# The Complex Form 23.6

## Introduction

In this Section we show how a Fourier series can be expressed more concisely if we introduce the complex number i where $\mathrm{i}^2 = -1$. By utilising the Euler relation:

$$e^{\mathrm{i}\theta} \equiv \cos\theta + \mathrm{i}\sin\theta$$

we can replace the trigonometric functions by complex exponential functions. By also combining the Fourier coefficients $a_n$ and $b_n$ into a complex coefficient $c_n$ through

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n)$$

we find that, for a given periodic signal, both sets of constants can be found in one operation.

We also obtain Parseval's theorem which has important applications in electrical engineering.

The complex formulation of a Fourier series is an important precursor of the Fourier transform which attempts to Fourier analyse non-periodic functions.

## Prerequisites

Before starting this Section you should ...

- know how to obtain a Fourier series
- be competent working with the complex numbers
- be familiar with the relation between the exponential function and the trigonometric functions

## Learning Outcomes

On completion you should be able to ...

- express a periodic function in terms of its Fourier series in complex form
- understand Parseval's theorem

# 1. Complex exponential form of a Fourier series

So far we have discussed the **trigonometric** form of a Fourier series i.e. we have represented functions of period $T$ in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right\}.$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

we obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals:

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_0 t\; dt \qquad n = 0, 1, 2, \ldots \qquad (1)$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_0 t\; dt \qquad n = 1, 2, \ldots \qquad (2)$$

An alternative, more concise form, of a Fourier series is available using **complex quantities**. This form is quite widely used by engineers, for example in Circuit Theory and Control Theory, and leads naturally into the Fourier Transform which is the subject of HELM 24.

# 2. Revision of the exponential form of a complex number

Recall that a complex number in **Cartesian** form which is written as

$$z = a + \mathrm{i}b,$$

where $a$ and $b$ are real numbers and $\mathrm{i}^2 = -1$, can be written in **polar** form as

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta$, the **argument** or **phase** of $z$, is such that

$$a = r\cos\theta \qquad\qquad b = r\sin\theta.$$

A more concise version of the polar form of $z$ can be obtained by defining a **complex exponential** quantity $e^{\,\mathrm{i}\theta}$ by Euler's relation

$$e^{\,\mathrm{i}\theta} \equiv \cos\theta + \mathrm{i}\sin\theta$$

The polar angle $\theta$ is normally expressed in **radians**. Replacing $\mathrm{i}$ by $-\mathrm{i}$ we obtain the alternative form

$$e^{-\,\mathrm{i}\theta} \equiv \cos\theta - \mathrm{i}\sin\theta$$

**Task**

Write down in $\cos\theta \pm \mathrm{i}\sin\theta$ form and also in Cartesian form (a) $e^{\,\mathrm{i}\pi/6}$ (b) $e^{-\,\mathrm{i}\pi/6}$.

Use Euler's relation:

**Your solution**

**Answer**

We have, by definition,

(a) $e^{\,\mathrm{i}\pi/6} = \cos\left(\frac{\pi}{6}\right) + \,\mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i}$ (b) $e^{-\,\mathrm{i}\pi/6} = \cos\left(\frac{\pi}{6}\right) - \,\mathrm{i}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\,\mathrm{i}$

**Task**

Write down (a) $\cos\left(\frac{\pi}{6}\right)$ (b) $\sin\left(\frac{\pi}{6}\right)$ in terms of $e^{\,\mathrm{i}\pi/6}$ and $e^{-\,\mathrm{i}\pi/6}$.

**Your solution**

**Answer**

We have, adding the two results from the previous task

$$e^{\,\mathrm{i}\pi/6} + e^{-\,\mathrm{i}\pi/6} = 2\cos\left(\frac{\pi}{6}\right) \qquad \text{or} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\left(e^{\,\mathrm{i}\pi/6} + e^{-\,\mathrm{i}\pi/6}\right)$$

Similarly, subtracting the two results,

$$e^{\,\mathrm{i}\pi/6} - e^{-\,\mathrm{i}\pi/6} = 2\,\mathrm{i}\sin\left(\frac{\pi}{6}\right) \qquad \text{or} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2\,\mathrm{i}}\left(e^{\,\mathrm{i}\pi/6} - e^{-\,\mathrm{i}\pi/6}\right)$$

(Don't forget the factor i in this latter case.)

