Representing Periodic Functions by Fourier Series

23.2

Introduction

In this Section we show how a periodic function can be expressed as a series of sines and cosines. We begin by obtaining some standard integrals involving sinusoids. We then **assume** that if $f(t)$ is a periodic function, of period 2π, then the Fourier series expansion takes the form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Our main purpose here is to show how the constants in this expansion, a_n (for $n = 0, 1, 2, 3 \ldots$ and b_n (for $n = 1, 2, 3, \ldots$), may be determined for any given function $f(t)$.

Prerequisites

Before starting this Section you should ...

- know what a periodic function is
- be able to integrate functions involving sinusoids
- have knowledge of integration by parts

Learning Outcomes

On completion you should be able to ...

- calculate Fourier coefficients of a function of period 2π
- calculate Fourier coefficients of a function of general period

1. Introduction

We recall first a simple trigonometric identity:

$$\cos 2t = -1 + 2\cos^2 t \qquad \text{or equivalently} \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t \tag{1}$$

Equation 1 can be interpreted as a simple **finite** Fourier series representation of the periodic function $f(t) = \cos^2 t$ which has period π. We note that the Fourier series representation contains a constant term and a period π term.

A more complicated trigonometric identity is

$$\sin^4 t = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t \tag{2}$$

which again can be considered as a finite Fourier series representation. (Do not worry if you are unfamiliar with the result (2).) Note that the function $f(t) = \sin^4 t$ (which has period π) is being written in terms of a constant function, a function of period π or frequency $\dfrac{1}{\pi}$ (the "first harmonic") and a function of period $\dfrac{\pi}{2}$ or frequency $\dfrac{2}{\pi}$ (the "second harmonic").

The reason for the constant term in both (1) and (2) is that each of the functions $\cos^2 t$ and $\sin^4 t$ is non-negative and hence each must have a positive average value. Any sinusoid of the form $\cos nt$ or $\sin nt$ has, by symmetry, zero average value. Therefore, so would a Fourier series containing only such terms. A constant term can therefore be expected to arise in the Fourier series of a function which has a non-zero average value.

2. Functions of period 2π

We now discuss how to represent periodic non-sinusoidal functions $f(t)$ of period 2π in terms of sinusoids, i.e. how to obtain Fourier series representations. As already discussed we expect such Fourier series to contain harmonics of frequency $\dfrac{n}{2\pi}$ $(n = 1, 2, 3, \ldots)$ and, if the periodic function has a non-zero average value, a constant term.

Thus we seek a Fourier series representation of the general form

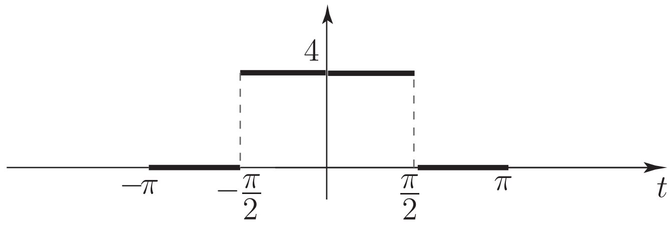
$$f(t) = \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + b_1 \sin t + b_2 \sin 2t + \ldots$$

The reason for labelling the constant term as $\dfrac{a_0}{2}$ will be discussed later. The amplitudes $a_1, a_2, \ldots$ $b_1, b_2, \ldots$ of the sinusoids are called **Fourier coefficients**.

Obtaining the Fourier coefficients for a given periodic function $f(t)$ is our main task and is referred to as Fourier Analysis. Before embarking on such an analysis it is instructive to establish, at least qualitatively, the plausibility of approximating a function by a few terms of its Fourier series.

Task

Consider the square wave of period 2π one period of which is shown in Figure 10.



(a) Write down the analytic description of this function,

(b) State whether you expect the Fourier series of this function to contain a constant term,

(c) List any other possible features of the Fourier series that you might expect from the graph of the square-wave function.

Your solution

Answer

(a) We have

$$\begin{aligned} f(t) &= \begin{cases} 4 & -\dfrac{\pi}{2} < t < \dfrac{\pi}{2} \\[2ex] 0 & -\pi < t < -\dfrac{\pi}{2}, \quad \dfrac{\pi}{2} < t < \pi \end{cases} \\ f(t+2\pi) &= f(t) \end{aligned}$$

(b) The Fourier series will contain a constant term since the square wave here is non-negative and cannot therefore have a zero average value. This constant term is often referred to as the d.c. (direct current) term by engineers.

