z-Transforms and Difference Equations 21.3

Introduction

In this we apply z-transforms to the solution of certain types of difference equation. We shall see that this is done by turning the difference equation into an ordinary algebraic equation. We investigate both first and second order difference equations.

A key aspect in this process in the inversion of the z-transform. As well as demonstrating the use of partial fractions for this purpose we show an alternative, often easier, method using what are known as residues.

Prerequisites

Before starting this Section you should ...

- have studied carefully Section 21.2
- be familiar with simple partial fractions

Learning Outcomes

On completion you should be able to ...

- invert z-transforms using partial fractions or residues where appropriate
- solve constant coefficient linear difference equations using z-transforms

1. Solution of difference equations using z-transforms

Using z-transforms, in particular the shift theorems discussed at the end of the previous Section, provides a useful method of solving certain types of difference equation. In particular linear constant coefficient difference equations are amenable to the z-transform technique although certain other types can also be tackled. In fact all the difference equations that we looked at in Section 21.1 were linear:

$$\begin{array}{ll} y_{n+1} = y_n + d & \text{(1st order)} \\ y_{n+1} = A\, y_n & \text{(1st order)} \\ y_{n+2} = y_{n+1} + y_n & \text{(2nd order)} \end{array}$$

Other examples of linear difference equations are

$$\begin{array}{ll} y_{n+2} + 4y_{n+1} - 3y_n = n^2 & \text{(2nd order)} \\ y_{n+1} + y_n = n\, 3^n & \text{(1st order)} \end{array}$$

The key point is that for a difference equation to be classified as linear the terms of the sequence $\{y_n\}$ arise only to power 1 or, more precisely, the highest subscript term is obtainable as a **linear combination** of the lower ones. All the examples cited above are consequently linear. Note carefully that the term n^2 in our fourth example does not imply non-linearity since linearity is determined by the y_n terms.

Examples of non-linear difference equations are

$$\begin{aligned} y_{n+1} &= \sqrt{y_n + 1} \\ y_{n+1}^2 + 2\, y_n &= 3 \\ y_{n+1} y_n &= n \\ \cos(y_{n+1}) &= y_n \end{aligned}$$

We shall not consider the problem of solving non-linear difference equations.

The five linear equations listed above also have **constant coefficients**; for example:

$$y_{n+2} + 4y_{n+1} - 3y_n = n^2$$

has the constant coefficients $1,\ 4,\ -3$.

The (linear) difference equation

$$n\, y_{n+2} - y_{n+1} + y_n = 0$$

has one variable coefficient viz n and so is not classified as a constant coefficient difference equation.

Solution of first order linear constant coefficient difference equations

Consider the first order difference equation

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

The equation could be solved in a step-by-step or **recursive** manner, provided that y_0 is known because

$y_1 = 4 + 3y_0 \qquad y_2 = 4 + 3y_1 \qquad y_3 = 4 + 3y_2$ and so on.

This process will certainly produce the terms of the solution sequence $\{y_n\}$ but the general term y_n may not be obvious.

So consider

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots \tag{1}$$

with initial condition $y_0 = 1$.

We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty}(y_{n+1} - 3y_n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n} \tag{2}$$

The three terms in (2) are clearly recognisable as z-transforms.

The right-hand side is the z-transform of the constant sequence $\{4,\ 4, \ldots\}$ which is $\dfrac{4z}{z-1}$.

If $Y(z) = \displaystyle\sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then $\displaystyle\sum_{n=0}^{\infty} y_{n+1}z^{-n} = z\ Y(z) - zy_0$ (by the left shift theorem).

Consequently (2) can be written

$$z\ Y(z) - zy_0 - 3\ Y(z) = \frac{4z}{z-1} \tag{3}$$

Equation (3) is the z-transform of the original difference equation (1). The intervening steps have been included here for explanation purposes but we shall omit them in future. The important point is that (3) is no longer a difference equation. It is an **algebraic** equation where the unknown, $Y(z)$, is the z-transform of the solution sequence $\{y_n\}$.

We now insert the initial condition $y_0 = 1$ and solve (3) for $Y(z)$:

$$\begin{aligned} (z-3)Y(z) - z &= \frac{4z}{(z-1)} \\ (z-3)Y(z) &= \frac{4z}{z-1} + z = \frac{z^2+3z}{z-1} \end{aligned}$$

so $$Y(z) = \frac{z^2+3z}{(z-1)(z-3)} \tag{4}$$

The final step consists of obtaining the sequence $\{y_n\}$ of which (4) is the z-transform. As it stands (4) is not recognizable as any of the standard transforms that we have obtained. Consequently, one method of 'inverting' (4) is to use a **partial fraction** expansion. (We assume that you are familiar with simple partial fractions. See HELM 3.6)

Thus

$$
\begin{aligned}
Y(z) &= z\frac{(z+3)}{(z-1)(z-3)} \\
&= z\left(\frac{-2}{z-1} + \frac{3}{z-3}\right) \qquad \text{(in partial fractions)}
\end{aligned}
$$

so $\quad Y(z) = \dfrac{-2z}{z-1} + \dfrac{3z}{z-3}$

Now, taking inverse z-transforms, the general term y_n is, using the linearity property,

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\ \mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

The symbolic notation $\mathbb{Z}^{-1}$ is common and is short for 'the inverse z-transform of'.