Clearly, similar calculations could be carried out for any angle $\theta$. The general results are summarised in the following Key Point.

## Key Point 8

**Euler's Relations**

$$\mathrm{e}^{\mathrm{i}\theta} \equiv \cos\theta + \mathrm{i}\sin\theta, \qquad \mathrm{e}^{-\mathrm{i}\theta} \equiv \cos\theta - \mathrm{i}\sin\theta$$

$$\cos\theta \equiv \frac{1}{2}\left(e^{\,\mathrm{i}\theta} + e^{-\,\mathrm{i}\theta}\right) \qquad \sin\theta \equiv \frac{1}{2\,\mathrm{i}}\left(e^{\,\mathrm{i}\theta} - e^{-\,\mathrm{i}\theta}\right)$$

Using these results we can redraft an expression of the form

$$a_n \cos n\theta + b_n \sin n\theta$$

in terms of complex exponentials.
(This expression, with $\theta = \omega_0 t$, is of course the $n^{\text{th}}$ harmonic of a trigonometric Fourier series.)

**Task**

Using the results from the Key Point 8 (with $n\theta$ instead of $\theta$) rewrite

$$a_n \cos n\theta + b_n \sin n\theta$$

in complex exponential form.

First substitute for $\cos n\theta$ and $\sin n\theta$ with exponential expressions using Key Point 8:

**Your solution**

**Answer**
We have

$$a_n \cos n\theta = \frac{a_n}{2}\left(e^{\,\mathrm{i}n\theta} + e^{-\,\mathrm{i}n\theta}\right) \qquad b_n \sin n\theta = \frac{b_n}{2\,\mathrm{i}}\left(e^{\,\mathrm{i}n\theta} - e^{-\,\mathrm{i}n\theta}\right)$$

so

$$a_n \cos n\theta + b_n \sin n\theta = \frac{a_n}{2}\left(e^{\,\mathrm{i}n\theta} + e^{-\,\mathrm{i}n\theta}\right) + \frac{b_n}{2\,\mathrm{i}}\left(e^{\,\mathrm{i}n\theta} - e^{-\,\mathrm{i}n\theta}\right)$$

Now collect the terms in $e^{\,\mathrm{i}n\theta}$ and in $e^{-\,\mathrm{i}n\theta}$ and use the fact that $\dfrac{1}{\mathrm{i}} = -\mathrm{i}$:

**Your solution**

**Answer**
We get

$$\frac{1}{2}\left(a_n + \frac{b_n}{\mathrm{i}}\right)e^{\,\mathrm{i}n\theta} + \frac{1}{2}\left(a_n - \frac{b_n}{\mathrm{i}}\right)e^{-\,\mathrm{i}n\theta}$$

or, since $\dfrac{1}{\mathrm{i}} = \dfrac{\mathrm{i}}{\mathrm{i}^2} = -\,\mathrm{i}$ $\quad \dfrac{1}{2}(a_n - \,\mathrm{i}b_n)e^{\,\mathrm{i}n\theta} + \dfrac{1}{2}(a_n + \,\mathrm{i}b_n)e^{-\,\mathrm{i}n\theta}$.

Now write this expression in more concise form by defining

$$c_n = \frac{1}{2}(a_n - \,\mathrm{i}b_n) \text{ which has complex conjugate } c_n^* = \frac{1}{2}(a_n + \,\mathrm{i}b_n).$$

Write the concise complex exponential expression for $a_n \cos n\theta + b_n \sin n\theta$:

**Your solution**

**Answer**

$$a_n \cos n\theta + b_n \sin n\theta = c_n e^{\,\mathrm{i}n\theta} + c_n^* e^{-\,\mathrm{i}n\theta}$$

Clearly, we can now rewrite the trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{as} \quad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n e^{\,\mathrm{i}n\omega_0 t} + c_n^* e^{-\,\mathrm{i}n\omega_0 t}\right) \qquad (3)$$

