Sampled Functions 21.5

Introduction

A sequence can be obtained by **sampling** a continuous function or signal and in this Section we show first of all how to extend our knowledge of z-transforms so as to be able to deal with sampled signals. We then show how the z-transform of a sampled signal is related to the Laplace transform of the unsampled version of the signal.

Prerequisites

Before starting this Section you should ...

- possess an outline knowledge of Laplace transforms and of z-transforms

Learning Outcomes

On completion you should be able to ...

- take the z-transform of a sequence obtained by sampling
- state the relation between the z-transform of a sequence obtained by sampling and the Laplace transform of the underlying continuous signal

1. Sampling theory

If a **continuous-time** signal $f(t)$ is sampled at terms $t = 0, T, 2T, \ldots nT, \ldots$ then a sequence of values

$$\{f(0),\ f(T),\ f(2T), \ldots f(nT), \ldots\}$$

is obtained. The quantity T is called the **sample interval** or **sample period**.

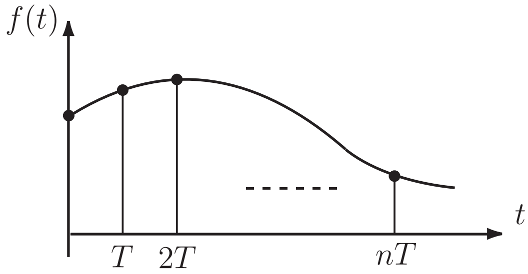


Figure 18

In the previous Sections of this Workbook we have used the simpler notation $\{f_n\}$ to denote a sequence. If the sequence has actually arisen by sampling then f_n is just a convenient notation for the sample value $f(nT)$.

Most of our previous results for z-transforms of sequences hold with only minor changes for sampled signals.

So consider a continuous signal $f(t)$; its z-transform is the z-transform of the sequence of sample values i.e.

$$\mathbb{Z}\{f(t)\} = \mathbb{Z}\{f(nT)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

We shall briefly obtain z-transforms of common sampled signals utilizing results obtained earlier. You may assume that all signals are sampled at $0, T, 2T, \ldots nT, \ldots$

Unit step function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Since the sampled values here are a sequence of 1's,

$$\begin{aligned} \mathbb{Z}\{u(t)\} = \mathbb{Z}\{u_n\} &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z-1} \qquad |z| > 1 \end{aligned}$$

where $\{u_n\} = \{\underset{\uparrow}{1},\ 1,\ 1,\ \ldots\}$ is the unit step sequence.

Ramp function

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The sample values here are

$$\{r(nT)\} = \{0,\ T,\ 2T,\ \ldots\}$$

The ramp sequence $\{r_n\} = \{0,\ 1,\ 2,\ \ldots\}$ has z-transform $\dfrac{z}{(z-1)^2}$.

Hence $\mathbb{Z}\{r(nT)\} = \dfrac{Tz}{(z-1)^2}$ since $\{r(nT)\} = T\{r_n\}$.

Task

Obtain the z-transform of the exponential signal

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

[Hint: use the z-transform of the geometric sequence $\{a^n\}$.]

Your solution

Answer

The sample values of the exponential are

$$\{1,\ e^{-\alpha T},\ e^{-\alpha 2T}, \ldots, e^{-\alpha nT}, \ldots\}$$

i.e. $f(nT) = e^{-\alpha nT} = (e^{-\alpha T})^n$.

But $\mathbb{Z}\{a^n\} = \dfrac{z}{z-a}$

$$\therefore \qquad \mathbb{Z}\{(e^{-\alpha T})^n\} = \frac{z}{z - e^{-\alpha T}} = \frac{1}{1 - e^{-\alpha T} z^{-1}}$$

Sampled sinusoids

Earlier in this Workbook we obtained the z-transform of the sequence $\{\cos \omega n\}$ i.e.

$$\mathbb{Z}\{\cos \omega n\} = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$$

Hence, since sampling the continuous sinusoid

$$f(t) = \cos \omega t$$

yields the sequence $\{\cos n\omega T\}$ we have, simply replacing ω by ωT in the z-transform:

$$\begin{aligned} \mathbb{Z}\{\cos \omega t\} &= \mathbb{Z}\{\cos n\omega T\} \\ &= \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1} \end{aligned}$$

Obtain the z-transform of the sampled version of the sine wave $f(t) = \sin \omega t$.

