Contents 24

Fourier Transforms

Learning outcomes

In this Workbook you will learn about the Fourier transform which has many applications in science and engineering. You will learn how to find Fourier transforms of some standard functions and some of the properties of the Fourier transform. You will learn about the inverse Fourier transform and how to find inverse transforms directly and by using a table of transforms. Finally, you will learn about some special Fourier transform pairs.

The Fourier Transform 24.1

Introduction

Fourier transforms have for a long time been a basic tool of applied mathematics, particularly for solving differential equations (especially partial differential equations) and also in conjunction with integral equations.

There are really three Fourier transforms, the Fourier Sine and Fourier Cosine transforms and a complex form which is usually referred to as *the* Fourier transform.

The last of these transforms in particular has extensive applications in Science and Engineering, for example in physical optics, chemistry (e.g. in connection with Nuclear Magnetic Resonance and Crystallography), Electronic Communications Theory and more general Linear Systems Theory.

Prerequisites

Before starting this Section you should . . .

- be familiar with basic Fourier series, particularly in the complex form

Learning Outcomes

On completion you should be able to . . .

- calculate simple Fourier transforms from the definition
- state how the Fourier transform of a function (signal) depends on whether that function is even or odd or neither

1. The Fourier transform

Unlike Fourier series, which are mainly useful for periodic functions, the Fourier transform permits alternative representations of mostly non-periodic functions.

We shall firstly derive the Fourier transform from the complex exponential form of the Fourier series and then study its various properties.

2. Informal derivation of the Fourier transform

Recall that if $f(t)$ is a period T function, which we will temporarily re-write as $f_T(t)$ for emphasis, then we can expand it in a complex Fourier series,

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} \tag{1}$$

where $\omega_0 = \dfrac{2\pi}{T}$. In words, harmonics of frequency $n\omega_0 = n\dfrac{2\pi}{T} \quad n = 0, \pm 1, \pm 2, \ldots$ are present in the series and these frequencies are separated by

$$n\omega_0 - (n-1)\omega_0 = \omega_0 = \frac{2\pi}{T}.$$

Hence, as T increases the frequency separation becomes smaller and can be conveniently written as $\Delta\omega$. This suggests that as $T \to \infty$, corresponding to a non-periodic function, then $\Delta\omega \to 0$ and the frequency representation contains **all** frequency harmonics.

To see this in a little more detail, we recall (HELM 23: Fourier series) that the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-\mathrm{i}n\omega_0 t}\,dt. \tag{2}$$

Putting $\dfrac{1}{T}$ as $\dfrac{\omega_0}{2\pi}$ and then substituting (2) in (1) we get

$$f_T(t) = \sum_{n=-\infty}^{\infty}\left\{\frac{\omega_0}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-\mathrm{i}n\omega_0 t}\,dt\right\}e^{\mathrm{i}n\omega_0 t}.$$

In view of the discussion above, as $T \to \infty$ we can put ω_0 as $\Delta\omega$ and replace the sum over the discrete frequencies $n\omega_0$ by an integral over all frequencies. We replace $n\omega_0$ by a general frequency variable ω. We then obtain the double integral representation

$$f(t) = \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,dt\right\}e^{\mathrm{i}\omega t}\,d\omega. \tag{3}$$

The inner integral (over all t) will give a function dependent only on ω which we write as $F(\omega)$. Then (3) can be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\,d\omega \tag{4}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,dt. \qquad (5)$$

The representation (4) of $f(t)$ which involves all frequencies ω can be considered as the equivalent for a non-periodic function of the complex Fourier series representation (1) of a periodic function.

The expression (5) for $F(\omega)$ is analogous to the relation (2) for the Fourier coefficients c_n.

The function $F(\omega)$ is called the **Fourier transform** of the function $f(t)$. Symbolically we can write

$$F(\omega) = \mathcal{F}\{f(t)\}.$$

Equation (4) enables us, in principle, to write $f(t)$ in terms of $F(\omega)$. $f(t)$ is often called the **inverse Fourier transform** of $F(\omega)$ and we denote this by writing

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}.$$

Looking at the basic relation (3) it is clear that the position of the factor $\dfrac{1}{2\pi}$ is somewhat arbitrary in (4) and (5). If instead of (5) we define

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,dt.$$

then (4) must be written

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\,d\omega.$$

A third, more symmetric, alternative is to write

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,dt$$

and, consequently:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\,d\omega.$$

We shall use (4) and (5) throughout this Section but you should be aware of these other possibilities which might be used in other texts.

