Contents 22

# Eigenvalues and Eigenvectors



## *Learning outcomes*

*In this Workbook you will learn about the matrix eigenvalue problem AX = kX where A is a square matrix and k is a scalar (number). You will learn how to determine the eigenvalues (k) and corresponding eigenvectors (X) for a given matrix A. You will learn of some of the applications of eigenvalues and eigenvectors. Finally you will learn how eigenvalues and eigenvectors may be determined numerically.*

# Basic Concepts

# 22.1

## Introduction

From an applications viewpoint, eigenvalue problems are probably the most important problems that arise in connection with matrix analysis. In this Section we discuss the basic concepts. We shall see that eigenvalues and eigenvectors are associated with square matrices of order $n \times n$. If $n$ is small (2 or 3), determining eigenvalues is a fairly straightforward process (requiring the solutiuon of a low order polynomial equation). Obtaining eigenvectors is a little strange initially and it will help if you read this preliminary Section first.

## Prerequisites

Before starting this Section you should ...

- have a knowledge of determinants and matrices
- have a knowledge of linear first order differential equations

## Learning Outcomes

On completion you should be able to ...

- obtain eigenvalues and eigenvectors of $2 \times 2$ and $3 \times 3$ matrices
- state basic properties of eigenvalues and eigenvectors

# 1. Basic concepts

## Determinants

A square matrix possesses an associated determinant. Unlike a matrix, which is an array of numbers, a determinant has a single value.

A two by two matrix $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ has an associated determinant

$$\text{det }(C) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}\, c_{22} - c_{21}\, c_{12}$$

(Note square or round brackets denote a matrix, straight vertical lines denote a determinant.)

A three by three matrix has an associated determinant

$$\det(C) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

Among other ways this determinant can be evaluated by an "expansion about the top row":

$$\det(C) = c_{11} \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - c_{12} \begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} + c_{13} \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}$$

Note the minus sign in the second term.

**Task**

Evaluate the determinants

$$\det(A) = \begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} \qquad \det(B) = \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} \qquad \det(C) = \begin{vmatrix} 6 & 5 & 4 \\ 2 & -1 & 7 \\ -3 & 2 & 0 \end{vmatrix}$$

**Your solution**

**Answer**

$$\det A = 4 \times 1 - 6 \times 3 = -14 \qquad \det B = 4 \times 2 - 8 \times 1 = 0$$

$$\det C = 6 \begin{vmatrix} -1 & 7 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & 7 \\ -3 & 0 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = 6 \times (-14) - 5(21) + 4(4 - 3) = -185$$

A matrix such as $B = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$ in the previous task which has zero determinant is called a **singular** matrix. The other two matrices $A$ and $C$ are **non-singular**. The key factor to be aware of is as follows:

**Key Point 1**

Any non-singular $n \times n$ matrix $C$, for which $\det(C) \neq 0$, possesses an inverse $C^{-1}$ i.e.

$$CC^{-1} = C^{-1}C = I \quad \text{where } I \text{ denotes the } n \times n \text{ identity matrix}$$

A singular matrix does **not** possess an inverse.

## Systems of linear equations

We first recall some basic results in linear (matrix) algebra. Consider a system of $n$ equations in $n$ unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{array}{ccccccccc}
c_{11}x_1 & + & c_{12}x_2 & + & \ldots & + & c_{1n}x_n & = & k_1 \\
c_{21}x_1 & + & c_{22}x_2 & + & \ldots & + & c_{2n}x_n & = & k_2 \\
\vdots & + & \vdots & + & \ldots & + & \vdots & = & \vdots \\
c_{n1}x_1 & + & c_{n2}x_2 & + & \ldots & + & c_{nn}x_n & = & k_n
\end{array}$$

We can write such a system in matrix form:

$$\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}, \qquad \text{or equivalently} \qquad CX = K.$$

We see that $C$ is an $n \times n$ matrix (called the coefficient matrix), $X = \{x_1, x_2, \ldots, x_n\}^T$ is the $n \times 1$ column vector of unknowns and $K = \{k_1, k_2, \ldots, k_n\}^T$ is an $n \times 1$ column vector of given constants.