Your solution

Answer

$$\mathbb{Z}\{\sin \omega n\} = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$$

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{\sin \omega t\} &= \mathbb{Z}\{\sin n\omega T\} \\ &= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \end{aligned}$$

Shift theorems

These are similar to those discussed earlier in this Workbook but for sampled signals the shifts are by integer multiples of the sample period T. For example a simple right shift, or delay, of a sampled signal by one sample period is shown in the following figure:

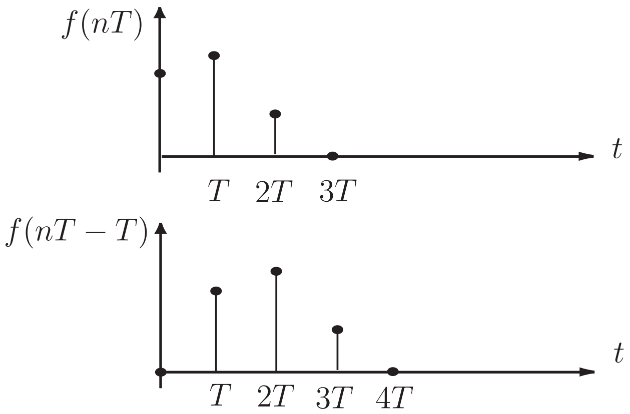


Figure 19

The right shift properties of z-transforms can be written down immediately. (Look back at the shift properties in Section 21.2 subsection 5, if necessary:)

If $y(t)$ has z-transform $Y(z)$ which, as we have seen, really means that its sample values $\{y(nT)\}$ give $Y(z)$, then for $y(t)$ shifted to the right by one sample interval the z-transform becomes

$$\mathbb{Z}\{y(t-T)\} = y(-T) + z^{-1}Y(z)$$

The proof is very similar to that used for sequences earlier which gave the result:

$$\mathbb{Z}\{y_{n-1}\} = y_{-1} + z^{-1}Y(z)$$

Task

Using the result

$$\mathbb{Z}\{y_{n-2}\} = y_{-2} + y_{-1}z^{-1} + z^{-2}Y(z)$$

write down the result for $\mathbb{Z}\{y(t-2T)\}$

Your solution

Answer

$$\mathbb{Z}\{y(t-2T)\} = y(-2T) + y(-T)z^{-1} + z^{-2}Y(z)$$

These results can of course be generalised to obtain $\mathbb{Z}\{y(t-mT)\}$ where m is any positive integer. In particular, for causal or one-sided signals $y(t)$ (i.e. signals which are zero for $t<0$):

$$\mathbb{Z}\{y(t-mT)\} = z^{-m}Y(z)$$

Note carefully here that the power of z is still z^{-m} **not** z^{-mT}.

Examples:
For the unit step function we saw that:

$$\mathbb{Z}\{u(t)\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

Hence from the shift properties above we have immediately, since $u(t)$ is certainly causal,

$$\begin{aligned} \mathbb{Z}\{u(t-T)\} &= \frac{zz^{-1}}{z-1} = \frac{z^{-1}}{1-z^{-1}} \\ \mathbb{Z}\{u(t-3T)\} &= \frac{zz^{-3}}{z-1} = \frac{z^{-3}}{1-z^{-1}} \end{aligned}$$

and so on.