Task

Using standard z-transforms write down y_n explicitly, where

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\ \mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

Your solution

Answer

$$y_n = -2 + 3\times 3^n = -2 + 3^{n+1} \qquad n = 0, 1, 2, \ldots \qquad (5)$$

Checking the solution:

From this solution (5)

$$y_n = -2 + 3^{n+1}$$

we easily obtain

$$y_0 = -2 + 3 = 1 \qquad \text{(as given)}$$

$$y_1 = -2 + 3^2 = 7$$

$$y_2 = -2 + 3^3 = 25$$

$$y_3 = -2 + 3^4 = 79 \qquad \text{etc.}$$

These agree with those obtained by recursive solution of the given problem (1):

$$y_{n+1} - 3y_n = 4 \qquad y_0 = 1$$

which yields

$$y_1 = 4 + 3y_0 = 7$$

$$y_2 = 4 + 3y_1 = 25$$

$$y_3 = 4 + 3y_2 = 79 \qquad \text{etc.}$$

More conclusively we can put the solution (5) back into the left-hand side of the difference equation (1).

If $y_n = -2 + 3^{n+1}$

then $\qquad 3y_n = -6 + 3^{n+2}$

and $\qquad y_{n+1} = -2 + 3^{n+2}$

So, on the left-hand side of (1),

$$y_{n+1} - 3y_n = -2 + 3^{n+2} - (-6 + 3^{n+2})$$

which does indeed equal 4, the given right-hand side, and so the solution has been verified.

Key Point 13

To solve a linear constant coefficient difference equation, three steps are involved:

1. Replace each term in the difference equation by its z-transform and insert the initial condition(s).
2. Solve the resulting algebraic equation. (Thus gives the z-transform $Y(z)$ of the solution sequence.)
3. Find the inverse z-transform of $Y(z)$.

The third step is usually the most difficult. We will consider the problem of finding inverse z-transforms more fully later.

Task

Solve the difference equation

$$y_{n+1} - y_n = d \qquad n = 0, 1, 2, \ldots \qquad y_0 = a \tag{6}$$

where a and d are constants.
(The solution will give the n th term of an arithmetic sequence with a constant difference d and initial term a.)

Start by replacing each term of (6) by its z-transform:

Your solution

Answer
If $Y(z) = \mathbb{Z}\{y_n\}$ we obtain the algebraic equation

$$z\,Y(z) - zy_0 - Y(z) = \frac{d \times z}{(z-1)}$$

Note that the right-hand side transform is that of a constant sequence $\{d, d, \ldots\}$. Note also the use of the left shift theorem.

Now insert the initial condition $y_0 = a$ and then solve for $Y(z)$:

Your solution

Answer

$$(z-1)Y(z) = \frac{d \times z}{(z-1)} + z \times a$$
$$Y(z) = \frac{d \times z}{(z-1)^2} + \frac{a \times z}{z-1}$$

Finally take the inverse z-transform of the right-hand side. [Hint: Recall the z-transform of the ramp sequence $\{n\}$.]

Your solution

Answer
We have

$$y_n = d \times \mathbb{Z}^{-1}\{\frac{z}{(z-1)^2}\} + a \times \mathbb{Z}^{-1}\{\frac{z}{z-1}\}$$
$$\therefore \qquad y_n = dn + a \qquad n = 0, 1, 2, \ldots \qquad (7)$$

using the known z-transforms of the ramp and unit step sequences. Equation (7) may well be a familiar result to you – an arithmetic sequence whose 'zeroth' term is $y_0 = a$ has general term $y_n = a + nd$.

i.e. $\{y_n\} = \{a,\ a+d, \ldots a+nd, \ldots\}$

This solution is of course readily obtained by direct recursive solution of (6) without need for z-transforms. In this case the general term $(a + nd)$ is readily seen from the form of the recursive solution: (Make sure you really do see it).

N.B. If the term a is labelled as the **first** term (rather than the zeroth) then

$$y_1 = a,\ y_2 = a + d,\ y_3 - a + 2d,$$

so in this case the n th term is

$$y_n = a + (n-1)d$$

rather than (7).

Use of the right shift theorem in solving difference equations

The problem just solved was given by (6), i.e.

$$y_{n+1} - y_n = d \qquad \text{with } y_0 = a \qquad n = 0, 1, 2, \ldots$$

We obtained the solution

$$y_n = a + nd \qquad n = 0, 1, 2, \ldots$$

Now consider the problem

$$y_n - y_{n-1} = d \qquad n = 0, 1, 2, \ldots \tag{8}$$

with $y_{-1} = a$.
The only difference between the two problems is that the 'initial condition' in (8) is given at $n = -1$ rather than at $n = 0$. Writing out the first few terms should make this clear.

$$\begin{array}{cc}
(6) & (8) \\
y_1 - y_0 = d & y_0 - y_{-1} = d \\
y_2 - y_1 = d & y_1 - y_0 = d \\
\vdots & \vdots \\
y_{n+1} - y_n = d & y_n - y_{n-1} = d \\
y_0 = a & y_{-1} = a
\end{array}$$

The solution to (8) must therefore be the same as for (6) but with every term in the solution (7) of (6) shifted 1 unit to the left.
Thus the solution to (8) is expected to be

$$y_n = a + (n+1)d \qquad n = -1, 0, 1, 2, \ldots$$

(replacing n by $(n+1)$ in the solution (7)).