(c) Since the square wave is an even function (i.e. the graph has symmetry about the y axis) then its Fourier series will contain cosine terms but not sine terms because only the cosines are even functions. (Well done if you spotted this at this early stage!)

It is possible to show, and we will do so later, that the Fourier series representation of this square wave is

$$2 + \frac{8}{\pi}\left\{\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \ldots\right\}$$

i.e. the Fourier coefficients are

$$\frac{a_0}{2} = 2, \qquad a_1 = \frac{8}{\pi}, \qquad a_2 = 0, \qquad a_3 = -\frac{8}{3\pi}, \qquad a_4 = 0, \qquad a_5 = \frac{8}{5\pi}\ , \ldots$$

Note, as well as the presence of the constant term and of the cosine (but not sine) terms, that only odd harmonics are present i.e. sinusoids of period 2π, $\frac{2\pi}{3}$, $\frac{2\pi}{5}$, $\frac{2\pi}{7}, \ldots$ or of frequency $1, 3, 5, 7, \ldots$ times the fundamental frequency $\frac{1}{2\pi}$.

We now show in Figure 8 graphs of

(i) the square wave

(ii) the first two terms of the Fourier series representing the square wave

(iii) the first three terms of the Fourier series representing the square wave

(iv) the first four terms of the Fourier series representing the square wave

(v) the first five terms of the Fourier series representing the square wave

Note: We show the graphs for $0 < t < \pi$ only since the square wave and its Fourier series are even.

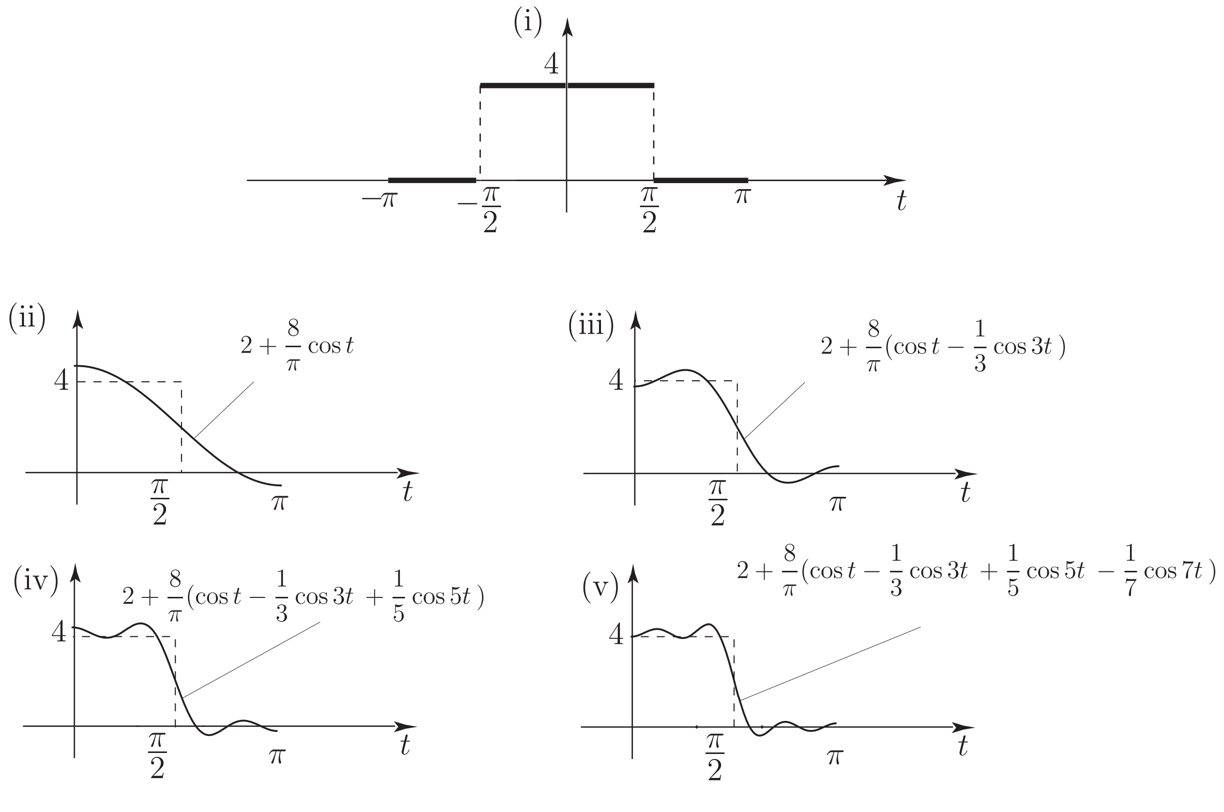


Figure 8

We can clearly see from Figure 8 that as the number of terms is increased the graph of the Fourier series gradually approaches that of the original square wave - the ripples increase in number but decrease in amplitude. (The behaviour near the **discontinuity**, at $t = \frac{\pi}{2}$, is slightly more complicated and it is possible to show that however many terms are taken in the Fourier series, some "overshoot" will always occur. This effect, which we do not discuss further, is known as the Gibbs Phenomenon.)