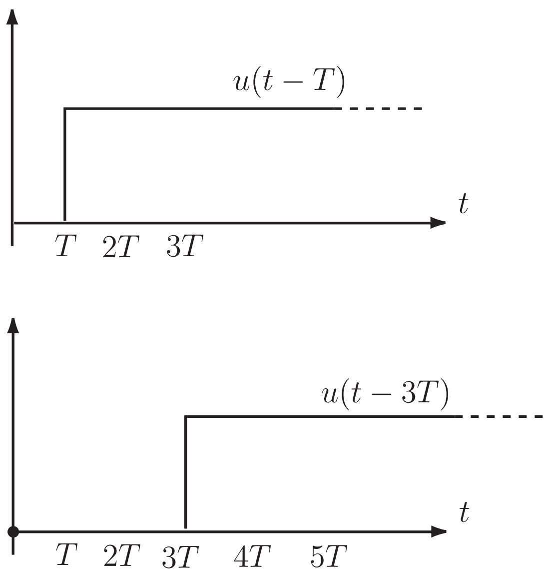


Figure 20

2. z-transforms and Laplace transforms

In this Workbook we have developed the theory and some applications of the z-transform from first principles. We mentioned much earlier that the z-transform plays essentially the same role for discrete systems that the Laplace transform does for continuous systems. We now explore the precise link between these two transforms. A brief knowledge of Laplace transform will be assumed.

At first sight it is not obvious that there is a connection. The z-transform is a **summation** defined, for a sampled signal $f_n \equiv f(nT)$, as

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

while the Laplace transform written symbolically as $\mathbb{L}\{f(t)\}$ is an **integral**, defined for a continuous time function $f(t),\ t \geq 0$ as

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt.$$

Thus, for example, if

$$f(t) = e^{-\alpha t} \qquad \text{(continuous time exponential)}$$

$$\mathbb{L}\{f(t)\} = F(s) = \frac{1}{s+\alpha}$$

which has a (simple) pole at $s = -\alpha = s_1$ say.
As we have seen, sampling $f(t)$ gives the sequence $\{f(nT)\} = \{e^{-\alpha nT}\}$ with z-transform

$$F(z) = \frac{1}{1 - e^{-\alpha T}z^{-1}} = \frac{z}{z - e^{-\alpha T}}.$$

The z-transform has a pole when $z = z_1$ where

$$z_1 = e^{-\alpha T} = e^{s_1 T}$$

[Note the abuse of notations in writing both $F(s)$ and $F(z)$ here since in fact these are **different** functions.]

Task

The continuous time function $f(t) = te^{-\alpha t}$ has Laplace transform $F(s) = \dfrac{1}{(s+\alpha)^2}$

Firstly write down the pole of this function and its order:

Your solution

Answer

$F(s) = \dfrac{1}{(s+\alpha)^2}$ has its pole at $s = s_1 = -\alpha$. The pole is second order.

Now obtain the z-transform $F(z)$ of the sampled version of $f(t)$, locate the pole(s) of $F(z)$ and state the order:

Your solution

Answer

Consider $f(nT) = nTe^{-\alpha nT} = (nT)(e^{-\alpha T})^n$

The ramp sequence $\{nT\}$ has z-transform $\dfrac{Tz}{(z-1)^2}$

$\therefore \quad f(nT)$ has z-transform

$$F(z) = \frac{Tze^{\alpha T}}{(ze^{\alpha T}-1)^2} = \frac{Tze^{-\alpha T}}{(z-e^{-\alpha T})^2} \qquad \text{(see Key Point 8)}$$

This has a (second order) pole when $z = z_1 = e^{-\alpha T} = e^{s_1 T}$.

We have seen in both the above examples a close link between the pole s_1 of the Laplace transform of $f(t)$ and the pole z_1 of the z-transform of the sampled version of $f(t)$ i.e.

$$z_1 = e^{s_1 T} \qquad (1)$$

where T is the sample interval.

Multiple poles lead to similar results i.e. if $F(s)$ has poles $s_1, s_2, \ldots$ then $F(z)$ has poles $z_1, z_2, \ldots$ where $z_i = e^{s_i T}$.

The relation (1) between the poles is, in fact, an example of a more general relation between the values of s and z as we shall now investigate.