Task

Use the right shift theorem of z-transforms to solve (8) with the initial condition $y_{-1} = a$.

(a) Begin by taking the z-transform of (8), inserting the initial condition and solving for $Y(z)$:

Your solution

Answer
We have, for the z-transform of (8)

$$\begin{aligned} Y(z) - (z^{-1}Y(z) + y_{-1}) &= \frac{dz}{z-1} \qquad \text{[Note that here } dz \text{ means } d \times z] \\ Y(z)(1 - z^{-1}) - a &= \frac{dz}{z-1} \\ Y(z)\left(\frac{z-1}{z}\right) &= \frac{dz}{(z-1)} + a \\ Y(z) &= \frac{dz^2}{(z-1)^2} + \frac{az}{z-1} \qquad (9) \end{aligned}$$

The second term of $Y(z)$ has the inverse z-transform $\{a\ u_n\} = \{a, a, a, \ldots\}$.
The first term is less straightforward. However, we have already reasoned that the other term in y_n here should be $(n+1)d$.

(b) Show that the z-transform of $(n+1)d$ is $\dfrac{dz^2}{(z-1)^2}$. Use the standard transform of the ramp and step:

Your solution

Answer
We have

$$\mathbb{Z}\{(n+1)d\} = d\mathbb{Z}\{n\} + d\mathbb{Z}\{1\}$$

by the linearity property

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{(n+1)d\} &= \frac{dz}{(z-1)^2} + \frac{dz}{z-1} \\ &= dz\left(\frac{1+z-1}{(z-1)^2}\right) \\ &= \frac{dz^2}{(z-1)^2} \end{aligned}$$

as expected.

(c) Finally, state y_n:

Your solution

Answer
Returning to (9) the inverse z-transform is

$$y_n = (n+1)d + a\ u_n \qquad \text{i.e.} \qquad y_n = a + (n+1)d \qquad n = -1, 0, 1, 2, \ldots$$

as we expected.

Task

Earlier in this Section (pages 37-39) we solved

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots \qquad \text{with } y_0 = 1.$$

Now solve $y_n - 3y_{n-1} = 4 \qquad n = 0, 1, 2, \ldots \qquad$ with $y_{-1} = 1.$ (10)

Begin by obtaining the z-transform of y_n:

Your solution

Answer
We have, taking the z-transform of (10),

$$Y(z) - 3(z^{-1}Y(z) + 1) = \frac{4z}{z-1}$$

(using the right shift property and inserting the initial condition.)

$$\therefore \qquad Y(z) - 3z^{-1}Y(z) = 3 + \frac{4z}{z-1}$$

$$Y(z)\frac{(z-3)}{z} = 3 + \frac{4z}{z-1} \qquad \text{so} \qquad Y(z) = \frac{3z}{z-3} + \frac{4z^2}{(z-1)(z-3)}$$

Write the second term as $4z\left(\dfrac{z}{(z-1)(z-3)}\right)$ and obtain the partial fraction expansion of the bracketed term. Then complete the z-transform inversion.

Your solution

Answer

$$\frac{z}{(z-1)(z-3)} = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z-3}$$

We now have

$$Y(z) = \frac{3z}{z-3} - \frac{2z}{z-1} + \frac{6z}{z-3}$$

so

$$y_n = 3 \times 3^n - 2 + 6 \times 3^n = -2 + 9 \times 3^n = -2 + 3^{n+2} \qquad (11)$$

Compare this solution (11) to that of the previous problem (5) on page 39:

Your solution

Answer

Solution (11) is just the solution sequence (5) moved 1 unit to the left. We anticipated this since the difference equation (10) and associated initial condition is the same as the difference equation (1) but shifted one unit to the left.

2. Second order difference equations

You will learn in this section about solving second order linear constant coefficient difference equations. In this case **two** initial conditions are required, typically either y_0 and y_1 or y_{-1} and y_{-2}. In the first case we use the left shift property of the z-transform, in the second case we use the right shift property. The same three basic steps are involved as in the first order case.

Task

By solving

$$y_{n+2} = y_{n+1} + y_n \qquad (12)$$

$$y_0 = y_1 = 1$$

obtain the general term y_n of the Fibonacci sequence.