The zero matrix will be denoted by $\underline{O}$.

If $K \neq \underline{O}$ the system is called **inhomogeneous**; if $K = \underline{O}$ the system is called **homogeneous**.

## Basic results in linear algebra

Consider the system of equations $CX = K$.

We are concerned with the nature of the solutions (if any) of this system. We shall see that this system only exhibits three solution types:

- The system is consistent and has a unique solution for $X$
- The system is consistent and has an infinite number of solutions for $X$
- The system is inconsistent and has no solution for $X$

There are two basic cases to consider:

det$(C) \neq 0$ or det$(C) = 0$

**Case 1**: det$(C) \neq 0$

In this case $C^{-1}$ exists and the **unique** solution to $CX = K$ is

$$X = C^{-1}K$$

**Case 2**: det$(C) = 0$

In this case $C^{-1}$ does not exist.

(a) If $K \neq \underline{O}$ the system $CA = K$ has **no solutions**.

(b) If $K = \underline{O}$ the system $CX = \underline{O}$ has an **infinite number of solutions.**

We note that a homogeneous system

$$CX = \underline{O}$$

has a unique solution $X = \underline{O}$ if det$(C) \neq 0$ (this is called the **trivial solution**) or an infinite number of solutions if det$(C) = 0$.

## Example 1

(Case 1) Solve the inhomogeneous system of equations

$$x_1 + x_2 = 1 \qquad 2x_1 + x_2 = 2$$

which can be expressed as $CX = K$ where

$$C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad K = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Solution**

Here $\det(C) = -1 \neq 0$.

The system of equations has the **unique solution**: $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

### Example 2

(Case 2a) Examine the following inhomogeneous system for solutions

$$x_1 + 2x_2 = 1$$
$$3x_1 + 6x_2 = 0$$

**Solution**

Here det $(C) = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$. In this case there are no solutions.

To see this we see the first equation of the system states $x_1 + 2x_2 = 1$ whereas the second equation (after dividing through by 3) states $x_1 + 2x_2 = 0$, a contradiction.

### Example 3

(Case 2b) Solve the homogeneous system

$$x_1 + x_2 = 0$$
$$2x_1 + 2x_2 = 0$$

**Solution**

Here $\det(C) = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$. The solutions are **any** pairs of numbers $\{x_1, x_2\}$ such that $x_1 = -x_2$,

i.e. $\quad X = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$ $\quad$ where $\alpha$ is arbitrary.

There are an **infinite number of solutions**.

## A simple eigenvalue problem

We shall be interested in simultaneous equations of the form:

$$AX = \lambda X,$$

where $A$ is an $n \times n$ matrix, $X$ is an $n \times 1$ column vector and $\lambda$ is a scalar (a constant) and, in the first instance, we examine some simple examples to gain experience of solving problems of this type.

**Example 4**

Consider the following system with $n = 2$:

$$\begin{aligned} 2x + 3y &= \lambda x \\ 3x + 2y &= \lambda y \end{aligned}$$

so that

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad \text{and} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

It appears that there are three unknowns $x, y, \lambda$. The obvious questions to ask are: can we find $x, y$? what is $\lambda$?

**Solution**

To solve this problem we firstly re-arrange the equations (take all unknowns onto one side);

$$(2 - \lambda)x + 3y = 0 \qquad (1)$$

$$3x + (2 - \lambda)y = 0 \qquad (2)$$

Therefore, from equation (2):

$$x = -\frac{(2 - \lambda)}{3}y. \qquad (3)$$

Then when we substitute this into (1)

$$-\frac{(2 - \lambda)^2}{3}y + 3y = 0 \quad \text{which simplifies to} \quad [-(2 - \lambda)^2 + 9]\, y = 0.$$

We conclude that either $y = 0$ or $9 = (2 - \lambda)^2$. There are thus two cases to consider:

**Case 1**

If $y = 0$ then $x = 0$ (from (3)) and we get the **trivial solution**. (We could have guessed this solution at the outset.)