A neater, and particularly concise, form of this expression can be obtained as follows:
Firstly write $\dfrac{a_0}{2} = c_0$ (which is consistent with the general definition of $c_n$ since $b_0 = 0$).
The second term in the summation

$$\sum_{n=1}^{\infty} c_n^* e^{-\,\mathrm{i}n\omega_0 t} = c_1^* e^{-\,\mathrm{i}\omega_0 t} + c_2^* e^{-2\,\mathrm{i}\omega_0 t} + \ldots$$

can be written, if we define $c_{-n} = c_n^* = \frac{1}{2}(a_n + \,\mathrm{i}b_n)$, as

$$c_{-1}e^{-\,\mathrm{i}\omega_0 t} + c_{-2}e^{-2\,\mathrm{i}\omega_0 t} + c_{-3}e^{-3\,\mathrm{i}\omega_0 t} + \ldots = \sum_{n=-1}^{-\infty} c_n e^{\,\mathrm{i}n\omega_0 t}$$

Hence (3) can be written $\;c_0 + \displaystyle\sum_{n=1}^{\infty} c_n e^{\,\mathrm{i}n\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{\,\mathrm{i}n\omega_0 t}$ or in the very concise form $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{\,\mathrm{i}n\omega_0 t}$.

The **complex Fourier coefficients** $c_n$ can be readily obtained as follows using (1) and (2) for $a_n, b_n$. Firstly

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt \qquad (4)$$

For $n = 1, 2, 3, \ldots$ we have

$$c_n = \frac{1}{2}(a_n - \ \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos n\omega_0 t - \ \mathrm{i}\sin n\omega_0 t)\,dt \quad \text{i.e.} \quad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\ \mathrm{i}n\omega_0 t}dt \qquad (5)$$

Also for $n = 1, 2, 3, \ldots$ we have

$$c_{-n} = c_n^* = \frac{1}{2}(a_n + \ \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\ \mathrm{i}n\omega_0 t}dt$$

This last expression is equivalent to stating that for $n = -1, -2, -3, \ldots$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\ \mathrm{i}n\omega_0 t}dt \qquad (6)$$

The three equations (4), (5), (6) can thus all be contained in the one expression

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\ \mathrm{i}n\omega_0 t}dt \qquad \text{for} \quad n = 0, \pm1, \pm2, \pm3, \ldots$$

The results of this discussion are summarised in the following Key Point.

### Key Point 9

**Fourier Series in Complex Form**

A function $f(t)$ of period $T$ has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\ \mathrm{i}n\omega_0 t} \qquad \text{where} \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\ \mathrm{i}n\omega_0 t}dt$$

For the special case $T = 2\pi$, so that $\omega_0 = 1$, these formulae become particularly simple:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\ \mathrm{i}nt} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\ \mathrm{i}nt}\,dt.$$

## 3. Properties of the complex Fourier coefficients

Using properties of the trigonometric Fourier coefficients $a_n$, $b_n$ we can readily deduce the following results for the $c_n$ coefficients:

1. $c_0 = \dfrac{a_0}{2}$ is always real.

2. Suppose the periodic function $f(t)$ is even so that all $b_n$ are zero. Then, since in the complex form the $b_n$ arise as the imaginary part of $c_n$, it follows that for $f(t)$ even the coefficients $c_n$ $(n = \pm 1, \pm 2, \ldots)$ are wholly real.

**Task**

If $f(t)$ is odd, what can you deduce about the Fourier coefficients $c_n$?

**Your solution**

**Answer**
Since, for an odd periodic function the Fourier coefficients $a_n$ (which constitute the real part of $c_n$) are zero, then in this case the complex coefficients $c_n$ are wholly imaginary.

3. Since

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}dt$$

then if $f(t)$ is even, $c_n$ will be real, and we have two possible methods for evaluating $c_n$:

(a) Evaluate the integral above **as it stands** i.e. over the full range $\left(-\dfrac{T}{2}, \dfrac{T}{2}\right)$. Note carefully that the second term in the integrand is neither an even nor an odd function so the integrand itself is

( even function) $\times$ ( neither even nor odd function) $=$ neither even nor odd function.