Engineers often refer to $F(\omega)$ (whichever precise definition is used!) as the **frequency domain** representation of a function or signal and $f(t)$ as the **time domain** representation. In what follows we shall use this language where appropriate. However, (5) is really a mathematical transformation for obtaining one function from another and (4) is then the inverse transformation for recovering the initial function. In some applications of Fourier transforms (which we shall not study) the time/frequency interpretations are not relevant. However, in engineering applications, such as communications theory, the frequency representation is often used very literally.

As can be seen above, notationally we will use capital letters to denote Fourier transforms: thus a function $f(t)$ has a Fourier transform denoted by $F(\omega)$, $g(t)$ has a Fourier transform written $G(\omega)$ and so on. The notation $F(\mathrm{i}\omega)$, $G(\mathrm{i}\omega)$ is used in some texts because ω occurs in (5) only in the term $e^{-\mathrm{i}\omega t}$.

3. Existence of the Fourier transform

We will discuss this question in a little detail at a later stage when we will also consider briefly the relation between the Fourier transform and the Laplace Transform (HELM 20). For now we will use (5) to obtain the Fourier transforms of some important functions.

Example 1

Find the Fourier transform of the one-sided exponential function

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant, shown below:

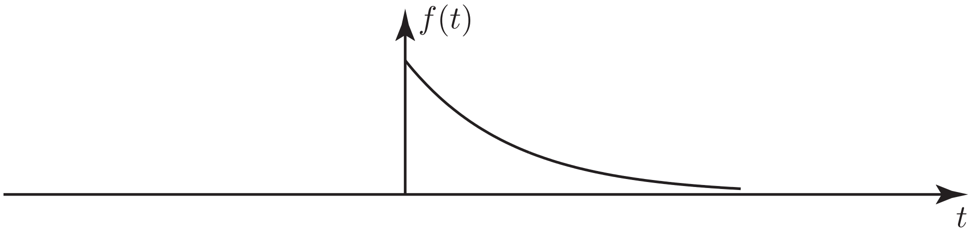


Figure 1

Solution

Using (5) then by straightforward integration

$$\begin{aligned} F(\omega) &= \int_0^\infty e^{-\alpha t} e^{-\mathrm{i}\omega t}\, dt \qquad \text{(since } f(t) = 0 \text{ for } t < 0\text{)} \\ &= \int_0^\infty e^{-(\alpha + \mathrm{i}\omega t)}\, dt \\ &= \left[\frac{e^{-(\alpha+\mathrm{i}\omega)t}}{-(\alpha + \mathrm{i}\omega)} \right]_0^\infty \\ &= \frac{1}{\alpha + \mathrm{i}\omega} \end{aligned}$$

since $e^{-\alpha t} \to 0$ as $t \to \infty$ for $\alpha > 0$.

This important Fourier transform is written in the following Key Point:

Key Point 1

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega}, \qquad \alpha > 0.$$

Note that this **real** function has a **complex** Fourier transform.

Note that if $u(t)$ is used to denote the **Heaviside unit step function**:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

then we can write the function in Example 1 as: $f(t) = e^{-\alpha t}u(t)$. We shall frequently use this concise notation for one-sided functions.

Task

Write down the Fourier transforms of

(a) $e^{-t}u(t)$ (b) $e^{-3t}u(t)$ (c) $e^{-\frac{t}{2}}u(t)$

Use Key Point 1:

Your solution

(a)

(b)

(c)

Answer

(a) $\alpha = 1$ so $\mathcal{F}\{e^{-t}u(t)\} = \dfrac{1}{1 + \mathrm{i}\omega}$

(b) $\alpha = 3$ so $\mathcal{F}\{e^{-3t}u(t)\} = \dfrac{1}{3 + \mathrm{i}\omega}$

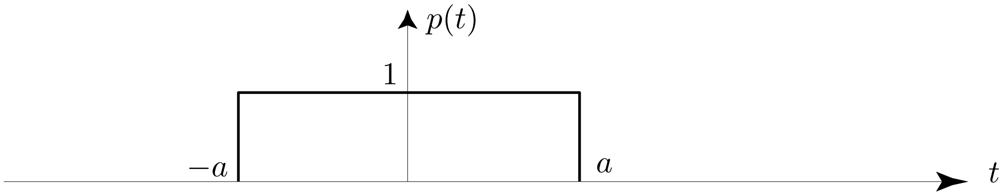
(c) $\alpha = \frac{1}{2}$ so $\mathcal{F}\{e^{-\frac{t}{2}}u(t)\} = \dfrac{1}{\frac{1}{2} + \mathrm{i}\omega}$

Task

Obtain, using the integral definition (5), the Fourier transform of the rectangular pulse

$$p(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases} .$$

Note that the pulse width is $2a$ as indicated in the diagram below.