Orthogonality properties of sinusoids

As stated earlier, a periodic function $f(t)$ with period 2π has a Fourier series representation

$$\begin{aligned} f(t) &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + b_1 \sin t + b_2 \sin 2t + \ldots, \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (3) \end{aligned}$$

To determine the Fourier coefficients $a_n,\ b_n$ and the constant term $\frac{a_0}{2}$ use has to be made of certain integrals involving sinusoids, the integrals being over a range $\alpha, \alpha + 2\pi$, where α is any number. (We will normally choose $\alpha = -\pi$.)

Task

Find $\displaystyle\int_{-\pi}^{\pi} \sin nt\, dt$ and $\displaystyle\int_{-\pi}^{\pi} \cos nt\, dt$ where n is an integer.

Your solution

Answer

In fact both integrals are zero for

$$\int_{-\pi}^{\pi} \sin nt\, dt = \left[-\frac{1}{n}\cos nt\right]_{-\pi}^{\pi} = \frac{1}{n}(-\cos n\pi + \cos n\pi) = 0 \qquad n \neq 0 \qquad (4)$$

$$\int_{-\pi}^{\pi} \cos nt\, dt = \left[\frac{1}{n}\sin nt\right]_{-\pi}^{\pi} = 0 \qquad n \neq 0 \qquad (5)$$

As special cases, if $n = 0$ the first integral is zero and the second integral has value 2π.

N.B. Any integration range $\alpha, \alpha + 2\pi$, would give these same (zero) answers.

These integrals enable us to calculate the constant term in the Fourier series (3) as in the following task.

Task

Integrate both sides of (3) from $-\pi$ to π and use the results from the previous Task. Hence obtain an expression for a_0.

Your solution

Answer

We get for the left-hand side

$$\int_{-\pi}^{\pi} f(t)dt$$

(whose value clearly depends on the function $f(t)$).

Integrating the right-hand side term by term we get

$$\frac{1}{2}\int_{-\pi}^{\pi} a_0\ dt + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nt\ dt + \int_{-\pi}^{\pi} b_n \sin nt\ dt\right\} = \frac{1}{2}\left[a_0\ t\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\{0+0\}$$

(using the integrals (4) and (5) shown above). Thus we get

$$\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2}(2a_0\pi) \qquad \text{or} \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \qquad (6)$$

Key Point 1

The constant term in a trigonometric Fourier series for a function of period 2π is

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \text{ average value of } f(t) \text{ over 1 period.}$$

This result ties in with our earlier discussion on the significance of the constant term. Clearly a signal whose average value is zero will have no constant term in its Fourier series. The following square wave (Figure 9) is an example.

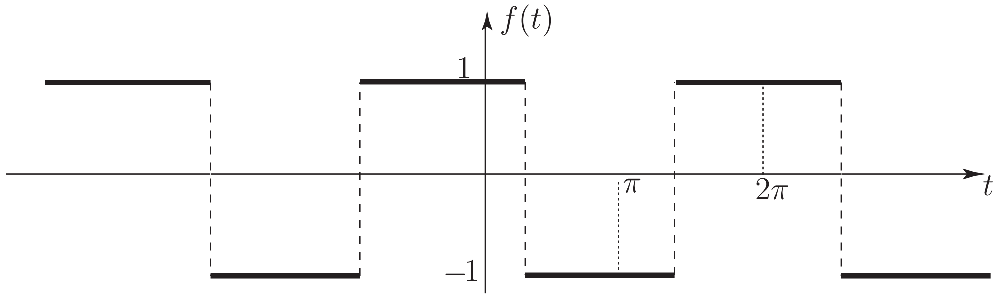


Figure 9

We now obtain further integrals, known as orthogonality properties, which enable us to find the remaining Fourier coefficients i.e. the amplitudes a_n and b_n $(n = 1, 2, 3, \ldots)$ of the sinusoids.

Task

Using the standard trigonometric identity that

$$\sin nt \cos mt \equiv \frac{1}{2}\{\sin(n+m)t + \sin(n-m)t\}$$

evaluate $\displaystyle\int_{-\pi}^{\pi} \sin nt \cos mt \; dt$ where n and m are any integers.