Begin by taking the z-transform of (12), using the left shift property. Then insert the initial conditions and solve the resulting algebraic equation for $Y(z)$, the z-transform of $\{y_n\}$:

Your solution

Answer

$$z^2Y(z) - z^2y_0 - zy_1 = zY(z) - zy_0 + Y(z) \qquad \text{(taking z-transforms)}$$

$$z^2Y(z) - z^2 - z = zY(z) - z + Y(z) \qquad \text{(inserting initial conditions)}$$

$$(z^2 - z - 1)Y(z) = z^2$$

so

$$Y(z) = \frac{z^2}{z^2 - z - 1} \qquad \text{(solving for } Y(z)\text{).}$$

Now solve the quadratic equation $z^2 - z - 1 = 0$ and hence factorize the denominator of $Y(z)$:

Your solution

Answer

$z^2 - z - 1 = 0$

$$\therefore \qquad z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

so if $a = \dfrac{1+\sqrt{5}}{2}, \quad b = \dfrac{1-\sqrt{5}}{2}$

$$Y(z) = \frac{z^2}{(z-a)(z-b)}$$

This form for $Y(z)$ often arises in solving second order difference equations. Write it in partial fractions and find y_n, leaving a and b as general at this stage:

Your solution

Answer

$$Y(z) = z\left(\frac{z}{(z-a)(z-b)}\right) = \frac{Az}{z-a} + \frac{Bz}{(z-b)} \qquad \text{in partial fractions}$$

where $A = \dfrac{a}{a-b}$ and $B = \dfrac{b}{b-a}$

Hence, taking inverse z-transforms

$$y_n = Aa^n + Bb^n = \frac{1}{(a-b)}\left(a^{n+1} - b^{n+1}\right) \qquad (13)$$

Now complete the Fibonacci problem:

Your solution

Answer

With $a = \dfrac{1+\sqrt{5}}{2}$ $\qquad b = \dfrac{1-\sqrt{5}}{2}$ $\qquad$ so $a - b = \sqrt{5}$

we obtain, using (13)

$$y_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right) \qquad n = 2, 3, 4, \ldots$$

for the n th term of the Fibonacci sequence.

With an appropriate computational aid you could (i) check that this formula does indeed give the familiar sequence

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots\}$$

and (ii) obtain, for example, y_{50} and y_{100}.

Key Point 14

The inverse z-transform of

$$Y(z) = \frac{z^2}{(z-a)(z-b)} \qquad a \neq b \qquad \text{is} \qquad y_n = \frac{1}{(a-b)}\left(a^{n+1} - b^{n+1}\right)$$

Task

Use the right shift property of z-transforms to solve the second order difference equation

$$y_n - 7y_{n-1} + 10\, y_{n-2} = 0 \quad \text{with } y_{-1} = 16 \quad \text{and } y_{-2} = 5.$$

[Hint: the steps involved are the same as in the previous Task]

Your solution

Answer

$$Y(z) - 7(z^{-1}Y(z) + 16) \quad + \quad 10(z^{-2}Y(z) + 16z^{-1} + 5) = 0$$

$$Y(z)(1 - 7z^{-1} + 10z^{-2}) \quad - \quad 112 + 160z^{-1} + 50 = 0$$

$$Y(z)\left(\frac{z^2 - 7z + 10}{z^2}\right) \quad = \quad 62 - 160z^{-1}$$

$$\begin{aligned} Y(z) &= \frac{62z^2}{z^2 - 7z + 10} - \frac{160z}{z^2 - 7z + 10} \\ &= z\frac{(62z - 160)}{(z-2)(z-5)} \\ &= \frac{12z}{z-2} + \frac{50z}{z-5} \qquad \text{in partial fractions} \\ \text{so} \qquad y_n &= 12 \times 2^n + 50 \times 5^n \qquad n = 0, 1, 2, \ldots \end{aligned}$$

We now give an Example where a quadratic equation with repeated solutions arises.

Example 1

(a) Obtain the z-transform of $\{f_n\} = \{na^n\}$.

(b) Solve

$$y_n - 6y_{n-1} + 9y_{n-2} = 0 \qquad n = 0, 1, 2, \ldots$$
$$y_{-1} = 1 \qquad y_{-2} = 0$$

[Hint: use the result from (a) at the inversion stage.]

Solution

(a) $Z\{n\} = \dfrac{z}{(z-1)^2} \qquad \therefore \qquad Z\{na^n\} = \dfrac{z/a}{(z/a-1)^2} = \dfrac{az}{(z-a)^2}$ where we have used the property $Z\{f_n\, a^n\} = F\left(\dfrac{z}{a}\right)$

(b) Taking the z-transform of the difference equation and inserting the initial conditions:

$$Y(z) - 6(z^{-1}Y(z) + 1) + 9(z^{-2}Y(z) + z^{-1}) = 0$$

$$Y(z)(1 - 6z^{-1} + 9z^{-2}) = 6 - 9z^{-1}$$

$$Y(z)(z^2 - 6z + 9) = 6z^2 - 9z$$

$$Y(z) = \frac{6z^2 - 9z}{(z-3)^2} = z\left(\frac{6z-9}{(z-3)^2}\right) = z\left(\frac{6}{z-3} + \frac{9}{(z-3)^2}\right) \qquad \text{in partial fractions}$$

from which, using the result (a) on the second term,

$$y_n = 6 \times 3^n + 3n \times 3^n = (6 + 3n)3^n$$

We shall re-do this inversion by an alternative method shortly.

