Engineering Applications of z-Transforms

21.4

Introduction

In this Section we shall apply the basic theory of z-transforms to help us to obtain the response or output sequence for a discrete system. This will involve the concept of the transfer function and we shall also show how to obtain the transfer functions of series and feedback systems. We will also discuss an alternative technique for output calculations using convolution. Finally we shall discuss the initial and final value theorems of z-transforms which are important in digital control.

Prerequisites

Before starting this Section you should ...

- be familiar with basic z-transforms, particularly the shift properties

Learning Outcomes

On completion you should be able to ...

- obtain transfer functions for discrete systems including series and feedback combinations
- state the link between the convolution summation of two sequences and the product of their z-transforms

1. Applications of z-transforms

Transfer (or system) function

Consider a first order linear constant coefficient difference equation

$$y_n + a\ y_{n-1} = bx_n \qquad n = 0, 1, 2, \ldots \tag{1}$$

where $\{x_n\}$ is a given sequence.
Assume an initial condition y_{-1} is given.

Task

Take the z-transform of (1), insert the initial condition and obtain $Y(z)$ in terms of $X(z)$.

Your solution

Answer
Using the right shift theorem

$$Y(z) + a(z^{-1}Y(z) + y_{-1}) = b\ X(z)$$

where $X(z)$ is the z-transform of the given or **input** sequence $\{x_n\}$ and $Y(z)$ is the z-transform of the **response** or **output** sequence $\{y_n\}$.

Solving for $Y(z)$

$$Y(z)(1 + az^{-1}) = bX(z) - ay_{-1}$$

so

$$Y(z) = \frac{bX(z)}{1 + az^{-1}} - \frac{ay_{-1}}{1 + az^{-1}} \tag{2}$$

The form of (2) shows us clearly that $Y(z)$ is made up of two components, $Y_1(z)$ and $Y_2(z)$ say, where

(i) $Y_1(z) = \dfrac{bX(z)}{1 + az^{-1}}$ which depends on the input $X(z)$

(ii) $Y_2(z) = \dfrac{-ay_{-1}}{1 + az^{-1}}$ which depends on the initial condition y_{-1}.

Clearly, from (2), if $y_{-1} = 0$ (zero initial condition) then

$$Y(z) = Y_1(z)$$

and hence the term **zero-state response** is sometimes used for $Y_1(z)$.
Similarly if $\{x_n\}$ and hence $X(z) = 0$ (zero input)

$$Y(z) = Y_2(z)$$

and hence the term **zero-input** response can be used for $Y_2(z)$.

In engineering the difference equation (1) is regarded as modelling a **system** or more specifically a linear **discrete** time-invariant system. The terms linear and time-invariant arise because the difference equation (1) is linear and has constant coefficients i.e. the coefficients do not involve the index n. The term 'discrete' is used because sequences of numbers, not continuous quantities, are involved. As noted above, the given sequence $\{x_n\}$ is considered to be the input sequence and $\{y_n\}$, the solution to (1), is regarded as the output sequence.

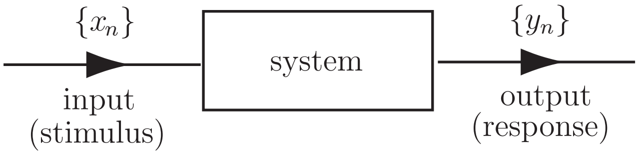
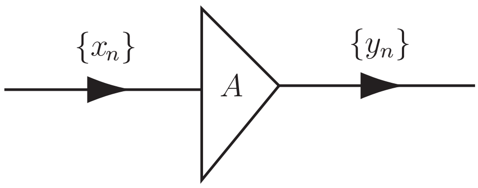


Figure 8

A more precise block diagram representation of a system can be easily drawn since only two operations are involved:

1. Multiplying the terms of a sequence by a constant.
2. Shifting to the right, or delaying, the terms of the sequence.

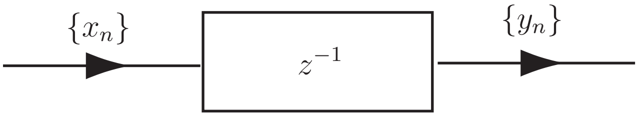
A system which consists of a single multiplier is denoted as shown by a triangular symbol:



$$y_n = Ax_n$$

Figure 9

As we have seen earlier in this workbook a system which consists of only a single delay unit is represented symbolically as follows



$$y_n = x_{n-1}$$

Figure 10

The system represented by the difference equation (1) consists of two multipliers and one delay unit. Because (1) can be written

$$y_n = bx_n - ay_{n-1}$$

a symbolic representation of (1) is as shown in Figure 11.