Thus we **cannot** write $\quad c_n = \dfrac{2}{T}\displaystyle\int_0^{T/2} f(t)e^{-\,\mathrm{i}n\omega_0 t}\,dt$

(b) Put $e^{-\,\mathrm{i}n\omega_0 t} = \cos n\omega_0 t - \,\mathrm{i}\sin n\omega_0 t$ so

$$\begin{aligned} f(t)e^{-\,\mathrm{i}n\omega_0 t} = f(t)\cos n\omega_0 t - \,\mathrm{i}f(t)\sin n\omega_0 t &= (\text{ even})(\text{ even}) - \,\mathrm{i}(\text{ even})(\text{ odd}) \\ &= (\text{ even}) - \,\mathrm{i}(\text{ odd}). \end{aligned}$$

Hence $\quad c_n = \dfrac{2}{T}\displaystyle\int_0^{\frac{T}{2}} f(t)\cos n\omega_0 t\,dt = \dfrac{a_n}{2}.$

4. If $f(t + \dfrac{T}{2}) = -f(t)$ then of course only **odd** harmonic coefficients $c_n\,(n = \pm 1, \pm 3, \pm 5, \ldots)$ will arise in the complex Fourier series just as with trigonometric series.

## Example 4

Find the complex Fourier series of the saw-tooth wave shown in Figure 24:

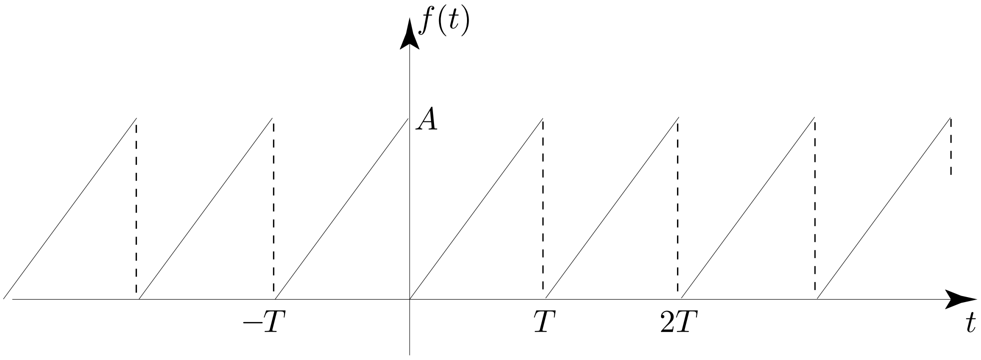



**Figure 24**

**Solution**

We have

$$f(t) = \frac{At}{T} \qquad 0 < t < T \qquad f(t+T) = f(t)$$

The period is $T$ in this case so $\omega_0 = \dfrac{2\pi}{T}$.

Looking at the graph of $f(t)$ we can say immediately

(a) the Fourier series will contain a constant term $c_0$

(b) if we imagine shifting the horizontal axis up to $\dfrac{A}{2}$ the signal can be written $f(t) = \dfrac{A}{2} + g(t)$, where $g(t)$ is an odd function with complex Fourier coefficients that are purely imaginary.

Hence we expect the required complex Fourier series of $f(t)$ to contain a constant term $\dfrac{A}{2}$ and complex exponential terms with purely imaginary coefficients. We have, from the general theory, and using $0 < t < T$ as the basic period for integrating,

$$c_n = \frac{1}{T}\int_0^T \frac{At}{T} e^{-\,\mathrm{i}n\omega_0 t}\,dt = \frac{A}{T^2}\int_0^T te^{-\,\mathrm{i}n\omega_0 t}\,dt$$

We can evaluate the integral using parts:

$$\begin{aligned}\int_0^T te^{-\,\mathrm{i}n\omega_0 t}\,dt &= \left[\frac{te^{-\,\mathrm{i}n\omega_0 t}}{(-\,\mathrm{i}n\omega_0)}\right]_0^T + \frac{1}{\mathrm{i}n\omega_0}\int_0^T e^{-\,\mathrm{i}n\omega_0 t}\,dt \\ &= \frac{Te^{\,\mathrm{i}n\omega_0 T}}{(-\,\mathrm{i}n\omega_0)} - \frac{1}{(\,\mathrm{i}n\omega_0)^2}\left[e^{-\,\mathrm{i}n\omega_0 t}\right]_0^T\end{aligned}$$