Key Point 19

The **unit impulse function** $\delta(t)$ can be defined informally as follows:

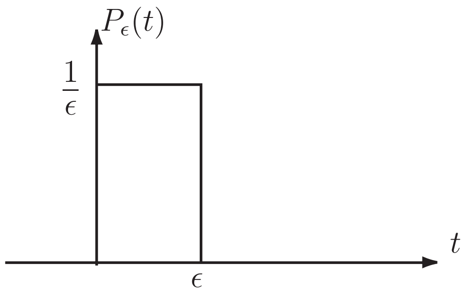


Figure 21

The rectangular pulse $P_\epsilon(t)$ of width ε and height $\dfrac{1}{\varepsilon}$ shown in Figure 21 encloses unit area and has Laplace transform

$$P_\varepsilon(s) = \int_0^\varepsilon \frac{1}{\varepsilon} e^{-st} = \frac{1}{\varepsilon s}(1 - e^{-\varepsilon s}) \qquad (2)$$

As ε becomes smaller $P_\varepsilon(t)$ becomes taller and narrower but still encloses unit area. The unit impulse function $\delta(t)$ (sometimes called the Dirac delta function) can be defined as

$$\delta(t) = \lim_{\varepsilon \to 0} P_\varepsilon(t)$$

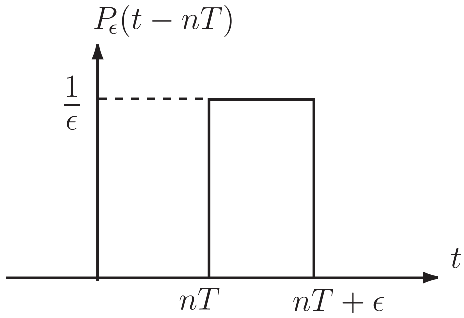
The Laplace transform, say $\Delta(s)$, of $\delta(t)$ can be obtained correspondingly by letting $\epsilon \to 0$ in (2), i.e.

$$\begin{aligned}\Delta(s) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon s}(1 - e^{-\varepsilon s}) \\ &= \lim_{\varepsilon \to 0} \frac{1 - (1 - \varepsilon s + \frac{(\varepsilon s)^2}{2!} - \ldots)}{\varepsilon s} \qquad \text{(Using the Maclaurin seies expansion of } e^{-\varepsilon s}\text{)} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon s - \frac{(\varepsilon s)^2}{2!} + \frac{(\varepsilon s)^3}{3!} + \ldots}{\varepsilon s} \\ &= 1\end{aligned}$$

i.e. $\mathbb{L}\delta(t) = 1$ (3)

Task

A shifted unit impulse $\delta(t - nT)$ is defined as $\lim_{\varepsilon \to 0} P_\varepsilon(t - nT)$ as illustrated below.



Obtain the Laplace transform of this rectangular pulse and, by letting $\varepsilon \to 0$, obtain the Laplace transform of $\delta(t - nT)$.

Your solution

Answer

$$\mathbb{L}\{P_\varepsilon(t-nT)\} = \int_{nT}^{nT+\varepsilon} \frac{1}{\varepsilon}e^{-st}dt = \frac{1}{\varepsilon s}\Big[-e^{-st}\Big]_{nT}^{nT+\varepsilon}$$

$$= \frac{1}{\varepsilon s}\left(e^{-snT} - e^{-s(nT+\varepsilon)}\right)$$

$$= \frac{1}{\varepsilon s}e^{-snT}(1-e^{-s\varepsilon}) \to e^{-snT} \quad \text{as } \varepsilon \to 0$$

Hence $\mathbb{L}\{\delta(t-nT)\} = e^{-snT}$ (4)

which reduces to the result (3)

$\mathbb{L}\{\delta(t)\} = 1$ when $n = 0$

These results (3) and (4) can be compared with the results

$$\mathbb{Z}\{\delta_n\} = 1$$

$$\mathbb{Z}\{\delta_{n-m}\} = z^{-m}$$

for discrete impulses of height 1.

Now consider a continuous function $f(t)$. Suppose, as usual, that this function is sampled at $t = nT$ for $n = 0, 1, 2, \ldots$

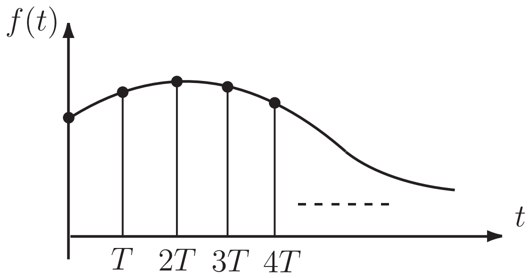


Figure 22

This sampled equivalent of $f(t)$, say $f_*(t)$ can be defined as a sequence of equidistant impulses, the 'strength' of each impulse being the sample value $f(nT)$i.e.