Your solution

Answer

We get

$$\int_{-\pi}^{\pi} \sin nt \cos mt \; dt = \frac{1}{2}\left\{\int_{-\pi}^{\pi} \sin(n+m)t \; dt + \int_{-\pi}^{\pi} \sin(n-m)t \; dt\right\} = \frac{1}{2}\{0+0\} = 0$$

using the results (4) and (5) since $n+m$ and $n-m$ are also integers.

This result holds for any interval of 2π.

Key Point 2

Orthogonality Relation

For any integers m, n, including the case $m = n$,

$$\int_{-\pi}^{\pi} \sin nt \cos mt \ dt = 0$$

We shall use this result shortly but need a few more integrals first.
Consider next

$$\int_{-\pi}^{\pi} \cos nt \cos mt \ dt \qquad \text{where } m \text{ and } n \text{ are integers.}$$

Using another trigonometric identity we have, for the case $n \neq m$,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nt \cos mt \ dt &= \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(n+m)t + \cos(n-m)t\}dt \\ &= \frac{1}{2}\{0+0\} = 0 \qquad \text{using the integrals (4) and (5).}\end{aligned}$$

For the case $n = m$ we must get a non-zero answer since $\cos^2 nt$ is non-negative. In this case:

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 nt \ dt &= \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nt)\, dt \\ &= \frac{1}{2}\left[t + \frac{1}{2n}\sin 2nt\right]_{-\pi}^{\pi} = \pi \qquad (\text{ provided } n \neq 0)\end{aligned}$$

For the case $n = m = 0$ we have $\int_{-\pi}^{\pi} \cos nt \cos mt \ dt = 2\pi$

Task

Proceeding in a similar way to the above, evaluate

$$\int_{-\pi}^{\pi} \sin nt \sin mt \quad dt$$

for integers m and n.

Again consider separately the three cases: (a) $n \neq m$, (b) $n = m \neq 0$ and (c) $n = m = 0$.

Your solution

Answer

(a) Using the identity $\sin nt \sin mt \equiv \frac{1}{2}\{\cos(n-m)t - \cos(n+m)t\}$ and integrating the right-hand side terms, we get, using (4) and (5)

$$\int_{-\pi}^{\pi} \sin nt \sin mt \; dt = 0 \qquad n, m \text{ integers} \qquad n \neq m$$

(b) Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$ with $\theta = nt$ gives for $n = m \neq 0$

$$\int_{-\pi}^{\pi} \sin^2 nt \; dt = \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos 2nt)dt = \pi$$

(c) When $n = m = 0$, $\displaystyle\int_{-\pi}^{\pi} \sin nt \sin mt \; dt = 0.$

We summarise these results in the following Key Point:

Key Point 3

For integers n, m

$$\int_{-\pi}^{\pi} \sin nt \cos mt \; dt = 0$$

$$\int_{-\pi}^{\pi} \cos nt \cos mt \; dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \\ 2\pi & n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt \; dt = \begin{cases} 0 & n \neq m, \; n = m = 0 \\ \pi & n = m \end{cases}$$

All these results hold for any integration range of width 2π.

3. Calculation of Fourier coefficients

Consider the Fourier series for a function $f(t)$ of period 2π:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad (7)$$

To obtain the coefficients a_n $(n = 1, 2, 3, \ldots)$, we multiply both sides by $\cos mt$ where m is some positive integer and integrate both sides from $-\pi$ to π.

For the left-hand side we obtain

$$\int_{-\pi}^{\pi} f(t) \cos mt\, dt$$

For the right-hand side we obtain

$$\frac{a_0}{2}\int_{-\pi}^{\pi} \cos mt\, dt + \sum_{n=1}^{\infty}\left\{a_n \int_{-\pi}^{\pi} \cos nt \cos mt\, dt + b_n \int_{-\pi}^{\pi} \sin nt \cos mt\, dt\right\}$$

The first integral is zero using (5).

Using the orthogonality relations all the integrals in the summation give zero except for the case $n = m$ when, from Key Point 3

$$\int_{-\pi}^{\pi} \cos^2 mt\, dt = \pi$$

Hence

$$\int_{-\pi}^{\pi} f(t) \cos mt\, dt = a_m \pi$$

from which the coefficient a_m can be obtained.

Rewriting m as n we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt \qquad \text{for } \; n = 1, 2, 3, \ldots \qquad (8)$$

Using (6), we see the formula also works for $n = 0$ (but we must remember that the constant term is $\dfrac{a_0}{2}$.)