**Case 2**

$$9 = (2 - \lambda)^2$$

which gives, on taking square roots:

$$\pm 3 = 2 - \lambda \quad \text{giving} \quad \lambda = 2 \pm 3 \qquad \text{so} \qquad \lambda = 5 \quad \text{or} \quad \lambda = -1.$$

Now, from equation (3), if $\lambda = 5$ then $x = +y$ and if $\lambda = -1$ then $x = -y$.

We have now completed the analysis. We have found values for $\lambda$ but we also see that we cannot obtain unique values for $x$ and $y$: all we can find is the ratio between these quantities. This behaviour is typical, as we shall now see, of an eigenvalue problem.

## 2. General eigenvalue problems

Consider a given square matrix $A$. If $X$ is a column vector and $\lambda$ is a scalar (a number) then the relation.

$$AX = \lambda X \qquad (4)$$

is called an **eigenvalue problem**. Our purpose is to carry out an analysis of this equation in a manner similar to the example above. However, we will attempt a more general approach which will apply to **all** problems of this kind.

Firstly, we can spot an obvious solution (for $X$) to these equations. The solution $X = 0$ is a possibility (for then both sides are zero). We will not be interested in these **trivial solutions** of the eigenvalue problem. Our main interest will be in the occurrence of **non-trivial solutions** for $X$. These may exist for special values of $\lambda$, called the **eigenvalues** of the matrix $A$. We proceed as in the previous example:

take all unknowns to one side:

$$(A - \lambda I)X = 0 \qquad (5)$$

where $I$ is a unit matrix with the same dimensions as $A$. (Note that $AX - \lambda X = 0$ does **not** simplify to $(A - \lambda)X = 0$ as you **cannot** subtract a scalar $\lambda$ from a matrix $A$). This equation (5) is a **homogeneous** system of equations. In the notation of the earlier discussion $C \equiv A - \lambda I$ and $K \equiv 0$. For such a system we know that non-trivial solutions will only exist if the determinant of the coefficient matrix is zero:

$$\det(A - \lambda I) = 0 \qquad (6)$$

Equation (6) is called the **characteristic equation** of the eigenvalue problem. We see that the characteristic equation only involves one unknown $\lambda$. The characteristic equation is generally a polynomial in $\lambda$, with degree being the same as the order of $A$ (so if $A$ is $2 \times 2$ the characteristic equation is a quadratic, if $A$ is a $3 \times 3$ it is a cubic equation, and so on). For each value of $\lambda$ that is obtained the corresponding value of $X$ is obtained by solving the original equations (4). These $X$'s are called **eigenvectors**.

N.B. We shall see that eigenvectors are only unique up to a multiplicative factor: i.e. if $X$ satisfies $AX = \lambda X$ then so does $kX$ when $k$ is any constant.

**Example 5**

Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

**Solution**

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{i.e.} \quad (A - \lambda I)X = 0.$$

Non-trivial solutions will exist if det $(A - \lambda I) = 0$

that is, det $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = 0, \qquad \therefore \qquad \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0,$

expanding this determinant: $(1-\lambda)(2-\lambda) = 0$. Hence the solutions for $\lambda$ are: $\lambda = 1$ and $\lambda = 2$.

So we have found two values of $\lambda$ for this $2 \times 2$ matrix $A$. Since these are unequal they are said to be **distinct** eigenvalues.

To each value of $\lambda$ there corresponds an eigenvector. We now proceed to find the eigenvectors.

## Case 1

$\lambda = 1$ (smaller eigenvalue). Then our original eigenvalue problem becomes: $AX = X$. In full this is

$$\begin{aligned} x &= x \\ x + 2y &= y \end{aligned}$$

Simplifying

$$\begin{aligned} x &= x \qquad \text{(a)} \\ x + y &= 0 \qquad \text{(b)} \end{aligned}$$

All we can deduce here is that $x = -y \qquad \therefore \qquad X = \begin{bmatrix} x \\ -x \end{bmatrix}$ for any $x \neq 0$

(We specify $x \neq 0$ as, otherwise, we would have the trivial solution.)

So the eigenvectors corresponding to eigenvalue $\lambda = 1$ are all proportional to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, e.g. $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ etc.