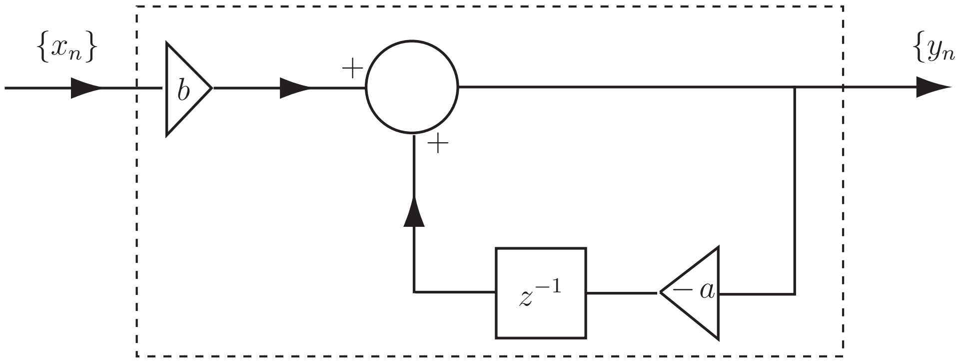


Figure 11

The circle symbol denotes an adder or summation unit whose output is the sum of the two (or more) sequences that are input to it.

We will now concentrate upon the zero state response of the system i.e. we will assume that the initial condition y_{-1} is zero.

Thus, using (2),

$$Y(z) = \frac{bX(z)}{1 + az^{-1}}$$

so

$$\frac{Y(z)}{X(z)} = \frac{b}{1 + az^{-1}} \qquad (3)$$

The quantity $\dfrac{Y(z)}{X(z)}$, the ratio of the output z-transform to the input z-transform, is called the **transfer function** of the discrete system. It is often denoted by $H(z)$.

Key Point 16

The transfer function $H(z)$ of a discrete system is defined by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\text{z-transform of output sequence}}{\text{z-transform of input sequence}}$$

when the initial conditions are zero.

(a) Write down the transfer function $H(z)$ of the system represented by (1)

(i) using negative powers of z
(ii) using positive powers of z.

(b) Write down the inverse z-transform of $H(z)$.

Your solution

Answer
(a) From (3)

(i) $H(z) = \dfrac{b}{1 + az^{-1}}$

(ii) $H(z) = \dfrac{bz}{z + a}$

(b) Referring to the Table of z-transforms at the end of the Workbook:

$\{h_n\} = b(-a)^n \qquad n = 0, 1, 2, \ldots$

We can represent any discrete system as follows

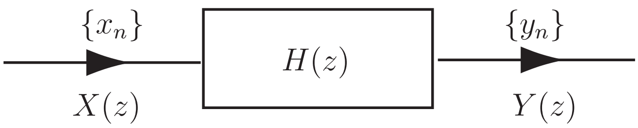


Figure 12

From the definition of the transfer function it follows that

$$Y(z) = X(z)H(z) \qquad \text{(at zero initial conditions).}$$

The corresponding relation between $\{y_n\}$, $\{x_n\}$ and the inverse z-transform $\{h_n\}$ of the transfer function will be discussed later; it is called a **convolution summation**.

The significance of $\{h_n\}$ is readily obtained.

Suppose $\quad \{x_n\} = \begin{cases} 1 & n = 0 \\ 0 & n = 1, 2, 3, \ldots \end{cases}$

i.e. $\{x_n\}$ is the unit impulse sequence that is normally denoted by δ_n. Hence, in this case,

$$X(z) = Z\{\delta_n\} = 1 \quad \text{so} \quad Y(z) = H(z) \quad \text{and} \quad \{y_n\} = \{h_n\}$$

In words: $\{h_n\}$ is the response or output of a system where the input is the unit impulse sequence $\{\delta_n\}$. Hence $\{h_n\}$ is called the **unit impulse response** of the system.