From (8)

$a_n = 2 \times$ average value of $f(t) \cos nt$ over one period.

Task

By multiplying (7) by $\sin mt$ obtain an expression for the Fourier Sine coefficients b_n, $n = 1, 2, 3, \ldots$

Your solution

Answer

A similar calculation to that performed to find the a_n gives

$$\int_{-\pi}^{\pi} f(t)\sin mt\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mt\,dt + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n\cos nt\sin mt\,dt + \int_{-\pi}^{\pi} b_n\sin nt\sin mt\,dt\right\}$$

All terms on the right-hand side integrate to zero except for the case $n = m$ where

$$\int_{-\pi}^{\pi} b_m\sin^2 mt\,dt = b_m\pi$$

Relabelling m as n gives

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad n = 1, 2, 3, \ldots \tag{9}$$

(There is no Fourier coefficient b_0.)

Clearly $b_n = 2\times$ average value of $f(t)\sin nt$ over one period.

Key Point 4

A function $f(t)$ with period 2π has a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

The Fourier coefficients are

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \qquad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt \qquad n = 1, 2, \ldots \end{aligned}$$

In the integrals any convenient integration range extending over an interval of 2π may be used.

4. Examples of Fourier series

We shall obtain the Fourier series of the "half-rectified" square wave shown in Figure 10.

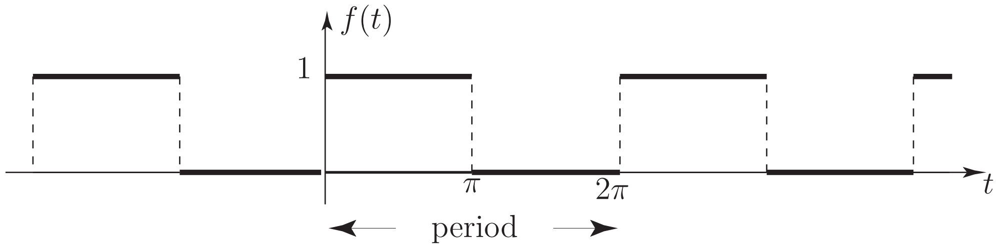


Figure 10

We have

$$\begin{aligned} f(t) &= \begin{cases} 1 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases} \\ f(t + 2\pi) &= f(t) \end{aligned}$$

The calculation of the Fourier coefficients is merely straightforward integration using the results already obtained:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$$

in general. Hence, for our square wave

$$a_n = \frac{1}{\pi}\int_0^{\pi}(1)\cos nt\, dt = \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_0^{\pi} = 0 \qquad \text{provided } n \neq 0$$

But $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(1)\,dt = 1$ so the constant term is $\dfrac{a_0}{2} = \dfrac{1}{2}$.

(The square wave takes on values 1 and 0 over equal length intervals of t so $\dfrac{1}{2}$ is clearly the mean value.)

Similarly

$$b_n = \frac{1}{\pi}\int_0^{\pi}(1)\sin nt\,dt = \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi}$$

Some care is needed now!

$$b_n = \frac{1}{n\pi}(1-\cos n\pi)$$

But $\quad \cos n\pi = +1 \qquad n = 2, 4, 6, \ldots,$

$$\therefore \qquad b_n = 0 \qquad n = 2, 4, 6, \ldots$$

However, $\quad \cos n\pi = -1 \qquad n = 1, 3, 5, \ldots$

$$\therefore \qquad b_n = \frac{1}{n\pi}(1-(-1)) = \frac{2}{n\pi} \qquad n = 1, 3, 5, \ldots$$

i.e. $\quad b_1 = \dfrac{2}{\pi},\ b_3 = \dfrac{2}{3\pi},\ b_5 = \dfrac{2}{5\pi}, \ldots$

Hence the required Fourier series is

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt) \qquad \text{in general}\\ f(t) &= \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right) \qquad \text{in this case} \end{aligned}$$

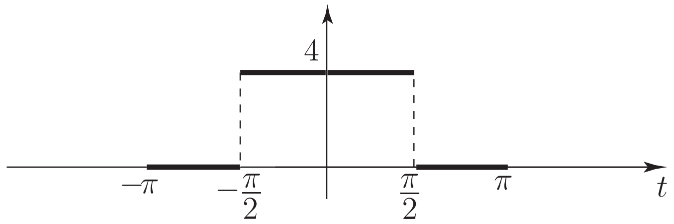
Note that the Fourier series for this particular form of the square wave contains a constant term and odd harmonic sine terms. We already know why the constant term arises (because of the non-zero mean value of the functions) and will explain later why the presence of any odd harmonic sine terms could have been predicted without integration.