First use (5) to write down the integral from which the transform will be calculated:

Your solution

Answer

$P(\omega) \equiv \mathcal{F}\{p(t)\} = \displaystyle\int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}\, dt$ using the definition of $p(t)$

Now evaluate this integral and write down the final Fourier transform in trigonometric, rather than complex exponential form:

Your solution

Answer

$$\begin{aligned} P(\omega) &= \int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}\, dt = \left[\frac{e^{-\mathrm{i}\omega t}}{(-\mathrm{i}\omega)}\right]_{-a}^{a} = \frac{e^{-\mathrm{i}\omega a} - e^{+\mathrm{i}\omega a}}{(-\mathrm{i}\omega)} \\ &= \frac{(\cos\omega a - \mathrm{i}\sin\omega a) - (\cos\omega a + \mathrm{i}\sin\omega a)}{(-\mathrm{i}\omega)} = \frac{2\mathrm{i}\sin\omega a}{\mathrm{i}\omega} \end{aligned}$$

i.e.

$$P(\omega) = \mathcal{F}\{p(t)\} = \frac{2\sin\omega a}{\omega} \qquad (6)$$

Note that in this case the Fourier transform is **wholly real**.

Engineers often call the function $\dfrac{\sin x}{x}$ the **sinc function**. Consequently if we write, the transform (6) of the rectangular pulse as

$$P(\omega) = 2a\frac{\sin \omega a}{\omega a},$$

we can say

$$P(\omega) = 2a\,\mathsf{sinc}(\omega a).$$

Using the result (6) in (4) we have the **Fourier integral representation** of the rectangular pulse.

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin \omega a}{\omega} e^{\mathrm{i}\omega t}\,d\omega.$$

As we have already mentioned, this corresponds to a Fourier series representation for a periodic function.

Key Point 2

The Fourier transform of a Rectangular Pulse

If $p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$ then:

$$\mathcal{F}\{p_a(t)\} = 2a\frac{\sin \omega a}{\omega a} = 2a\,\mathsf{sinc}(\omega a)$$

Clearly, if the rectangular pulse has width 2, corresponding to $a = 1$ we have:

$$P_1(\omega) \equiv \mathcal{F}\{p_1(t)\} = 2\frac{\sin \omega}{\omega}.$$

As $\omega \to 0$, then $2\dfrac{\sin \omega}{\omega} \to 2$. Also, the function $2\dfrac{\sin \omega}{\omega}$ is an even function being the product of two odd functions $2\sin\omega$ and $\dfrac{1}{\omega}$. The graph of $P_1(\omega)$ is as follows:

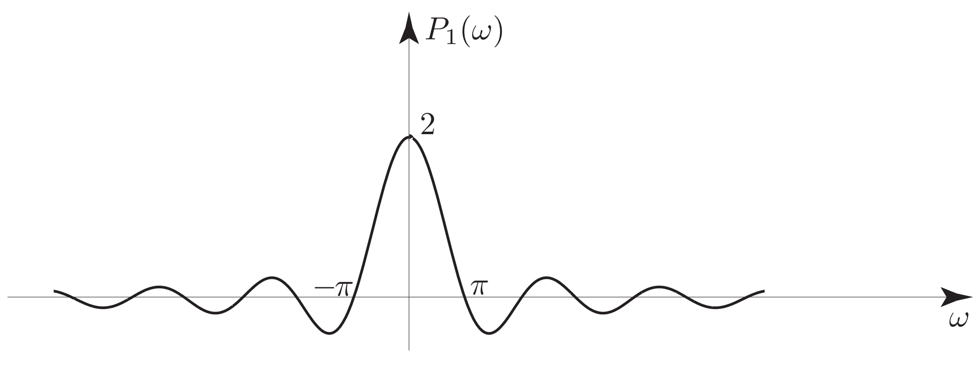


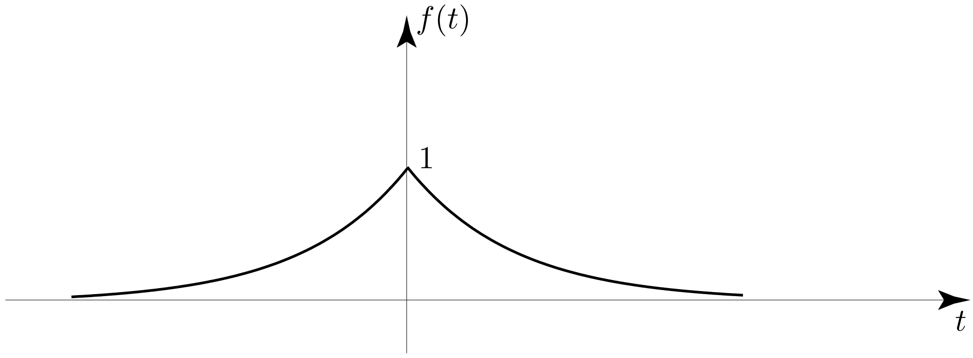
Figure 2

Task

Obtain the Fourier transform of the **two sided** exponential function

$$f(t) = \begin{cases} e^{\alpha t} & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant.



Your solution

Answer

We must separate the range of the integrand into $[-\infty, 0]$ and $[0, \infty]$ since the function $f(t)$ is defined separately in these two regions: then

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{0} e^{\alpha t} e^{-\mathrm{i}\omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-\mathrm{i}\omega t}\, dt = \int_{-\infty}^{0} e^{(\alpha - \mathrm{i}\omega)t}\, dt + \int_{0}^{\infty} e^{-(\alpha + \mathrm{i}\omega)t}\, dt \\ &= \left[\frac{e^{(\alpha - \mathrm{i}\omega)t}}{(\alpha - \mathrm{i}\omega)} \right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha + \mathrm{i}\omega)t}}{-(\alpha + \mathrm{i}\omega)} \right]_{0}^{\infty} \\ &= \frac{1}{\alpha - \mathrm{i}\omega} + \frac{1}{\alpha + \mathrm{i}\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}. \end{aligned}$$

Note that, as in the case of the rectangular pulse, we have here a real even function of t giving a Fourier transform which is wholly real. Also, in both cases, the Fourier transform is an **even** (as well as real) function of ω.

Note also that it follows from the above calculation that

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} \quad \text{(as we have already found)}$$

and

$$\mathcal{F}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - \mathrm{i}\omega} \quad \text{where} \quad e^{\alpha t}u(-t) = \begin{cases} e^{\alpha t} & t < 0 \\ 0 & t > 0 \end{cases} .$$

4. Basic properties of the Fourier transform

Real and imaginary parts of a Fourier transform

Using the definition (5) we have,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\, dt.$$

If we write $e^{-\mathrm{i}\omega t} = \cos\omega t - \mathrm{i}\sin\omega t$, then

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\ dt - \mathrm{i}\int_{-\infty}^{\infty} f(t)\sin\omega t\ dt$$

where both integrals are real, assuming that $f(t)$ is real. Hence the real and imaginary parts of the Fourier transform are:

$$\text{Re}\ (F(\omega)) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt \qquad\qquad \text{Im}\ (F(\omega)) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt.$$

Task

Recalling that if $h(t)$ is even and $g(t)$ is odd then $\displaystyle\int_{-a}^{a} h(t)\, dt = 2\int_{0}^{a} h(t)\, dt$ and $\displaystyle\int_{-a}^{a} g(t)\, dt = 0$, deduce $\text{Re}(F(\omega))$ and $\text{Im}(F(\omega))$ if

(a) $f(t)$ is a real even function

(b) $f(t)$ is a real odd function.

Your solution

(a)

Answer
If $f(t)$ is real and even

$$R(\omega) \equiv \text{Re } F(\omega) = 2\int_0^\infty f(t)\cos\omega t\,dt \quad \text{(because the integrand is even)}$$

$$I(\omega) \equiv \text{Im } F(\omega) = -\int_{-\infty}^\infty f(t)\sin\omega t\,dt = 0 \quad \text{(because the integrand is odd).}$$

Thus, any real even function $f(t)$ has a wholly real Fourier transform. Also since

$$\cos((-\omega)t) = \cos(-\omega t) = \cos\omega t$$

the Fourier transform in this case will be a real even function.