Key Point 17

For a linear, time invariant discrete system, the unit impulse response and the system transfer function are a z-transform pair:

$$H(z) = \mathbb{Z}\{h_n\} \qquad \{h_n\} = \mathbb{Z}^{-1}\{H(z)\}$$

It follows from the previous Task that for the first order system (1)

$H(z) = \dfrac{b}{1 + az^{-1}} = \dfrac{bz}{z + a}$ is the transfer function and

$\{h_n\} = \{b(-a)^n\}$ is the unit impulse response sequence.

Task

Write down the transfer function of

(a) a single multiplier unit (b) a single delay unit.

Your solution

Answer

(a) $\{y_n\} = \{A\, x_n\}$ if the multiplying factor is A

$\therefore$ using the linearity property of z-transform

$Y(z) = AX(z)$

so $H(z) = \dfrac{Y(z)}{X(z)} = A$ is the required transfer function.

(b) $\{y_n\} = \{x_{n-1}\}$

so $Y(z) = z^{-1}X(z)$ (remembering that initial conditions are zero)

$\therefore$ $H(z) = z^{-1}$ is the transfer function of the single delay unit.

Task

Obtain the transfer function of the system.

$$y_n + a_1 y_{n-1} = b_0 x_n + b_1 x_{n-1} \qquad n = 0, 1, 2, \ldots$$

where $\{x_n\}$ is a known sequence with $x_n = 0$ for $n = -1, -2, \ldots$.
[Remember that the transfer function is only defined at zero initial condition i.e. assume $y_{-1} = 0$ also.]

Your solution

Answer
Taking z-transforms

$$\begin{aligned} Y(z) + a_1 z^{-1} Y(z) &= b_0 X(z) + b_1 z^{-1} X(z) \\ Y(z)(1 + a_1 z^{-1}) &= (b_0 + b_1 z^{-1}) X(z) \end{aligned}$$

so the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = \frac{b_0 z + b_1}{z + a_1}$$

Second order systems

Consider the system whose difference equation is

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b x_n \qquad n = 0, 1, 2, \ldots \tag{4}$$

where the input sequence $x_n = 0, \quad n = -1, -2, \ldots$

In exactly the same way as for first order systems it is easy to show that the system response has a z-transform with two components.

Task

Take the z-transform of (4), assuming given initial values $y_{-1},\ y_{-2}$. Show that $Y(z)$ has two components. Obtain the transfer function of the system (4).

Your solution

Answer

From (4)

$$Y(z) + a_1(z^{-1}Y(z) + y_{-1}) + a_2(z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}) = bX(z)$$
$$Y(z)(1 + a_1z^{-1} + a_2z^{-2}) + a_1y_{-1} + a_2z^{-1}y_{-1} + a_2y_{-2} = bX(z)$$

$$\therefore \qquad Y(z) = \frac{bX(z)}{1 + a_1z^{-1} + a_2z^{-2}} - \frac{(a_1y_{-1} + a_2z^{-1}y_{-1} + a_2y_{-2})}{1 + a_1z^{-1} + a_2z^{-2}} = Y_1(z) + Y_2(z) \quad \text{say.}$$

At zero initial conditions, $Y(z) = Y_1(z)$ so the transfer function is

$$H(z) = \frac{b}{1 + a_1z^{-1} + a_2z^{-2}} = \frac{bz^2}{z^2 + a_1z + a_2}.$$

Example

Obtain (i) the unit impulse response (ii) the unit step response of the system specified by the second order difference equation

$$y_n - \frac{3}{4}y_{n-1} + \frac{1}{8}y_{n-2} = x_n \qquad (5)$$

Note that both these responses refer to the case of zero initial conditions. Hence it is convenient to first obtain the transfer function $H(z)$ of the system and then use the relation $Y(z) = X(z)H(z)$ in each case.