The Fourier series we have found can be written in summation notation in various ways:

$\dfrac{1}{2} + \dfrac{2}{\pi}\displaystyle\sum_{\substack{n=1\\(n\ \text{odd})}}^{\infty}\dfrac{1}{n}\sin nt$ or, since n is odd, we may write $n = 2k-1 \quad k = 1, 2, \ldots$ and write the Fourier series as $\dfrac{1}{2} + \dfrac{2}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{1}{(2k-1)}\sin(2k-1)t$

Task

Obtain the Fourier series of the square wave one period of which is shown:



Your solution

Answer

We have, since the function is non-zero only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$,

$$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\,dt = 4$$

$\therefore \qquad \dfrac{a_0}{2} = 2$ is the constant term as we would expect. Also

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos nt\,dt = \frac{4}{\pi}\left[\frac{\sin nt}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{4}{n\pi}\left\{\sin\left(\frac{n\pi}{2}\right\} - \sin\left(-\frac{n\pi}{2}\right)\right) = \frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right) \qquad n = 1, 2, 3, \ldots \end{aligned}$$

It follows from a knowledge of the sine function that

$$a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\ \dfrac{8}{n\pi} & n = 1, 5, 9, \ldots \\ -\dfrac{8}{n\pi} & n = 3, 7, 11, \ldots \end{cases}$$

Also

$$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sin nt\,dt = \frac{4}{\pi}\left[-\frac{\cos nt}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{4}{n\pi}\left\{\cos\left(\frac{n\pi}{2}\right) - \cos\left(-\frac{n\pi}{2}\right)\right\} = 0$$

Hence, the required Fourier series is

$$f(t) = 2 + \frac{8}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \ldots\right)$$

which, like the previous square wave, contains a constant term and odd harmonics, but in this case odd harmonic cosine terms rather than sine.

You may recall that this particular square wave was used earlier and we have already sketched the form of the Fourier series for 2, 3, 4 and 5 terms in Figure 8.

Clearly, in finding the Fourier series of square waves, the integration is particularly simple because $f(t)$ takes on piecewise constant values. For other functions, such as saw-tooth waves this will not be the case. Before we tackle such functions however we shall generalise our formulae for the Fourier coefficients a_n, b_n to the case of a periodic function of arbitrary period, rather than confining ourselves to period 2π.

5. Fourier series for functions of general period

This is a straightforward extension of the period 2π case that we have already discussed.
Using x (instead of t) temporarily as the variable. We have seen that a 2π periodic function $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad n = 0, 1, 2, \ldots \qquad\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \qquad n = 1, 2, \ldots$$

Suppose we now change the variable to t where $x = \dfrac{2\pi}{T}t$.

Thus $x = \pi$ corresponds to $t = T/2$ and $x = -\pi$ corresponds to $t = -T/2$.
Hence regarded as a function of t, we have a function with period T.
Making the substitution $x = \dfrac{2\pi}{T}t$, and hence $dx = \dfrac{2\pi}{T}\,dt$, in the expressions for a_n and b_n we obtain

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2n\pi t}{T}\right) dt \qquad n = 0, 1, 2\ldots \\ b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi t}{T}\right) dt \qquad n = 1, 2\ldots \end{aligned}$$

These integrals give the Fourier coefficients for a function of period T whose Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right\}$$

Various other notations are commonly used in this case e.g. it is sometimes convenient to write the period $T = 2\ell$. (This is particularly useful when Fourier series arise in the solution of partial differential equations.) Another alternative is to use the angular frequency ω and put $T = 2\pi/\omega$.

Write down the form of the Fourier series and expressions for the coefficients if
(a) $T = 2\ell$ (b) $T = 2\pi/\omega$.

Your solution

Answer

(a) $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi t}{\ell}\right) + b_n\sin\left(\frac{n\pi t}{\ell}\right)\right\}$ with $a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{n\pi t}{\ell}\right)\,dt$ and similarly for b_n.

(b) $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos(n\omega t) + b_n\sin(n\omega t)\}$ with $a_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\cos(n\omega t)\,dt$ and similarly for b_n.

You should note that, as usual, any convenient integration range of length T (or 2ℓ or $\frac{2\pi}{\omega}$) can be used in evaluating a_n and b_n.

Example 1

Find the Fourier series of the function shown in Figure 11 which is a saw-tooth wave with alternate portions removed.