Sometimes we write the eigenvector in **normalised** form that is, with modulus or magnitude 1. Here, the normalised form of $X$ is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{which } \textbf{is unique}.$$

**Solution (contd.)**

**Case 2** Now we consider the larger eigenvalue $\lambda = 2$. Our original eigenvalue problem $AX = \lambda X$ becomes $AX = 2X$ which gives the following equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.

$$\begin{aligned} x &= 2x \\ x + 2y &= 2y \end{aligned}$$

These equations imply that $x = 0$ whilst the variable $y$ may take any value whatsoever (except zero as this gives the trivial solution).

Thus the eigenvector corresponding to eigenvalue $\lambda = 2$ has the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$, e.g. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ etc.

The normalised eigenvector here is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In conclusion: the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ has two eigenvalues and two associated normalised eigenvectors:

$$\lambda_1 = 1, \qquad \lambda_2 = 2$$

$$X_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

## Example 6

Find the eigenvalues and eigenvectors of the $3 \times 3$ matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

**Solution**

The eigenvalues and eigenvectors are found by solving the eigenvalue problem

$$AX = \lambda X \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proceeding as in Example 5:

$(A - \lambda I)X = 0$ and non-trivial solutions for $X$ will exist if det $(A - \lambda I) = 0$

**Solution (contd.)**

that is,

$$\det\left\{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = 0$$

$$\text{i.e.} \qquad \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

Expanding this determinant we find:

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

that is,

$$(2-\lambda)\left\{(2-\lambda)^2 - 1\right\} - (2-\lambda) = 0$$

Taking out the common factor $(2-\lambda)$:

$$(2-\lambda)\left\{4 - 4\lambda + \lambda^2 - 1 - 1\right\}$$

which gives: $\qquad (2-\lambda)\left[\lambda^2 - 4\lambda + 2\right] = 0.$

This is easily solved to give: $\lambda = 2$ or $\lambda = \dfrac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}.$

So (typically) we have found three possible values of $\lambda$ for this $3 \times 3$ matrix $A$.

To each value of $\lambda$ there corresponds an eigenvector.

**Case 1:** $\lambda = 2 - \sqrt{2}$ (lowest eigenvalue)

Then $AX = (2-\sqrt{2})X$ implies

$$\begin{aligned} 2x - y &= (2-\sqrt{2})x \\ -x + 2y - z &= (2-\sqrt{2})y \\ -y + 2z &= (2-\sqrt{2})z \end{aligned}$$

Simplifying

$$\begin{aligned} \sqrt{2}x - y &= 0 && \text{(a)} \\ -x + \sqrt{2}y - z &= 0 && \text{(b)} \\ -y + \sqrt{2}z &= 0 && \text{(c)} \end{aligned}$$

We conclude the following:

(c) $\Rightarrow y = \sqrt{2}z \qquad$ (a) $\Rightarrow y = \sqrt{2}x$

$\therefore \qquad$ these two relations give $\quad x = z \quad$ then $\qquad$ (b) $\Rightarrow -x + 2x - x = 0$

The last equation gives us no information; it simply states that $0 = 0$.

**Solution (contd.)**

$\therefore \qquad X = \begin{bmatrix} x \\ \sqrt{2}x \\ x \end{bmatrix}$ for any $x \neq 0$ (otherwise we would have the trivial solution). So the eigenvectors corresponding to eigenvalue $\lambda = 2 - \sqrt{2}$ are all proportional to $\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$.

In normalised form we have an eigenvector $\dfrac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$.

**Case 2:** $\lambda = 2$

Here $AX = 2X$ implies $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e.

$$\begin{aligned} 2x - y &= 2x \\ -x + 2y - z &= 2y \\ -y + 2z &= 2z \end{aligned}$$

After simplifying the equations become:

$$\begin{aligned} -y &= 0 \qquad \text{(a)} \\ -x - z &= 0 \qquad \text{(b)} \\ -y &= 0 \qquad \text{(c)} \end{aligned}$$

(a), (c) imply $y = 0$: (b) implies $x = -z$

$\therefore \qquad$ eigenvector has the form $\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix}$ for any $x \neq 0$.