We write down the transfer function of (5), using positive powers of z. Taking the z-transform of (5) at zero initial conditions we obtain

$$\begin{aligned} Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) &= X(z) \\ Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) &= X(z) \\ \therefore \qquad H(z) = \frac{Y(z)}{X(z)} &= \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \end{aligned}$$

We now complete the problem for inputs (i) $x_n = \delta_n$ (ii) $x_n = u_n$, the unit step sequence, using partial fractions.

$$H(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$

(i) With $x_n = \delta_n$ so $X(z) = 1$ the response is, as we saw earlier,

$$Y(z) = H(z)$$

so $y_n = h_n$

where $h_n = \mathbb{Z}^{-1}H(z) = 2 \times \left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \qquad n = 0, 1, 2, \ldots$

(ii) The z-transform of the unit step is $\dfrac{z}{z-1}$ so the unit step response has z-transform

$$\begin{aligned} Y(z) &= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\frac{z}{(z-1)} \\ &= -\frac{2z}{z-\frac{1}{2}} + \frac{\frac{1}{3}z}{z-\frac{1}{4}} + \frac{\frac{8}{3}z}{z-1} \end{aligned}$$

Hence, taking inverse z-transforms, the unit step response of the system is

$$y_n = (-2) \times \left(\frac{1}{2}\right)^n + \frac{1}{3} \times \left(\frac{1}{4}\right)^n + \frac{8}{3} \qquad n = 0, 1, 2, \ldots$$

Notice carefully the form of this unit step response - the first two terms decrease as n increases and are called **transients**. Thus

$$y_n \to \frac{8}{3} \quad \text{as} \quad n \to \infty$$

and the term $\dfrac{8}{3}$ is referred to as the **steady state** part of the unit step response.

Combinations of systems

The concept of transfer function enables us to readily analyse combinations of discrete systems.

Series combination

Suppose we have two systems S_1 and S_2 with transfer functions $H_1(z)$, $H_2(z)$ in series with each other. i.e. the output from S_1 is the input to S_2.

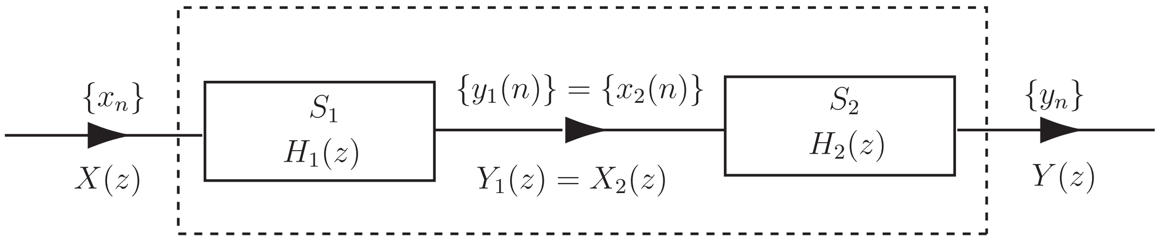


Figure 13

Clearly, at zero initial conditions,

$$\begin{aligned} Y_1(z) &= H_1(z)X(z) \\ Y(z) &= H_2(z)X_2(z) \\ &= H_2(z)Y_1(z) \\ \therefore \quad Y(z) &= H_2(z)H_1(z)X(z) \end{aligned}$$

so the ratio of the final output transform to the input transform is

$$\frac{Y(z)}{X(z)} = H_2(z)\ H_1(z) \qquad (6)$$

i.e. the series system shown above is equivalent to a single system with transfer function $H_2(z)\ H_1(z)$

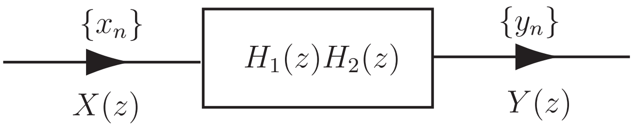


Figure 14

Task

Obtain (a) the transfer function (b) the governing difference equation of the system obtained by connecting two first order systems S_1 and S_2 in series. The governing equations are:

$$S_1: \quad y_n - ay_{n-1} = bx_n$$

$$S_2: \quad y_n - cy_{n-1} = dx_n$$

(a) Begin by finding the transfer function of S_1 and S_2 and then use (6):

Your solution

Answer

S_1: $\quad Y(z) - az^{-1}Y(z) = bX(z) \qquad$ so $\qquad H_1(z) = \dfrac{b}{1 - az^{-1}}$

S_2: $\quad H_2(z) = \dfrac{d}{1 - cz^{-1}}$

so the series arrangement has transfer function

$$\begin{aligned} H(z) &= \frac{bd}{(1 - az^{-1})(1 - cz^{-1})} \\ &= \frac{bd}{1 - (a + c)z^{-1} + acz^{-2}} \end{aligned}$$