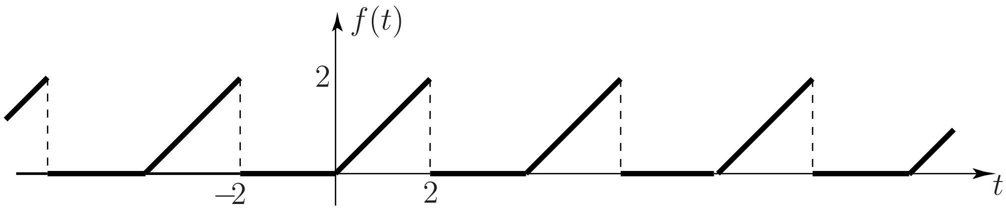


Figure 11

Solution

Here the period $T = 2\ell = 4$ so $\ell = 2$. The Fourier series will have the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi t}{2}\right) + b_n\sin\left(\frac{n\pi t}{2}\right)\right\}$$

The coefficients a_n are given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{n\pi t}{2}\right)\,dt$$

where

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases} \qquad f(t+4) = f(t)$$

Hence $a_n = \frac{1}{2}\int_{0}^{2} t\cos\left(\frac{n\pi t}{2}\right)\,dt.$

Solution (contd.)

The integration is readily performed using integration by parts:

$$\begin{aligned}\int_0^2 t\cos\left(\frac{n\pi t}{2}\right)\,dt &= \left[t\,\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi t}{2}\right)\,dt \\ &= \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi t}{2}\right)\right]_0^2 \qquad n \neq 0 \\ &= \frac{4}{n^2\pi^2}(\cos n\pi - 1).\end{aligned}$$

Hence, since $a_n = \dfrac{1}{2}\displaystyle\int_0^2 t\cos(\frac{n\pi t}{2})dt$

$$a_n = \begin{cases} 0 & n = 2,4,6,\ldots \\ -\dfrac{4}{n^2\pi^2} & n = 1,3,5,\ldots \end{cases}$$

The constant term is $\dfrac{a_0}{2}$ where $\quad a_0 = \dfrac{1}{2}\displaystyle\int_0^2 t\,dt = 1.$

Similarly

$$b_n = \frac{1}{2}\int_0^2 t\sin\left(\frac{n\pi t}{2}\right)\,dt$$

where

$$\int_0^2 t\sin\left(\frac{n\pi t}{2}\right)\,dt = \left[-t\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi t}{2}\right)\,dt.$$

The second integral gives zero. Hence

$$b_n = -\frac{2}{n\pi}\cos n\pi = \begin{cases} -\dfrac{2}{n\pi} & n = 2,4,6,\ldots \\ +\dfrac{2}{n\pi} & n = 1,3,5,\ldots \end{cases}$$

Hence, using all these results for the Fourier coefficients, the required Fourier series is

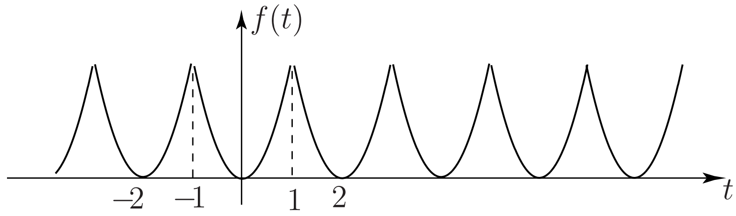
$$\begin{aligned} f(t) = \frac{1}{2} \quad &- \quad \frac{4}{\pi^2}\left\{\cos\left(\frac{\pi t}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi t}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi t}{2}\right) + \ldots\right\} \\ &+ \quad \frac{2}{\pi}\left\{\sin\left(\frac{\pi t}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right)\ldots\right\}\end{aligned}$$

Notice that because the Fourier coefficients depend on $\dfrac{1}{n^2}$ (rather than $\dfrac{1}{n}$ as was the case for the square wave) the sinusoidal components in the Fourier series have quite rapidly decreasing amplitudes. We would therefore expect to be able to approximate the original saw-tooth function using only a quite small number of terms in the series.

Task

Obtain the Fourier series of the function

$$\begin{aligned} f(t) &= t^2 \qquad -1 < t < 1 \\ f(t+2) &= f(t) \end{aligned}$$



First write out the form of the Fourier series in this case:

Your solution

Answer
Since $T = 2\ell = 2$ and since the function has a non-zero average value, the form of the Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n(\cos n\pi t) + b_n \sin(n\pi t)\}$$

Now write out integral expressions for a_n and b_n. Will there be a constant term in the Fourier series?