**Solution (contd.)**

But $\omega_0 = \dfrac{2\pi}{T}$ so

$$\begin{aligned} e^{-\,\mathrm{i}n\omega_0 T} = e^{-\,\mathrm{i}n2\pi} &= \cos 2n\pi - \,\mathrm{i}\sin 2n\pi \\ &= 1 - 0\,\mathrm{i} = 1 \end{aligned}$$

Hence the integral becomes

$$\frac{T}{-\,\mathrm{i}n\omega_0} - \frac{1}{(\,\mathrm{i}n\omega_0)^2}\left(e^{-\,\mathrm{i}n\omega_0 T} - 1\right)$$

Hence

$$c_n = \frac{A}{T^2}\left(\frac{T}{-\,\mathrm{i}n\omega_0}\right) = \frac{\mathrm{i}A}{2\pi n} \qquad n = \pm 1, \pm 2, \ldots$$

Note that

$$c_{-n} = \frac{\mathrm{i}A}{2\pi(-n)} = \frac{-\,\mathrm{i}A}{2\pi n} = c_n^* \quad \text{as it must}$$

Also $c_0 = \dfrac{1}{T}\displaystyle\int_0^T \frac{At}{T}dt = \frac{A}{2}$ as expected.

Hence the required complex Fourier series is

$$f(t) = \frac{A}{2} + \frac{\mathrm{i}A}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{\,\mathrm{i}n\omega_0 t}}{n}$$

which could be written, showing only the constant and the first two harmonics, as

$$f(t) = \frac{A}{2\pi}\left\{\ldots - \,\mathrm{i}\frac{e^{-\,\mathrm{i}2\omega_0 t}}{2} - \,\mathrm{i}e^{-\,\mathrm{i}\omega_0 t} + \pi + \,\mathrm{i}e^{\,\mathrm{i}\omega_0 t} + \,\mathrm{i}\frac{e^{\,\mathrm{i}2\omega_0 t}}{2} + \ldots\right\}.$$

The corresponding trigonometric Fourier series for the function can be readily obtained from this complex series by combining the terms in $\pm n$, $n = 1, 2, 3, \ldots$
For example this first harmonic is

$$\begin{aligned} \frac{A}{2\pi}\left\{-\,\mathrm{i}e^{-\,\mathrm{i}\omega_0 t} + \,\mathrm{i}e^{\,\mathrm{i}\omega_0 t}\right\} &= \frac{A}{2\pi}\{-\,\mathrm{i}(\cos\omega_0 t - \,\mathrm{i}\sin\omega_0 t) + \,\mathrm{i}(\cos\omega_0 t + \,\mathrm{i}\sin\omega_0 t)\} \\ &= \frac{A}{2\pi}(-2\sin\omega_0 t) = -\frac{A}{\pi}\sin\omega_0 t \end{aligned}$$

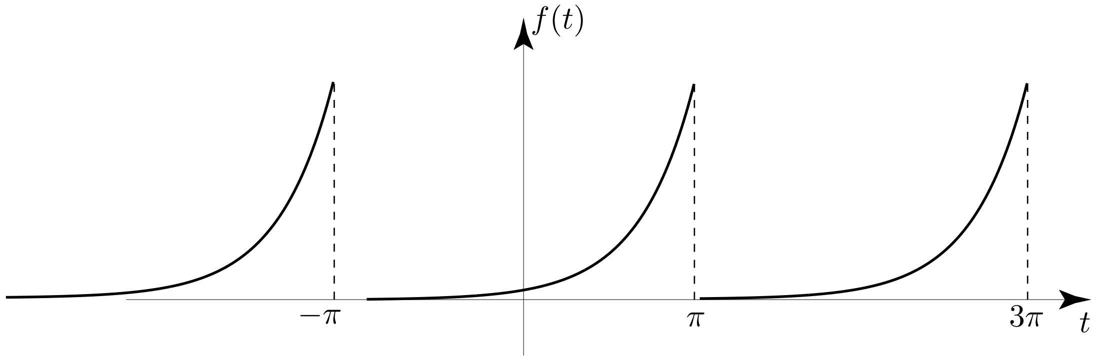
Performing similar calculations on the other harmonics we obtain the trigonometric form of the Fourier series

$$f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega_0 t}{n}.$$

Find the complex Fourier series of the periodic function:

$$f(t) = e^t \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$




Firstly write down an integral expression for the Fourier coefficients $c_n$:

**Your solution**

**Answer**
We have, since $T = 2\pi$, so $\omega_0 = 1$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t e^{-\,\mathrm{i}nt}dt$$

Now combine the real exponential and the complex exponential as one term and carry out the integration:

**Your solution**

**Answer**
We have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-\,\mathrm{i}n)t}dt = \frac{1}{2\pi}\left[\frac{e^{(1-\,\mathrm{i}n)t}}{(1-\,\mathrm{i}n)}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{(1-\,\mathrm{i}n)}\left(e^{(1-\,\mathrm{i}n)\pi} - e^{-(1-\,\mathrm{i}n)\pi}\right)$$

Now simplify this as far as possible and write out the Fourier series:

**Your solution**

**Answer**

$$e^{(1-\,\mathrm{i}n)\pi} = e^{\pi}\ e^{-\,\mathrm{i}n\pi} = e^{\pi}(\cos n\pi - \ \mathrm{i}\sin n\pi) = e^{\pi}\cos n\pi$$

$$e^{-(1-\,\mathrm{i}n)\pi} = e^{-\pi}e^{\,\mathrm{i}n\pi} = e^{-\pi}\cos n\pi$$

Hence $\quad c_n = \dfrac{1}{2\pi}\dfrac{1}{(1-\ \mathrm{i}n)}(e^{\pi}-e^{-\pi})\cos n\pi = \dfrac{\sinh\pi}{\pi}\dfrac{(1+\ \mathrm{i}n)}{(1+n^2)}\cos n\pi$

Note that the coefficients $c_n$ $n=\pm 1,\pm 2,\ldots$ have both real and imaginary parts in this case as the function being expanded is neither even nor odd.

Also $\quad c_{-n} = \dfrac{\sinh\pi}{\pi}\dfrac{(1-\ \mathrm{i}n)}{(1+(-n)^2)}\cos(-n\pi) = \dfrac{\sinh\pi}{\pi}\dfrac{(1-\ \mathrm{i}n)}{(1+n^2)}\cos n\pi = c_n^*$ as required.

This includes the constant term $c_0 = \dfrac{\sinh\pi}{\pi}$. Hence the required Fourier series is

$$f(t) = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{(1+\ \mathrm{i}n)}{(1+n^2)}e^{\,\mathrm{i}nt} \qquad \text{since} \quad \cos n\pi = (-1)^n.$$

# 4. Parseval's theorem

This is essentially a mathematical theorem but has, as we shall see, an important engineering interpretation particularly in electrical engineering. Parseval's theorem states that if $f(t)$ is a periodic function with period $T$ and if $c_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ denote the complex Fourier coefficients of $f(t)$, then

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

In words the theorem states that the mean square value of the signal $f(t)$ over one period equals the sum of the squared magnitudes of all the complex Fourier coefficients.

## Proof of Parseval's theorem.

Assume $f(t)$ has a complex Fourier series of the usual form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\,\mathrm{i}n\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}dt$$

Then

$$f^2(t) = f(t)f(t) = f(t)\sum c_n e^{\,\mathrm{i}n\omega_0 t} = \sum c_n f(t) e^{\,\mathrm{i}n\omega_0 t}$$

Hence

$$\begin{aligned}
\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum c_n f(t) e^{\,\mathrm{i}n\omega_0 t}dt \\
&= \frac{1}{T}\sum c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\,\mathrm{i}n\omega_0 t}dt \\
&= \sum c_n c_n^* \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

which completes the proof.