If $X(z)$ and $Y(z)$ are the input and output transforms for the series arrangement, then

$$Y(z) = H(z)\ X(z) = \frac{bdX(z)}{1 - (a + c)z^{-1} + acz^{-2}}$$

(b) By transfering the denominator from the right-hand side to the left-hand side and taking inverse z-transforms obtain the required difference equation of the series arrangement:

Your solution

Answer
We have

$$Y(z)(1 - (a + c)z^{-1} + acz^{-2}) = bdX(z)$$

$$Y(z) - (a + c)z^{-1}Y(z) + acz^{-2}Y(z) = bdX(z)$$

from which, using the right shift theorem,

$$y_n - (a + c)y_{n-1} + acy_{n-2} = bd\ x_n.$$

which is the required difference equation.

You can see that the two first order systems in series have an equivalent second order system.

Feedback combination

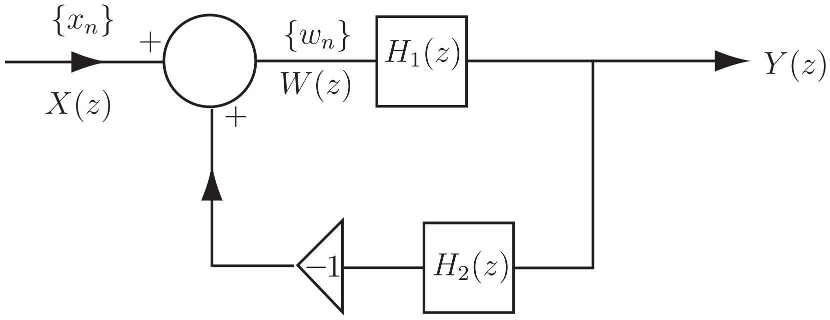


Figure 15

For the above **negative feedback** arrangement of two discrete systems with transfer functions $H_1(z),\ H_2(z)$ we have, at zero initial conditions,

$$Y(z) = W(z)H_1(z) \qquad \text{where} \qquad W(z) = X(z) - H_2(z)Y(z)$$

Task

Eliminate $W(z)$ and hence obtain the transfer function of the feedback system.

Your solution

Answer

$$\begin{aligned} Y(z) &= (X(z) - H_2(z)Y(z))H_1(z) \\ &= X(z)H_1(z) - H_2(z)H_1(z)Y(z) \end{aligned}$$

so

$$Y(z)(1 + H_2(z)H_1(z)) = X(z)H_1(z)$$

$$\therefore \qquad \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_2(z)H_1(z)}$$

This is the required transfer function of the negative feedback system.

2. Convolution and z-transforms

Consider a discrete system with transfer function $H(z)$

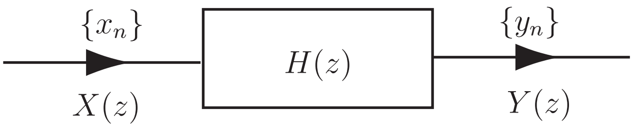


Figure 16

We know, from the definition of the transfer function that at zero initial conditions

$$Y(z) = X(z)H(z) \tag{7}$$

We now investigate the corresponding relation between the input sequence $\{x_n\}$ and the output sequence $\{y_n\}$. We have seen earlier that the system itself can be characterised by its unit impulse response $\{h_n\}$ which is the inverse z-transform of $H(z)$.

We are thus seeking the inverse z-transform of the product $X(z)H(z)$. We emphasize immediately that this is **not** given by the product $\{x_n\}\{h_n\}$, a point we also made much earlier in the workbook. We go back to basic definitions of the z-transform:

$$Y(z) = y_0 + y_1z^{-1} + y_2z^{-2} + y_3z^{-3} + \dots$$

$$X(z) = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \dots$$

$$H(z) = h_0 + h_1z^{-1} + h_2z^{-2} + h_3z^{-3} + \dots$$

Hence, multiplying $X(z)$ by $H(z)$ we obtain, collecting the terms according to the powers of z^{-1}:

$$x_0h_0 + (x_0h_1 + x_1h_0)z^{-1} + (x_0h_2 + x_1h_1 + x_2h_0)z^{-2} + \dots$$

Task

Write out the terms in z^{-3} in the product $X(z)H(z)$ and, looking at the emerging pattern, deduce the coefficient of z^{-n}.