Your solution

Answer
Because the function is non-negative there will be a constant term. Since $T = 2\ell = 2$ then $\ell = 1$ and we have

$$\begin{aligned} a_n &= \int_{-1}^{1} t^2 \cos(n\pi t)\, dt \qquad n = 0, 1, 2, \ldots \\ b_n &= \int_{-1}^{1} t^2 \sin(n\pi t)\, dt \qquad n = 1, 2, \ldots \end{aligned}$$

The constant term will be $\dfrac{a_0}{2}$ where $a_0 = \displaystyle\int_{-1}^{1} t^2\, dt$.

Now evaluate the integrals. Try to spot the value of the integral for b_n so as to avoid integration. Note that the integrand is an even functions for a_n and an odd functon for b_n.

Your solution

Answer

The integral for b_n is zero for all n because the integrand is an odd function of t. Since the integrand is even in the integrals for a_n we can write

$$a_n = 2\int_0^1 t^2 \cos n\pi t\, dt \qquad n = 0, 1, 2, \ldots$$

The constant term will be $\dfrac{a_o}{2}$ where $\quad a_0 = 2\displaystyle\int_0^1 t^2\, dt = \dfrac{2}{3}.$

For $n = 1, 2, 3, \ldots$ we must integrate by parts (twice)

$$\begin{aligned} a_n &= 2\left\{\left[\frac{t^2}{n\pi}\sin(n\pi t)\right]_0^1 - \frac{2}{n\pi}\int_0^1 t\sin(n\pi t)\, dt\right\} \\ &= -\frac{4}{n\pi}\left\{\left[-\frac{t}{n\pi}\cos(n\pi t)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)\, dt\right\}. \end{aligned}$$

The integral in the second term gives zero so $\quad a_n = \dfrac{4}{n^2\pi^2}\cos n\pi.$

Now writing out the final form of the Fourier series we have

$$f(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}\cos(n\pi t) = \frac{1}{3} + \frac{4}{\pi^2}\left\{-\cos(\pi t) + \frac{1}{4}\cos(2\pi t) - \frac{1}{9}\cos(3\pi t) + \ldots\right\}$$

Exercises

For each of the following periodic signals

- sketch the given function over a few periods
- find the trigonometric Fourier coefficients
- write out the first few terms of the Fourier series.

1. $f(t) = \begin{cases} 1 & 0 < t < \pi/2 \\ 0 & \pi/2 < t < 2\pi \end{cases}$ $\qquad f(t + 2\pi) = f(t)$ $\qquad$ square wave

2. $f(t) = t^2 \quad -1 < t < 1$ $\qquad f(t + 2) = f(t)$

3. $f(t) = \begin{cases} -1 & -T/2 < t < 0 \\ 1 & 0 < t < T/2 \end{cases}$ $\qquad f(t + T) = f(t)$ $\qquad$ square wave

4. $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t^2 & 0 < t < \pi \end{cases}$ $\qquad f(t + 2\pi) = f(t)$

5. $f(t) = \begin{cases} 0 & -T/2 < t < 0 \\ A\ \text{sin}\,\dfrac{2\pi t}{T} & 0 < t < T/2 \end{cases}$ $\qquad f(t + T) = f(t)$ $\qquad$ half-wave rectifier

Answers

1.

$$\frac{1}{4} + \frac{1}{\pi}\left\{\cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \dots\right\} + \frac{1}{\pi}\left\{\sin t + \frac{2\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{2\sin 6t}{6} + \dots\right\}$$

2. $$\frac{1}{3} - \frac{4}{\pi^2}\left\{\cos \pi t - \frac{\cos 2\pi t}{4} + \frac{\cos 3\pi t}{9} - \frac{\cos 4\pi t}{16} + \dots\right\}$$

3. $$\frac{4}{\pi}\left\{\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \dots\right\}$$ where $\omega = 2\pi/T$.

4.

$$\frac{\pi^2}{6} - 2\left\{\cos t - \frac{\cos 2t}{2^2} + \frac{\cos 3t}{3^2} - \dots\right\} + \left\{\left(\pi - \frac{4}{\pi}\right)\sin t - \frac{\pi}{2}\sin 2t + \left(\frac{\pi}{3} - \frac{4}{3^3\pi}\right)\sin 3t - \frac{\pi}{4}\sin 4t + \dots\right\}$$

5. $$\frac{A}{\pi} + \frac{A}{2}\sin \omega t - \frac{2A}{\pi}\left\{\frac{\cos 2\omega t}{(1)(3)} + \frac{\cos 4\omega t}{(3)(5)} + \dots\right\}$$