Task

Solve the difference equation

$$y_{n+2} + y_n = 0 \quad \text{with } y_0,\ y_1 \text{ arbitrary.} \qquad (14)$$

Start by obtaining $Y(z)$ using the left shift theorem:

Your solution

Answer

$$\begin{aligned} z^2Y(z) - z^2y_0 - zy_1 + Y(z) &= 0 \\ (z^2+1)Y(z) &= z^2y_0 + zy_1 \\ Y(z) &= \frac{z^2}{z^2+1}y_0 + \frac{z}{z^2+1}y_1 \end{aligned}$$

To find the inverse z-transforms recall the results for $Z\{\cos\omega n\}$ and $Z\{\sin\omega n\}$ from Key Point 6 (page 21) and some of the particular cases discussed in Section 21.2. Hence find y_n here:

Your solution

Answer

Taking $Z\{\cos\omega n\}$ and $Z\{\sin\omega n\}$ with $\omega = \dfrac{\pi}{2}$

$$\begin{aligned} Z\left\{\cos\left(\frac{n\pi}{2}\right)\right\} &= \frac{z^2}{z^2+1} \\ Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\} &= \frac{z}{z^2+1} \end{aligned}$$

Hence $\quad y_n = y_0\mathbb{Z}^{-1}\{\dfrac{z^2}{z^2+1}\} + y_1\mathbb{Z}^{-1}\{\dfrac{z}{z^2+1}\} = y_0\cos\left(\dfrac{n\pi}{2}\right) + y_1\sin\left(\dfrac{n\pi}{2}\right) \qquad (15)$

Those of you who are familiar with **differential** equations may know that

$$\frac{d^2y}{dt^2} + y = 0 \qquad y(0) = y_0, \quad y'(0) = y'_0 \qquad (16)$$

has solutions $y_1 = \cos t$ and $y_2 = \sin t$ and a general solution

$$y = c_1\cos t + c_2\sin t \qquad (17)$$

where $c_1 = y_0$ and $c_2 = y'_0$.

This differential equation is a model for simple harmonic oscillations. The difference equation (14) and its solution (15) are the discrete counterparts of (16) and (17).

3. Inversion of z-transforms using residues

This method has its basis in a branch of mathematics called complex integration. You may recall that the 'z' quantity of z-transforms is a complex quantity, more specifically a **complex** variable. However, it is not necessary to delve deeply into the theory of complex variables in order to obtain simple inverse z-transforms using what are called **residues**. In many cases inversion using residues is easier than using partial fractions. Hence reading on is strongly advised.

Pole of a function of a complex variable

If $G(z)$ is a function of the complex variable z and if

$$G(z) = \frac{G_1(z)}{(z-z_0)^k}$$

where $G_1(z_0)$ is non-zero and finite then $G(z)$ is said to have a **pole** of order k at $z = z_0$.
For example if

$$G(z) = \frac{6(z-2)}{z(z-3)(z-4)^2}$$

then $G(z)$ has the following 3 poles.

(i) pole of order 1 at $z = 0$
(ii) pole of order 1 at $z = 3$
(iii) pole of order 2 at $z = 4$.

(Poles of order 1 are sometimes known as simple poles.)

Note that when $z = 2$, $G(z) = 0$. Hence $z = 2$ is said to be a **zero** of $G(z)$. (It is the only zero in this case).

Task

Write down the poles and zeros of

$$G(z) = \frac{3(z+4)}{z^2(2z+1)(3z-9)} \qquad (18)$$

State the order of each pole.

Your solution

Answer
$G(z)$ has a zero when $z = -4$.
$G(z)$ has first order poles at $z = -1/2$, $z = 3$.
$G(z)$ has a second order pole at $z = 0$.

Residue at a pole

The residue of a complex function $G(z)$ at a first order pole z_0 is

$$\text{Res } (G(z),\ z_0) = [G(z)(z - z_0)]_{z_0} \tag{19}$$

The residue at a second order pole z_0 is

$$\text{Res } (G(z),\ z_0) = \left[\frac{d}{dz}(G(z)(z - z_0)^2)\right]_{z_0} \tag{20}$$

You need not worry about how these results are obtained or their full mathematical significance. (Any textbook on Complex Variable Theory could be consulted by interested readers.)