Your solution

Answer

$(x_0h_3 + x_1h_2 + x_2h_1 + x_3h_0)z^{-3}$

which suggests that the coefficient of z^{-n} is

$x_0h_n + x_1h_{n-1} + x_2h_{n-2} + \ldots + x_{n-1}h_1 + x_nh_0$

Hence, comparing corresponding terms in $Y(z)$ and $X(z)H(z)$

$$\left.\begin{array}{lll} z^0: & y_0 & = x_0h_0 \\ z^{-1}: & y_1 & = x_0h_1 + x_1h_0 \\ z^{-2}: & y_2 & = x_0h_2 + x_1h_1 + x_2h_0 \\ z^{-3}: & y_3 & = x_0h_3 + x_1h_2 + x_2h_1 + x_3h_0 \end{array}\right\} \qquad (8)$$

$$\vdots \qquad \vdots$$

$$z^{-n}: \quad y_n = x_0h_n + x_1h_{n-1} + x_2h_{n-2} + \ldots + x_{n-1}h_1 + x_nh_0 \qquad (9)$$

$$= \sum_{k=0}^{n} x_k h_{n-k} \qquad (10a)$$

$$= \sum_{k=0}^{n} h_k x_{n-k} \qquad (10b)$$

(Can you see why (10b) also follows from (9)?)

The sequence $\{y_n\}$ whose n th term is given by (9) and (10) is said to be the **convolution** (or more precisely the **convolution summation**) of the sequences $\{x_n\}$ and $\{h_n\}$,

The convolution of two sequences is usually denoted by an asterisk symbol $(*)$.

We have shown therefore that

$$\mathbb{Z}^{-1}\{X(z)H(z)\} = \{x_n\} * \{h_n\} = \{h_n\} * \{x_n\}$$

where the general term of $\{x_n\} * \{h_n\}$ is in (10a) and that of $\{h_n\} * \{x_n\}$ is in (10b).

In words: the output sequence $\{y_n\}$ from a linear time invariant system is given by the convolution of the input sequence with the unit impulse response sequence of the system.

This result only holds if initial conditions are zero.

Key Point 18

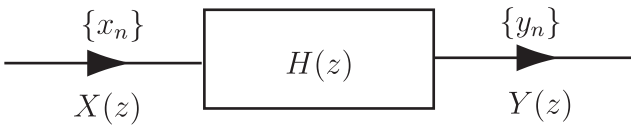


Figure 17

We have, at zero initial conditions

$$Y(z) = X(z)H(z) \qquad \text{(definition of transfer function)}$$

$$\{y_n\} = \{x_n\} * \{h_n\} \qquad \text{(convolution summation)}$$

where y_n is given in general by (9) and (10) with the first four terms written out explicitly in (8).

Although we have developed the convolution summation in the context of linear systems the proof given actually applies to any sequences i.e. for arbitrary causal sequences say $\{v_n\}$ $\{w_n\}$ with z-transforms $V(z)$ and $W(z)$ respectively:

$$\mathbb{Z}^{-1}\{V(z)W(z)\} = \{v_n\} * \{w_n\} \quad \text{or, equivalently,} \quad \mathbb{Z}(\{v_n\} * \{w_n\}) = V(z)W(z).$$

Indeed it is simple to prove this second result from the definition of the z-transform for any causal sequences $\{v_n\} = \{v_0, v_1, v_2, \ldots\}$ and $\{w_n\} = \{w_0, w_1, w_2, \ldots\}$

Thus since the general term of $\{v_n\} * \{w_n\}$ is $\displaystyle\sum_{k=0}^{n} v_k w_{n-k}$

we have

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} v_k w_{n-k}\right\} z^{-n}$$

or, since $w_{n-k} = 0$ if $k > n$,

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} v_k w_{n-k} z^{-n}$$

Putting $m = n - k$ or $n = m + k$ we obtain

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} v_k w_m z^{-(m+k)} \qquad \text{(Why is the lower limit } m = 0 \text{ correct?)}$$

Finally,

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{m=0}^{\infty} w_m z^{-m} \sum_{k=0}^{\infty} v_k z^{-k} = W(z)V(z)$$

which completes the proof.

Example 2

Calculate the convolution $\{y_n\}$ of the sequences

$$\{v_n\} = \{a^n\} \qquad \{w_n\} = \{b^n\} \qquad a \neq b$$

(i) directly (ii) using z-transforms.