That is, eigenvectors corresponding to $\lambda = 2$ are all proportional to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

In normalised form we have an eigenvector $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

**Solution (contd.)**

**Case 3:** $\lambda = 2 + \sqrt{2}$ (largest eigenvalue)

Proceeding along similar lines to cases 1,2 above we find that the eigenvectors corresponding to $\lambda = 2 + \sqrt{2}$ are each proportional to $\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ with normalised eigenvector $\frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$.

In conclusion the matrix $A$ has three distinct eigenvalues:

$$\lambda_1 = 2 - \sqrt{2}, \qquad \lambda_2 = 2 \qquad \lambda_3 = 2 + \sqrt{2}$$

and three corresponding normalised eigenvectors:

$$X_1 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \qquad X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad X_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

## Exercise

Find the eigenvalues and eigenvectors of each of the following matrices $A$:

(a) $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & -2 & 4 \\ -20 & 4 & -10 \\ -30 & 6 & -13 \end{bmatrix}$

**Answer** (eigenvectors are written in normalised form)

(a) 3 and 2; $\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

(b) 3 and 9; $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(c) 1, 4 and 6; $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(d) 0, $-1$ and 2; $\frac{1}{\sqrt{26}}\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$; $\frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$; $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

## 3. Properties of eigenvalues and eigenvectors

There are a number of general properties of eigenvalues and eigenvectors which you should be familiar with. You will be able to use them as a check on some of your calculations.

**Property 1: Sum of eigenvalues**

For any square matrix $A$:

sum of eigenvalues = sum of diagonal terms of $A$ (called the **trace** of $A$)

Formally, for an $n \times n$ matrix $A$: $$\sum_{i=1}^{n} \lambda_i = \text{trace}(A)$$

(Repeated eigenvalues must be counted according to their multiplicity.)

Thus if $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 1$ then $\displaystyle\sum_{i=1}^{3} \lambda_i = 9$).

**Property 2: Product of eigenvalues**

For any square matrix $A$:

product of eigenvalues = determinant of $A$

Formally: $$\lambda_1\lambda_2\lambda_3 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i = \det(A)$$

The symbol $\prod$ simply denotes multiplication, as $\sum$ denotes summation.

### Example 7

Verify Properties 1 and 2 for the $3 \times 3$ matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

whose eigenvalues were found earlier.

**Solution**

The three eigenvalues of this matrix are:

$$\lambda_1 = 2 - \sqrt{2}, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2 + \sqrt{2}$$

Therefore

$$\lambda_1 + \lambda_2 + \lambda_3 = (2 - \sqrt{2}) + 2 + (2 + \sqrt{2}) = 6 = \text{trace}(A)$$

whilst $$\lambda_1\lambda_2\lambda_3 = (2 - \sqrt{2})(2)(2 + \sqrt{2}) = 4 = \det(A)$$

**Property 3: Linear independence of eigenvectors**

Eigenvectors of a matrix $A$ corresponding to distinct eigenvalues are **linearly independent** i.e. one eigenvector **cannot** be written as a linear sum of the other eigenvectors. The proof of this result is omitted but we illustrate this property with two examples.

We saw earlier that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

has distinct eigenvalues $\lambda_1 = 1 \qquad \lambda_2 = 2$ with associated eigenvectors

$$X^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad X^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

respectively.

Clearly $X^{(1)}$ is **not** a constant multiple of $X^{(2)}$ and these eigenvectors are **linearly independent**.

We also saw that the $3 \times 3$ matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

had the following distinct eigenvalues $\lambda_1 = 2 - \sqrt{2},\ \lambda_2 = 2,\ \lambda_3 = 2 + \sqrt{2}$ with corresponding eigenvectors of the form shown:

$$X^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \qquad X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad X^{(3)} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Clearly none of these eigenvectors is a constant multiple of any other. Nor is any one obtainable as a linear combination of the other two. The three eigenvectors are **linearly independent**.