$$f_*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t-nT)$$

This function is a continuous-time signal i.e. is defined for all t. Using (4) it has a Laplace transform

$$F_*(s) = \sum_{n=0}^{\infty} f(nT)e^{-snT} \qquad (5)$$

If, in this sum (5) we replace e^{sT} by z we obtain the z-transform of the sequence $\{f(nT)\}$ of samples:

$$\sum_{n=0}^{\infty} f(nT)z^{-n}$$

Key Point 20

The Laplace transform

$$F(s) = \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

of a sampled function is equivalent to the z-transform $F(z)$ of the sequence $\{f(nT)\}$ of sample values with $z = e^{sT}$.

Table 2: z-transforms of some sampled signals

This table can be compared with the table of the z-transforms of sequences on the following page.

$f(t)$ $t \geq 0$	$f(nT)$ $n = 0, 1, 2, \ldots$	$F(z)$	Radius of convergence R
1	1	$\dfrac{z}{z-1}$	1
t	nT	$\dfrac{z}{(z-1)^2}$	1
t^2	$(nT)^2$	$\dfrac{T^2 z(z+1)}{(z-1)^3}$	1
$e^{-\alpha t}$	$e^{-\alpha nT}$	$\dfrac{z}{z - e^{-\alpha T}}$	$\lvert e^{-\alpha T}\rvert$
$\sin \omega t$	$\sin n\omega T$	$\dfrac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$	1
$\cos \omega t$	$\cos n\omega T$	$\dfrac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$	1
$te^{-\alpha t}$	$nTe^{-\alpha nT}$	$\dfrac{Tze^{-\alpha T}}{(z - e^{-\alpha T})^2}$	$\lvert e^{-\alpha T}\rvert$
$e^{-\alpha t} \sin \omega t$	$e^{-\alpha nT} \sin \omega nT$	$\dfrac{e^{-\alpha T} z^{-1} \sin \omega T}{1 - 2e^{-\alpha T} z^{-1} \cos \omega T + e^{-2\alpha T} z^{-2}}$	$\lvert e^{-\alpha T}\rvert$
$e^{-\alpha T} \cos \omega t$	$e^{-\alpha nT} \cos \omega nT$	$\dfrac{1 - e^{-\alpha T} z^{-1} \cos \omega T}{1 - 2e^{-\alpha T} z^{-1} \cos \omega T + e^{-2\alpha T} z^{-2}}$	$\lvert e^{-\alpha T}\rvert$

Note: R is such that the closed forms of $F(z)$ (those listed in the above table) are valid for $|z| > R$.

Table of z-transforms

f_n	$F(z)$	Name
δ_n	1	unit impulse
δ_{n-m}	z^{-m}	
u_n	$\dfrac{z}{z-1}$	unit step sequence
a^n	$\dfrac{z}{z-a}$	geometric sequence
$e^{\alpha n}$	$\dfrac{z}{z-e^{\alpha}}$	
$\sinh \alpha n$	$\dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$	
$\cosh \alpha n$	$\dfrac{z^2-z\cosh\alpha}{z^2-2z\cosh\alpha+1}$	
$\sin \omega n$	$\dfrac{z\sin\omega}{z^2-2z\cos\omega+1}$	
$\cos \omega n$	$\dfrac{z^2-z\cos\omega}{z^2-2z\cos\omega+1}$	
$e^{-\alpha n}\sin \omega n$	$\dfrac{ze^{-\alpha}\sin\omega}{z^2-2ze^{-\alpha}\cos\omega+e^{-2\alpha}}$	
$e^{-\alpha n}\cos \omega n$	$\dfrac{z^2-ze^{-\alpha}\cos\omega}{z^2-2ze^{-\alpha}\cos\omega+e^{-2\alpha}}$	
n	$\dfrac{z}{(z-1)^2}$	ramp sequence
n^2	$\dfrac{z(z+1)}{(z-1)^3}$	
n^3	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	
$a^n f_n$	$F\left(\dfrac{z}{a}\right)$	
$n\ f_n$	$-z\dfrac{dF}{dz}$	