An Application of Fourier Series

23.7

Introduction

In this Section we look at a typical application of Fourier series. The problem we study is that of a differential equation with a periodic (but non-sinusoidal) forcing function. The differential equation chosen models a lightly damped vibrating system.

Prerequisites

Before starting this Section you should ...

- know how to obtain a Fourier series
- be competent to use complex numbers
- be familiar with the relation between the exponential function and the trigonometric functions

Learning Outcomes

On completion you should be able to ...

- solve a linear differential equation with a periodic forcing function using Fourier series

1. Modelling vibration by differential equation

Vibration problems are often modelled by ordinary differential equations with constant coefficients. For example the motion of a spring with stiffness k and damping constant c is modelled by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad (1)$$

where $y(t)$ is the displacement of a mass m connected to the spring. It is well-known that if $c^2 < 4mk$, usually referred to as the lightly damped case, then

$$y(t) = \mathrm{e}^{-\alpha t}(A\cos\omega t + B\sin\omega t) \qquad (2)$$

i.e. the motion is sinusoidal but damped by the negative exponential term. In (2) we have used the notation

$$\alpha = \frac{c}{2m} \qquad \omega = \frac{1}{2m}\sqrt{4km - c^2} \quad \text{to simplify the equation.}$$

The values of A and B depend upon initial conditions.

The system represented by (1), whose solution is (2), is referred to as an **unforced damped harmonic oscillator**.

A lightly damped oscillator driven by a time-dependent forcing function $F(t)$ is modelled by the differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \qquad (3)$$

The solution or **system response** in (3) has two parts:

(a) A **transient** solution of the form (2),

(b) A **forced** or **steady state** solution whose form, of course, depends on $F(t)$.

If $F(t)$ is sinusoidal such that

$$F(t) = A\sin(\Omega t + \phi) \quad \text{where } \Omega \text{ and } \phi \text{ are constants,}$$

then the steady state solution is fairly readily obtained by standard techniques for solving differential equations. If $F(t)$ is periodic but non-sinusoidal then Fourier series may be used to obtain the steady state solution. The method is based on the **principle of superposition** which is actually applicable to any linear (homogeneous) differential equation. (Another engineering application is the series LCR circuit with an applied periodic voltage.)

The principle of superposition is easily demonstrated:-
Let $y_1(t)$ and $y_2(t)$ be the steady state solutions of (3) when $F(t) = F_1(t)$ and $F(t) = F_2(t)$ respectively. Then

$$m\frac{d^2y_1}{dt^2} + c\frac{dy_1}{dt} + ky_1 = F_1(t)$$

$$m\frac{d^2y_2}{dt^2} + c\frac{dy_2}{dt} + ky_2 = F_2(t)$$

Simply adding these equations we obtain

$$m\frac{d^2}{dt^2}(y_1 + y_2) + c\frac{d}{dt}(y_1 + y_2) + k(y_1 + y_2) = F_1(t) + F_2(t)$$

from which it follows that if $F(t) = F_1(t) + F_2(t)$ then the system response is the sum $y_1(t) + y_2(t)$. This, in its simplest form, is the principle of superposition. More generally if the forcing function is

$$F(t) = \sum_{n=1}^{N} F_n(t)$$

then the response is $y(t) = \sum_{n=1}^{N} y_n(t)$ where $y_n(t)$ is the response to the forcing function $F_n(t)$.

Returning to the specific case where $F(t)$ is periodic, the solution procedure for the steady state response is as follows:

Step 1: Obtain the Fourier series of $F(t)$.

Step 2: Solve the differential equation (3) for the response $y_n(t)$ corresponding to the n th harmonic in the Fourier series. (The response y_o to the constant term, if any, in the Fourier series may have to be obtained separately.)

Step 3: Superpose the solutions obtained to give the overall steady state motion:

$$y(t) = y_0(t) + \sum_{n=1}^{N} y_n(t)$$

The procedure can be lengthy but the solution is of great engineering interest because if the frequency of one harmonic in the Fourier series is close to the **natural frequency** $\sqrt{\dfrac{k}{m}}$ of the undamped system then the response to that harmonic will dominate the solution.

2. Applying Fourier series to solve a differential equation

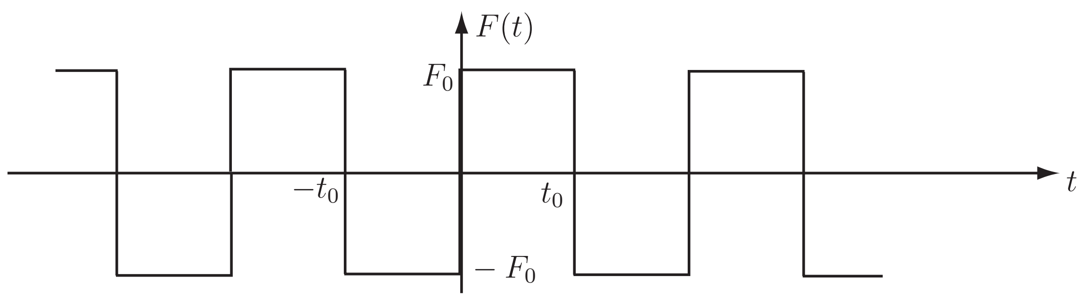
The following Task which is quite long will provide useful practice in applying Fourier series to a practical problem. Essentially you should follow Steps 1 to 3 above carefully.