Parseval's theorem can also be written in terms of the Fourier coefficients $a_n, b_n$ of the trigonometric Fourier series. Recall that

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - \mathrm{i}b_n}{2} \quad n = 1, 2, 3, \ldots \qquad\qquad c_n = \frac{a_n + \mathrm{i}b_n}{2} \quad n = -1, -2, -3, \ldots$$

so

$$|c_n|^2 = \frac{a_n^2 + b_n^2}{4} \qquad\qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

so

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + 2\sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{4}$$

and hence Parseval's theorem becomes

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (7)$$

The engineering interpretation of this theorem is as follows. Suppose $f(t)$ denotes an electrical signal (current or voltage), then from elementary circuit theory $f^2(t)$ is the instantaneous power (in a 1 ohm resistor) so that

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt$$

is the energy dissipated in the resistor during one period.
Now a sinusoid wave of the form

$$A\cos\omega t \qquad (\text{ or } A\sin\omega t)$$

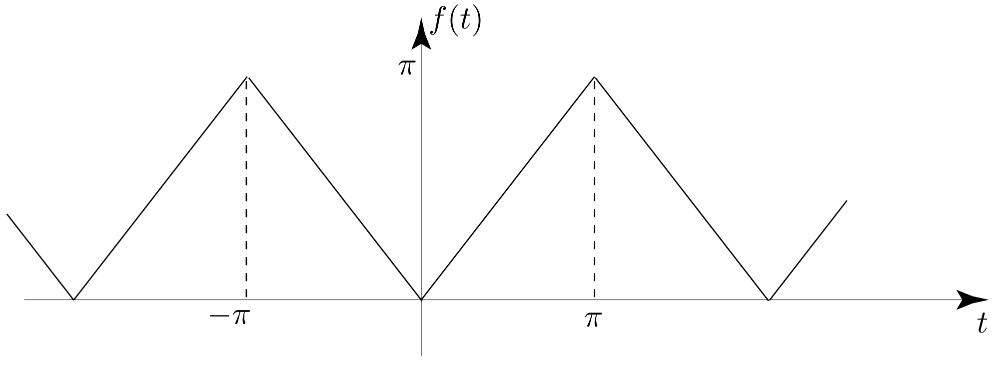
has a mean square value $\dfrac{A^2}{2}$ so a purely sinusoidal signal would dissipate a power $\dfrac{A^2}{2}$ in a 1 ohm resistor. Hence Parseval's theorem in the form (7) states that the average power dissipated over 1 period equals the sum of the powers of the constant (or d.c.) components and of all the sinusoidal (or alternating) components.

**Task**

The triangular signal shown below has trigonometric Fourier series

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\(\text{ odd } n)}}^{\infty} \frac{\cos nt}{n^2}.$$

[This was deduced in the Task in Section 23.3, page 39.]




Use Parseval's theorem to show that $\displaystyle\sum_{\substack{n=1\\(n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}.$

First, identify $a_0$, $a_n$ and $b_n$ for this situation and write down the definition of $f(t)$ for this case:

**Your solution**

**Answer**

We have $\dfrac{a_0}{2} = \dfrac{\pi}{2}$

$$a_n = \begin{cases} -\dfrac{4}{n^2\pi} & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = 0 \qquad\qquad n = 1, 2, 3, 4, \ldots$$

Also

$$f(t) = |t| \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$

Now evaluate the integral on the left hand side of Parseval's theorem and hence complete the problem:

**Your solution**

**Answer**

We have $f^2(t) = t^2$ so

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

The right-hand side of Parseval's theorem is

$$\frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 = \frac{\pi^2}{4} + \frac{1}{2}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{16}{n^4\pi^2}$$

Hence

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} \qquad \therefore \qquad \frac{8}{\pi^2}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{12} \qquad \therefore \qquad \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

## Exercises

Obtain the complex Fourier series for each of the following functions of period $2\pi$.

1. $f(t) = t \qquad -\pi \leq t \leq \pi$
2. $f(t) = t \qquad 0 \leq t \leq 2\pi$
3. $f(t) = \mathrm{e}^t \qquad -\pi \leq t \leq \pi$

**Answers**

1. $\mathrm{i}\sum \dfrac{(-1)^n}{n}\mathrm{e}^{\mathrm{i}nt}$ (sum from $-\infty$ to $\infty$ excluding $n = 0$).
2. $\pi + \mathrm{i}\sum \dfrac{1}{n}\mathrm{e}^{\mathrm{i}nt}$ (sum from $-\infty$ to $\infty$ excluding $n = 0$).
3. $\dfrac{\sinh\pi}{\pi}\sum(-1)^n\dfrac{(1+\mathrm{i}n)}{(1+n^2)}\mathrm{e}^{\mathrm{i}nt}$ (sum from $-\infty$ to $\infty$).