Your solution
(b)

Answer
Now

$$\text{Re } F(\omega) = \int_{-\infty}^\infty f(t)\cos\omega t\,dt = \int_{-\infty}^\infty \text{(odd)} \times \text{(even)}\,dt = \int_{-\infty}^\infty \text{(odd)}\,dt = 0$$

and

$$\text{Im } F(\omega) = -\int_{-\infty}^\infty f(t)\sin\omega t\,dt = -2\int_0^\infty f(t)\sin\omega t\,dt$$

(because the integrand is (odd)×(odd)=(even)).

Also since $\sin((-\omega)t) = -\sin\omega t$, the Fourier transform in this case is an odd function of ω.

These results are summarised in the following Key Point:

Key Point 3

$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
real and even	real and even
real and odd	purely imaginary and odd
neither even nor odd	complex, $F(\omega) = R(\omega) + \mathrm{i}I(\omega)$

Polar form of a Fourier transform

Task

The one-sided exponential function $f(t) = e^{-\alpha t}u(t)$ has Fourier transform $F(\omega) = \dfrac{1}{\alpha + \mathrm{i}\omega}$. Find the real and imaginary parts of $F(\omega)$.

Your solution

Answer

$$F(\omega) = \frac{1}{\alpha + \mathrm{i}\omega} = \frac{\alpha - \mathrm{i}\omega}{\alpha^2 + \omega^2}.$$

Hence $R(\omega) = \text{Re } F(\omega) = \dfrac{\alpha}{\alpha^2 + \omega^2}$ $\qquad I(\omega) = \text{Im } F(\omega) = \dfrac{-\omega}{\alpha^2 + \omega^2}$

We can rewrite $F(\omega)$, like any other complex quantity, in **polar** form by calculating the magnitude and the argument (or phase). For the Fourier transform in the last Task

$$|F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)} = \sqrt{\frac{\alpha^2 + \omega^2}{(\alpha^2 + \omega^2)^2}} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$

and $\qquad \arg F(\omega) = \tan^{-1} \dfrac{I(\omega)}{R(\omega)} = \tan^{-1}\left(\dfrac{-\omega}{\alpha}\right).$

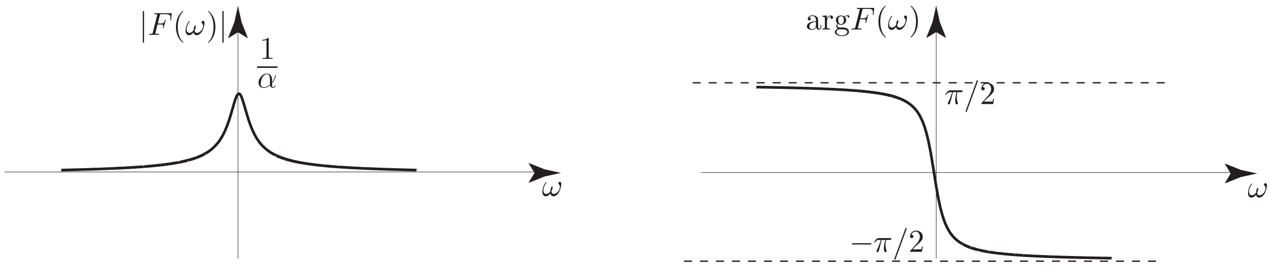


Figure 3

In general, a Fourier transform whose Cartesian form is $F(\omega) = R(\omega) + \mathrm{i}I(\omega)$ has a polar form $F(\omega) = |F(\omega)|e^{\mathrm{i}\phi(\omega)}$ where $\phi(\omega) \equiv \arg F(\omega)$.

Graphs, such as those shown in Figure 3, of $|F(\omega)|$ and arg $F(\omega)$ plotted against ω, are often referred to as **magnitude** spectra and **phase** spectra, respectively.

Exercises

1. Obtain the Fourier transform of the rectangular pulses

(a) $f(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$ (b) $f(t) = \begin{cases} \frac{1}{4} & |t| \leq 3 \\ 0 & |t| > 3 \end{cases}$

2. Find the Fourier transform of

$$f(t) = \begin{cases} 1 - \frac{t}{2} & 0 \leq t \leq 2 \\ 1 + \frac{t}{2} & -2 \leq t \leq 0 \\ 0 & |t| > 2 \end{cases}$$

Answers

1.(a) $F(\omega) = \frac{2}{\omega}\sin\omega$

(b) $F(\omega) = \frac{\sin 3\omega}{2\omega}$

2. $\frac{1 - \cos 2\omega}{\omega^2}$