Example

Consider again the function (18) in the previous guided exercise.

$$\begin{aligned} G(z) &= \frac{3(z+4)}{z^2(2z+1)(3z-9)} \\ &= \frac{(z+4)}{2z^2\left(z+\frac{1}{2}\right)(z-3)} \end{aligned}$$

The second form is the more convenient for the residue formulae to be used.
Using (19) at the two first order poles:

$$\begin{aligned} \text{Res } \left(G(z),\ -\frac{1}{2}\right) &= \left[G(z)\left(z - \left(-\frac{1}{2}\right)\right)\right]_{\frac{1}{2}} \\ &= \left[\frac{(z+4)}{2z^2(z-3)}\right]_{\frac{1}{2}} = -\frac{18}{5} \\ \text{Res } [G(z),\ 3] &= \left[\frac{(z+4)}{2z^2\left(z+\dfrac{1}{2}\right)}\right]_3 = \frac{1}{9} \end{aligned}$$

Using (20) at the second order pole

$$\text{Res } (G(z),\ 0) = \left[\frac{d}{dz}(G(z)(z-0)^2)\right]_0$$

The differentiation has to be carried out before the substitution of $z = 0$ of course.

$$\begin{aligned} \therefore \qquad \text{Res } (G(z),\ 0) &= \left[\frac{d}{dz}\left(\frac{(z+4)}{2\left(z+\dfrac{1}{2}\right)(z-3)}\right)\right]_0 \\ &= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{z+4}{z^2 - \dfrac{5}{2}z - \dfrac{3}{2}}\right)\right]_0 \end{aligned}$$

Task

Carry out the differentiation shown on the last line of the previous page, then substitute $z = 0$ and hence obtain the required residue.

Your solution

Answer

Differentiating by the quotient rule then substituting $z = 0$ gives

$$\text{Res}\ (G(z),\ 0) = \frac{17}{9}$$

Key Point 15

Residue at a Pole of Order k

If $G(z)$ has a k^{th} order pole at $z = z_0$

i.e. $G(z) = \dfrac{G_1(z)}{(z - z_0)^k}$ $\qquad G_1(z_0) \neq 0$ and finite

$$\text{Res}\ (G(z),\ z_0) = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}(G(z)\ (z - z_0)^k)\right]_{z_0} \qquad (21)$$

This formula reduces to (19) and (20) when $k = 1$ and 2 respectively.

Inverse z-transform formula

Recall that, by definition, the z-transform of a sequence $\{f_n\}$ is

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots f_n z^{-n} + \dots$$

If we multiply both sides by z^{n-1} where n is a positive integer we obtain

$$F(z)z^{n-1} = f_0 z^{n-1} + f_1 z^{n-2} + f_2 z^{n-3} + \dots f_n z^{-1} + f_{n+1} z^{-2} + \dots$$

Using again a result from complex integration it can be shown from this expression that the general term f_n is given by

$$f_n = \text{sum of residues of } F(z)\ z^{n-1} \text{ at its poles} \qquad (22)$$

The poles of $F(z)z^{n-1}$ will be those of $F(z)$ with possibly additional poles at the origin.

To illustrate the residue method of inversion we shall re-do some of the earlier examples that were done using partial fractions.

Example:

$$Y(z) = \frac{z^2}{(z-a)(z-b)} \qquad a \neq b$$

so

$$Y(z)z^{n-1} = \frac{z^{n+1}}{(z-a)(z-b)} = G(z), \text{ say.}$$

$G(z)$ has first order poles at $z = a$, $z = b$ so using (19).

$$\begin{aligned}
\text{Res } (G(z),\ a) &= \left[\frac{z^{n+1}}{z-b}\right]_a = \frac{a^{n+1}}{a-b} \\
\text{Res } (G(z),\ b) &= \left[\frac{z^{n+1}}{z-a}\right]_b = \frac{b^{n+1}}{b-a} = \frac{-b^{n+1}}{a-b}
\end{aligned}$$

We need simply add these residues to obtain the required inverse z-transform

$$\therefore \qquad f_n = \frac{1}{(a-b)}(a^{n+1} - b^{n+1})$$

as before.

Task

Obtain, using (22), the inverse z-transform of

$$Y(z) = \frac{6z^2 - 9z}{(z-3)^2}$$

Firstly, obtain the pole(s) of $G(z) = Y(z)z^{n-1}$ and deduce the order:

Your solution

Answer

$$G(z) = Y(z)z^{n-1} = \frac{6z^{n+1} - 9z^n}{(z-3)^2}$$

whose only pole is one of second order at $z = 3$.

Now calculate the residue of $G(z)$ at $z = 3$ using (20) and hence write down the required inverse z-transform y_n:

Your solution

Answer

$$\begin{aligned}\text{Res }(G(z),\ 3) &= \left[\frac{d}{dz}(6z^{n+1} - 9z^n)\right]_3 \\ &= \left[6(n+1)z^n - 9nz^{n-1}\right]_3 \\ &= 6(n+1)3^n - 9n3^{n-1} \\ &= 6 \times 3^n + 3n3^n\end{aligned}$$

This is the same as was found by partial fractions, but there is considerably less labour by the residue method.

In the above examples all the poles of the various functions $G(z)$ were real. This is the easiest situation but the residue method will cope with complex poles.

Example

We showed earlier that

$$\frac{z^2}{z^2+1} \text{ and } \cos\left(\frac{n\pi}{2}\right)$$

formed a z-transform pair.