Solution

(i) We have from (10)

$$\begin{aligned} y_n &= \sum_{k=0}^{n} v_k w_{n-k} = \sum_{k=0}^{n} a^k b^{n-k} \\ &= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \\ &= b^n \left(1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \ldots \left(\frac{a}{b}\right)^n\right) \end{aligned}$$

The bracketed sum involves $n+1$ terms of a geometric series of common ratio $\dfrac{a}{b}$.

$$\therefore \qquad \begin{aligned} y_n &= b^n \frac{\left(1 - \left(\frac{a}{b}\right)^{n+1}\right)}{1 - \frac{a}{b}} \\ &= \frac{(b^{n+1} - a^{n+1})}{(b-a)} \end{aligned}$$

(ii) The z-transforms are

$$V(z) = \frac{z}{z-a}$$

$$W(z) = \frac{z}{z-b}$$

so

$$\therefore \qquad \begin{aligned} y_n &= \mathbb{Z}^{-1}\{\frac{z^2}{(z-a)(z-b)}\} \\ &= \frac{b^{n+1} - a^{n+1}}{(b-a)} \qquad \text{using partial fractions or residues} \end{aligned}$$

Task

Obtain by two methods the convolution of the causal sequence

$\{2^n\} = \{1, 2, 2^2, 2^3, \ldots\}$

with itself.

Your solution

Answer

(a) By direct use of (10) if $\{y_n\} = \{2^n\} * \{2^n\}$

$$y_n = \sum_{k=0}^{n} 2^k 2^{n-k} = 2^n \sum_{k=0}^{n} 1 = (n+1)2^n$$

(b) Using z-transforms:

$$\mathbb{Z}\{2^n\} = \frac{z}{z-2}$$

so $\qquad \{y_n\} = \mathbb{Z}^{-1}\{\dfrac{z^2}{(z-2)^2}\}$

We will find this using the residue method. $Y(z)z^{n-1}$ has a second order pole at $z = 2$.

$$\begin{aligned} \therefore \qquad y_n &= \text{Res}\left(\frac{z^{n+1}}{(z-2)^2},\ 2\right) \\ &= \left[\frac{d}{dz} z^{n+1}\right]_2 = (n+1)2^n \end{aligned}$$

3. Initial and final value theorems of z-transforms

These results are important in, for example, Digital Control Theory where we are sometimes particularly interested in the initial and ultimate behaviour of systems.

Initial value theorem

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If f_n is a sequence with z-transform $F(z)$ then the 'initial value' f_0 is given by

$$f_0 = \lim_{z\to\infty} F(z) \qquad \text{(provided, of course, that this limit exists).}$$

This result follows, at least informally, from the definition of the z-transform:

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots$$

from which, taking limits as $z \to \infty$ the required result is obtained.

Task

Obtain the z-transform of

$$f(n) = 1 - a^n, \qquad 0 < a < 1$$

Verify the initial value theorem for the z-transform pair you obtain.

Your solution

Answer

Using standard z-transforms we obtain

$$\begin{aligned} Z\{f_n\} = F(z) &= \frac{z}{z-1} - \frac{z}{z-a} \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-az^{-1}} \end{aligned}$$

hence, as $z \to \infty : \; F(z) \to 1 - 1 = 0$

Similarly, as $n \to 0$

$$f_n \to 1 - 1 = 0$$

so the initial value theorem is verified for this case.

Final value theorem

Suppose again that $\{f_n\}$ is a sequence with z-transform $F(z)$. We further assume that all the poles of $F(z)$ lie inside the unit circle in the $z-$plane (i.e. have magnitude less than 1) apart possibly from a first order pole at $z = 1$.