Task

The problem is to find the steady state response $y(t)$ of a spring/mass/damper system modelled by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \qquad (4)$$

where $F(t)$ is the **periodic square wave** function shown in the diagram.



Step 1: Obtain the Fourier series of $F(t)$ noting that it is an odd function:

Your solution

Answer

The calculation is similar to those you have performed earlier in this Workbook.

Since $F(t)$ is an odd function and has period $2t_0$ so that $\omega = \dfrac{2\pi}{2t_0} = \dfrac{\pi}{t_0}$, it has Fourier coefficients:

$$\begin{aligned}
b_n &= \frac{2}{t_0}\int_0^{t_o} F_0 \sin\left(\frac{n\pi t}{t_0}\right)\,dt \qquad n = 1, 2, 3, \ldots \\
&= \left(\frac{2F_0}{t_0}\right)\left(\frac{t_0}{n\pi}\right)\left[-\cos\frac{n\pi t}{t_0}\right]_0^{t_0} \\
&= \frac{2F_0}{n\pi}(1-\cos n\pi) = \begin{cases} \dfrac{4F_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}$$

so $\quad F(t) = \dfrac{4F_0}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin n\omega t}{n}$ $\qquad$ (where the sum is over odd n only).

Step 2(a):

Since each term in the Fourier series is a sine term you must now solve (4) to find the steady state response y_n to the n th harmonic input: $\quad F_n(t) = b_n \sin n\omega t \qquad n = 1, 3, 5, \ldots$

From the basic theory of linear differential equations this response has the form

$$y_n = A_n \cos n\omega t + B_n \sin n\omega t \tag{5}$$

where A_n and B_n are coefficients to be determined by substituting (5) into (4) with $F(t) = F_n(t)$. Do this to obtain simultaneous equations for A_n and B_n:

Your solution

Answer

We have, differentiating (5),

$$\begin{aligned} y_n' &= n\omega(-A_n \sin n\omega t + B_n \cos n\omega t) \\ y_n'' &= (n\omega)^2(-A_n \cos n\omega t - B_n \sin n\omega t) \end{aligned}$$

from which, substituting into (4) and collecting terms in $\cos n\omega t$ and $\sin n\omega t$,

$$(-m(n\omega)^2A_n + cn\omega B_n + kA_n)\cos n\omega t + (-m(n\omega)^2B_n - cn\omega A_n + kB_n)\sin n\omega t = b_n \sin n\omega t$$

Then, by comparing coefficients of $\cos n\omega t$ and $\sin n\omega t$, we obtain the simultaneous equations:

$$(k - m(n\omega)^2)A_n + c(n\omega)B_n = 0 \qquad (6)$$

$$-c(n\omega)A_n + (k - m(n\omega)^2)B_n = b_n \qquad (7)$$

Step 2(b):

Now solve (6) and (7) to obtain A_n and B_n:

Your solution

Answer

$$A_n = -\frac{c\omega_n b_n}{(k - m\omega_n^2)^2 + \omega_n^2 c^2} \qquad (8)$$

$$B_n = \frac{(k - m\omega_n^2) b_n}{(k - m\omega_n^2)^2 + \omega_n^2 c^2} \qquad (9)$$

where we have written ω_n for $n\omega$ as the frequency of the n th harmonic

It follows that the steady state response y_n to the n th harmonic of the Fourier series of the forcing function is given by (5). The amplitudes A_n and B_n are given by (8) and (9) respectively in terms of the systems parameters k, c, m, the frequency ω_n of the harmonic and its amplitude b_n. In practice it is more convenient to represent y_n in the so-called **amplitude/phase** form:

$$y_n = C_n \sin(\omega_n t + \phi_n) \qquad (10)$$

where, from (5) and (10),

$$A_n \cos\omega_n t + B_n \sin\omega_n t = C_n(\cos\phi_n \sin\omega_n t + \sin\phi_n \cos\omega_n t).$$

Hence

$$C_n \sin\phi_n = A_n \qquad C_n \cos\phi_n = B_n$$

so

$$\tan\phi_n = \frac{A_n}{B_n} = \frac{c\omega_n}{(m\omega_n^2 - k)^2} \qquad (11)$$

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{b_n}{\sqrt{(m\omega_n^2 - k)^2 + \omega_n^2 c^2}} \qquad (12)$$

Step 3:

Finally, use the superposition principle, to state the complete steady state response of the system to the periodic square wave forcing function:

Your solution

Answer

$$y(t) = \sum_{n=1}^{\infty} y_n(t) = \sum_{\substack{n=1 \\ (n \text{ odd})}} C_n(\sin\omega_n t + \phi_n)$$ where C_n and ϕ_n are given by (11) and (12).

In practice, since $b_n = \dfrac{4F_0}{n\pi}$ it follows that the amplitude C_n also decreases as $\dfrac{1}{n}$. However, if one of the harmonic frequencies say ω_n' is close to the natural frequency $\sqrt{\dfrac{k}{m}}$ of the undamped oscillator then that particular frequency harmonic will dominate in the steady state response. The particular value ω_n' will, of course, depend on the values of the system parameters k and m.