We will now obtain y_n if $Y(z) = \dfrac{z^2}{z^2+1}$ using residues.
Using residues with, from (22),

$$G(z) = \frac{z^{n+1}}{z^2+1} = \frac{z^{n+1}}{(z-\mathsf{i})(z+\mathsf{i})} \text{ where } \mathsf{i}^2 = -1.$$

we see that $G(z)$ has first order poles at the complex conjugate points $\pm$ i.
Using (19)

$$\text{Res }(G(z),\ i) = \left[\frac{z^{n+1}}{z+\mathsf{i}}\right]_{\mathsf{i}} = \frac{\mathsf{i}^{n+1}}{2\mathsf{i}} \qquad \text{Res }(G(z),\ -\mathsf{i}) = \frac{(-\mathsf{i})^{n+1}}{(-2\mathsf{i})}$$

(Note the complex conjugate residues at the complex conjugate poles.)
Hence $\mathbb{Z}^{-1}\{\dfrac{z^2}{z^2+1}\} = \dfrac{1}{2\mathsf{i}}\left(\mathsf{i}^{n+1} - (-\mathsf{i})^{n+1}\right)$

But $\mathsf{i} = e^{\mathsf{i}\pi/2}$ and $-\mathsf{i} = e^{-\mathsf{i}\pi/2}$, so the inverse z-transform is

$$\frac{1}{2\mathsf{i}}\left(e^{\mathsf{i}(n+1)\pi/2} - e^{-\mathsf{i}(n+1)\pi/2}\right) = \sin(n+1)\frac{\pi}{2} = \cos\left(\frac{n\pi}{2}\right) \qquad \text{as expected.}$$

Task

Show, using residues, that

$$\mathbb{Z}^{-1}\{\frac{z}{z^2+1}\} = \sin\left(\frac{n\pi}{2}\right)$$

Your solution

Answer

Using (22):

$$\begin{aligned}
G(z) &= z^{n-1}\frac{z}{z^2+1} = \frac{z^n}{z^2+1} = \frac{z^n}{(z+\mathrm{i})(z-\mathrm{i})} \\
\text{Res}\ (G(z),\ \mathrm{i}) &= \frac{\mathrm{i}^n}{2\mathrm{i}} \\
\text{Res}\ (G(z),\ -\mathrm{i}) &= \frac{(-\mathrm{i})^n}{-2\mathrm{i}} \\
\mathbb{Z}^{-1}\{\frac{z}{z^2+1}\} &= \frac{1}{2\mathrm{i}}(\mathrm{i}^n - (-\mathrm{i})^n) \\
&= \frac{1}{2\mathrm{i}}(e^{\mathrm{i}n\pi/2} - e^{-\mathrm{i}n\pi/2}) \\
&= \sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

4. An application of difference equations – currents in a ladder network

The application we will consider is that of finding the electric currents in each loop of the ladder resistance network shown, which consists of $(N+1)$ loops. The currents form a sequence $\{i_0, i_1, \ldots i_N\}$

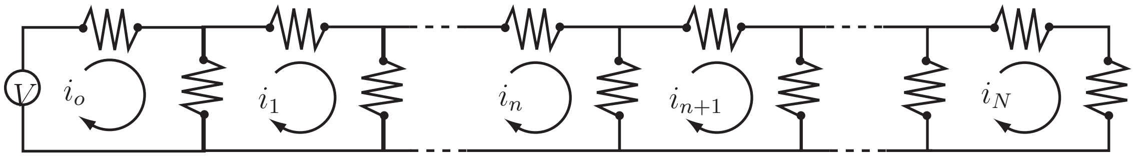

Figure 7

All the resistors have the same resistance R so loops 1 to N are identical. The zero'th loop contains an applied voltage V. In this zero'th loop, Kirchhoff's voltage law gives

$$V = Ri_0 + R(i_0 - i_1)$$

from which

$$i_1 = 2i_0 - \frac{V}{R} \tag{23}$$

Similarly, applying the Kirchhoff law to the $(n+1)^{th}$ loop where there is no voltage source and 3 resistors

$$0 = Ri_{n+1} + R(i_{n+1} - i_{n+2}) + R(i_{n+1} - i_n)$$

from which

$$i_{n+2} - 3i_{n+1} + i_n = 0 \qquad n = 0, 1, 2, \ldots (N-2) \tag{24}$$

(24) is the basic difference equation that has to be solved.

Task

Using the left shift theorems obtain the z-transform of equation (24). Denote by $I(z)$ the z-transform of $\{i_n\}$. Simplify the algebraic equation you obtain.

Your solution

Answer
We obtain

$$z^2I(z) - z^2i_0 - zi_1 - 3(zI(z) - zi_0) + I(z) = 0$$

Simplifying

$$(z^2 - 3z + 1)I(z) = z^2i_0 + zi_1 - 3zi_0 \qquad (25)$$

If we now eliminate i_1 using (23), the right-hand side of (25) becomes

$$z^2i_0 + z\left(2i_0 - \frac{V}{R}\right) - 3zi_0 = z^2i_0 - zi_0 - z\frac{V}{R} = i_0\left(z^2 - z - z\frac{V}{i_0R}\right)$$

Hence from (25)

$$I(z) = \frac{i_0\left(z^2 - \left(1 + \frac{V}{i_0R}\right)z\right)}{z^2 - 3z + 1} \qquad (26)$$

Our final task is to find the inverse z-transform of (26).