The 'final value' of f_n i.e. $\lim_{n\to\infty} f_n$ is then given by $\quad \lim_{n\to\infty} f_n = \lim_{z\to 1}(1 - z^{-1})F(z)$

Proof: Recalling the left shift property

$$Z\{f_{n+1}\} = zF(z) - zf_0$$

we have

$$Z\{f_{n+1} - f_n\} = \lim_{k\to\infty} \sum_{n=0}^{k} (f_{n+1} - f_n)z^{-n} = zF(z) - zf_0 - F(z)$$

or, alternatively, dividing through by z on both sides:

$$(1 - z^{-1})F(z) - f_0 = \lim_{k\to\infty} \sum_{n=0}^{k} (f_{n+1} - f_n)z^{-(n+1)}$$

Hence $\quad (1 - z^{-1})F(z) = f_0 + (f_1 - f_0)z^{-1} + (f_2 - f_1)z^{-2} + \ldots$

or as $z \to 1$

$$\begin{aligned} \lim_{z\to 1}(1 - z^{-1})F(z) &= f_0 + (f_1 - f_0) + (f_2 - f_1) + \ldots \\ &= \lim_{k\to\infty} f_k \end{aligned}$$

Example

Again consider the sequence $\quad f_n = 1 - a^n \qquad 0 < a < 1 \quad$ and its z-transform

$$F(z) = \frac{z}{z-1} - \frac{z}{z-a} = \frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}}$$

Clearly as $n \to \infty$ then $f_n \to 1$.
Considering the right-hand side

$$(1 - z^{-1})F(z) = 1 - \frac{(1 - z^{-1})}{1 - az^{-1}} \to 1 - 0 = 1 \quad \text{as } z \to 1.$$

Note carefully that

$$F(z) = \frac{z}{z-1} - \frac{z}{z-a}$$

has a pole at a $(0 < a < 1)$ and a simple pole at $z = 1$.

The final value theorem does not hold for z-transform poles outside the unit circle
e.g. $\qquad f_n = 2^n \qquad F(z) = \dfrac{z}{z-2}$
Clearly $f_n \to \infty$ as $n \to \infty$
whereas

$$(1 - z^{-1})F(z) = \left(\frac{z-1}{z}\right)\frac{z}{(z-2)} \qquad \to 0 \text{ as } z \to 1$$

Exercises

1. A low pass digital filter is characterised by

$$y_n = 0.1x_n + 0.9y_{n-1}$$

Two such filters are connected in series. Deduce the transfer function and governing difference equation for the overall system. Obtain the response of the series system to (i) a unit step and (ii) a unit alternating input. Discuss your results.

2. The two systems

$$y_n = x_n - 0.7x_{n-1} + 0.4y_{n-1}$$

$$y_n = 0.9x_{n-1} - 0.7y_{n-1}$$

are connected in series. Find the difference equation governing the overall system.

3. A system S_1 is governed by the difference equation

$$y_n = 6x_{n-1} + 5y_{n-1}$$

It is desired to stabilise S_1 by using a feedback configuration. The system S_2 in the feedback loop is characterised by

$$y_n = \alpha x_{n-1} + \beta y_{n-1}$$

Show that the feedback system S_3 has an overall transfer function

$$H_3(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

and determine values for the parameters α and β if $H_3(z)$ is to have a second order pole at $z = 0.5$. Show briefly why the feedback systems S_3 *stabilizes* the original system.

4. Use z-transforms to find the sum of squares of all integers from 1 to n:

$$y_n = \sum_{k=1}^{n} k^2$$

[Hint: $y_n - y_{n-1} = n^2$]

5. Evaluate each of the following convolution summations (i) directly (ii) using z-transforms:

(a) $a^n * b^n \quad a \neq b$ (b) $a^n * a^n$ (c) $\delta_{n-3} * \delta_{n-5}$

(d) $x_n * x_n$ where $x_n = \begin{cases} 1 & n = 0, 1, 2, 3 \\ 0 & n = 4, 5, 6, 7 \ldots \end{cases}$

Answers

1. Step response: $y_n = 1 - (0.99)(0.9)^n - 0.09n(0.9)^n$

   Alternating response: $y_n = \dfrac{1}{361}(-1)^n + \dfrac{2.61}{361}(0.9)^n + \dfrac{1.71}{361}n(0.9)^n$

2. $y_n + 0.3y_{n-1} - 0.28y_{n-2} = 0.9x_{n-1} - 0.63x_{n-2}$

3. $\alpha = 3.375 \qquad \beta = -4$

4. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{(2n+1)(n+1)n}{6}$

5. (a) $\dfrac{1}{(a-b)}(a^{n+1} - b^{n+1})$ (b) $(n+1)a^n$ (c) δ_{n-8} (d) $\{1, 2, 3, 4, 3, 2, 1\}$