}{t^3}$

3. Show that

$$\mathcal{F}\{\sin\omega_0 t\} = \mathrm{i}\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

   Verify your result using inverse Fourier transform properties.

Answers

2 (a) $\mathcal{F}\{\dfrac{1}{t}\} = -\pi\mathrm{i}\ \mathsf{sgn}(\omega)$ (by the duality property)

(b) $\mathcal{F}\{|t|\} = -\dfrac{2}{\omega^2}$

(c) $\mathcal{F}\{-\dfrac{1}{t^2}\} = \pi\omega\ \mathsf{sgn}(\omega) = \begin{cases} \pi\omega, & \omega > 0 \\ -\pi\omega, & \omega < 0 \end{cases}$

(d) $\mathcal{F}\{\dfrac{1}{t^3}\} = \dfrac{\mathrm{i}\pi\omega^2}{2}\mathsf{sgn}(\omega)$

(Using time differentiation property in (b), (c) and (d).)