**Property 4: Eigenvalues of diagonal matrices**

A $2 \times 2$ diagonal matrix $D$ has the form

$$D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

The characteristic equation

$$|D - \lambda I| = 0 \quad \text{is} \quad \begin{vmatrix} a - \lambda & 0 \\ 0 & d - \lambda \end{vmatrix} = 0$$

i.e. $\qquad (a - \lambda)(d - \lambda) = 0$

So the eigenvalues are simply the diagonal elements $a$ and $d$.

Similarly a $3 \times 3$ diagonal matrix has the form

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

having characteristic equation

$$|D - \lambda I| = (a - \lambda)(b - \lambda)(c - \lambda) = 0$$

so again the diagonal elements **are** the eigenvalues.

We can see that a diagonal matrix is a particularly simple matrix to work with. In addition to the eigenvalues being obtainable immediately by inspection it is exceptionally easy to multiply diagonal matrices.

**Task**

Obtain the products $D_1D_2$ and $D_2D_1$ of the diagonal matrices

$$D_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad D_2 = \begin{bmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix}$$

**Your solution**

**Answer**

$$D_1D_2 = D_2D_1 = \begin{bmatrix} ae & 0 & 0 \\ 0 & bf & 0 \\ 0 & 0 & cg \end{bmatrix}$$

which of course is also a diagonal matrix.

## Exercise

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of a matrix $A$, prove the following:

(a) $A^T$ has eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$.

(b) If $A$ is upper triangular, then its eigenvalues are exactly the main diagonal entries.

(c) The inverse matrix $A^{-1}$ has eigenvalues $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \ldots \dfrac{1}{\lambda_n}$.

(d) The matrix $A - kI$ has eigenvalues $\lambda_1 - k, \lambda_2 - k, \ldots \lambda_n - k$.

(e) (Harder) The matrix $A^2$ has eigenvalues $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$.

(f) (Harder) The matrix $A^k$ ($k$ a non-negative integer) has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$.

Verify the above results for any $2 \times 2$ matrix and any $3 \times 3$ matrix found in the previous Exercises on page 13.

N.B. Some of these results are useful in the numerical calculation of eigenvalues which we shall consider later.

**Answer**

(a) Using the property that for any square matrix $A$, $\det(A) = \det(A^T)$ we see that if

$$\det(A - \lambda I) = 0 \qquad \text{then} \qquad \det(A - \lambda I)^T = 0$$

This immediately tells us that $\det(A^T - \lambda I) = 0$ which shows that $\lambda$ is also an eigenvalue of $A^T$.

(b) Here simply write down a typical upper triangular matrix $U$ which has terms on the leading diagonal $u_{11}, u_{22}, \ldots, u_{nn}$ and above it. Then construct $(U - \lambda I)$. Finally imagine how you would then obtain $\det(U - \lambda I) = 0$. You should see that the determinant is obtained by multiplying together those terms on the leading diagonal. Here the characteristic equation is:

$$(u_{11} - \lambda)(u_{22} - \lambda) \ldots (u_{nn} - \lambda) = 0$$

This polynomial has the obvious roots $\lambda_1 = u_{11},\ \lambda_2 = u_{22},\ \ldots, \lambda_n = u_{nn}$.

(c) Here we begin with the usual eigenvalue problem $AX = \lambda X$. If $A$ has an inverse $A^{-1}$ we can multiply both sides by $A^{-1}$ on the left to give

$$A^{-1}(AX) = A^{-1}\lambda X \qquad \text{which gives} \qquad X = \lambda A^{-1} X$$

or, dividing through by the scalar $\lambda$ we get

$A^{-1}X = \dfrac{1}{\lambda}X$ which shows that if $\lambda$ and $X$ are respectively eigenvalue and eigenvector of $A$ then $\lambda^{-1}$ and $X$ are respectively eigenvalue and eigenvector of $A^{-1}$.

As an example consider $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. This matrix has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 5$ with corresponding eigenvectors $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The reader should verify (by direct multiplication) that $A^{-1} = -\dfrac{1}{5}\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ has eigenvalues $-1$ and $\dfrac{1}{5}$ with respective eigenvectors $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) (e) and (f) are proved in similar way to the proof outlined in (c).