Task

Look at the table of z-transforms on page 35 (or at the back of the Workbook) and suggest what sequences are likely to arise by inverting $I(z)$ as given in (26).

Your solution

Answer
The most likely candidates are hyperbolic sequences because both $\{\cosh \alpha n\}$ and $\{\sinh \alpha n\}$ have z-transforms with denominator

$$z^2 - 2z\cosh\alpha + 1$$

which is of the same form as the denominator of (26), remembering that $\cosh\alpha \geq 1$. (Why are the trigonometric sequences $\{\cos\omega n\}$ and $\{\sin\omega n\}$ not plausible here?)

To proceed, we introduce a quantity α such that α is the positive solution of $2\cosh\alpha = 3$ from which (using $\cosh^2\alpha - \sinh^2\alpha \equiv 1$) we get

$$\sinh\alpha = \sqrt{\frac{9}{4} - 1} = \frac{\sqrt{5}}{2}$$

Hence (26) can be written

$$I(z) = i_0 \frac{\left(z^2 - \left(1 + \frac{V}{i_0 R}\right) z\right)}{z^2 - 2z\cosh\alpha + 1} \qquad (27)$$

To further progress, bearing in mind the z-transforms of $\{\cosh\alpha n\}$ and $\{\sinh\alpha n\}$, we must subtract and add $z\cosh\alpha$ to the numerator of (27), where $\cosh\alpha = \frac{3}{2}$.

$$\begin{aligned} I(z) &= i_0 \left[\frac{z^2 - z\cosh\alpha + \frac{3z}{2} - \left(1 + \frac{V}{i_0 R}\right) z}{z^2 - 2z\cosh\alpha + 1} \right] \\ &= i_0 \left[\frac{(z^2 - z\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1} + \frac{\left(\frac{3}{2} - 1\right) z - \frac{Vz}{i_0 R}}{z^2 - 2z\cosh\alpha + 1} \right] \end{aligned}$$

The first term in the square bracket is the z-transform of $\{\cosh\alpha n\}$.

The second term is

$$\frac{\left(\frac{1}{2} - \frac{V}{i_0 R}\right) z}{z^2 - 2z\cosh\alpha + 1} = \frac{\left(\frac{1}{2} - \frac{V}{i_0 R}\right) \frac{2}{\sqrt{5}} z \frac{\sqrt{5}}{2}}{z^2 - 2z\cosh\alpha + 1}$$

which has inverse z-transform

$$\left(\frac{1}{2} - \frac{V}{i_0 R}\right) \frac{2}{\sqrt{5}} \sinh\alpha n$$

Hence we have for the loop currents

$$i_n = i_0 \cosh(\alpha n) + \left(\frac{i_0}{2} - \frac{V}{R}\right) \frac{2}{\sqrt{5}} \sinh(\alpha n) \qquad n = 0, 1, \ldots N \qquad (27)$$

where $\cosh\alpha = \dfrac{3}{2}$ determines the value of α.

Finally, by Kirchhoff's law applied to the rightmost loop

$$3i_N = i_{N-1}$$

from which, with (27), we could determine the value of i_0.

Exercises

1. Deduce the inverse z-transform of each of the following functions:

(a) $\dfrac{2z^2 - 3z}{z^2 - 3z - 4}$

(b) $\dfrac{2z^2 + z}{(z-1)^2}$

(c) $\dfrac{2z^2 - z}{2z^2 - 2z + 2}$

(d) $\dfrac{3z^2 + 5}{z^4}$

2. Use z-transforms to solve each of the following difference equations:

(a) $y_{n+1} - 3y_n = 4^n \qquad y_0 = 0$

(b) $y_n - 3y_{n-1} = 6 \qquad y_{-1} = 4$

(c) $y_n - 2y_{n-1} = n \qquad y_{-1} = 0$

(d) $y_{n+1} - 5y_n = 5^{n+1} \qquad y_0 = 0$

(e) $y_{n+1} + 3y_n = 4\delta_{n-2} \qquad y_0 = 2$

(f) $y_n - 7y_{n-1} + 10y_{n-2} = 0 \qquad y_{-1} = 16, \quad y_{-2} = 5$

(g) $y_n - 6y_{n-1} + 9y_{n-2} = 0 \qquad y_{-1} = 1, \quad y_{-2} = 0$

Answers

1 (a) $(-1)^n + 4^n$ (b) $2 + 3n$ (c) $\cos(n\pi/3)$ (d) $3\delta_{n-2} + 5\delta_{n-4}$

2 (a) $y_n = 4^n - 3^n$ (b) $y_n = 21 \times 3^n - 3$ (c) $y_n = 2 \times 2^n - 2 - n$ (d) $y_n = n5^n$

(e) $y_n = 2 \times (-3)^n + 4 \times (-3)^{n-3}u_{n-2}$ (f) $y_n = 12 \times 2^n + 50 \times 5^n$ (g) $y_n = (6 + 3n)3^n$