# Contents 21

# z-Transforms



## Learning outcomes

*In this Workbook you will learn about the properties and applications of the z-transform, a major mathematical tool for the analysis and design of discrete systems including digital control systems.*

# The z-Transform

# 21.1

## Introduction

The z-transform is the major mathematical tool for analysis in such topics as digital control and digital signal processing. In this introductory Section we lay the foundations of the subject by briefly discussing sequences, shifting of sequences and difference equations. Readers familiar with these topics can proceed directly to Section 21.2 where z-transforms are first introduced.

## Prerequisites

Before starting this Section you should ...

- have competence with algebra

## Learning Outcomes

On completion you should be able to ...

- explain what is meant by a sequence and by a difference equation
- distinguish between first and second order difference equations
- shift sequences to the left or right

# 1. Preliminaries: Sequences and Difference Equations

## Sequences

A sequence is a set of numbers formed according to some definite rule. For example the sequence

$$\{1, 4, 9, 16, 25, \ldots\} \qquad (1)$$

is formed by the **squares** of the positive integers.
If we write

$$y_1 = 1, \quad y_2 = 4, \quad y_3 = 9, \ldots$$

then the **general** or $\boldsymbol{n^{th}}$ **term** of the sequence (1) is $y_n = n^2$. The notations $y(n)$ and $y[n]$ are also used sometimes to denote the general term. The notation $\{y_n\}$ is used as an abbreviation for a whole sequence.

An alternative way of considering a sequence is to view it as being obtained by **sampling** a continuous function. In the above example the sequence of squares can be regarded as being obtained from the function

$$y(t) = t^2$$

by sampling the function at $t = 1, 2, 3, \ldots$ as shown in Figure 1.

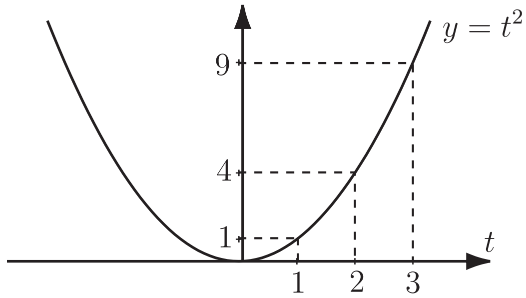



**Figure 1**

The notation $y(n)$, as opposed to $y_n$, for the general term of a sequence emphasizes this sampling aspect.

**Task**

Find the general term of the sequence $\{2, 4, 8, 16, 32, \ldots\}$.

**Your solution**

**Answer**

The terms of the sequence are the integer powers of 2: $y_1 = 2 = 2^1 \quad y_2 = 4 = 2^2$

$y_3 = 8 = 2^3 \ldots$ so $y_n = 2^n$.

Here the sequence $\{2^n\}$ are the sample values of the continuous function $y(t) = 2^t$ at $t = 1, 2, 3, \ldots$

An alternative way of defining a sequence is as follows:

(i) give the first term $y_1$ of the sequence
(ii) give the rule for obtaining the $(n+1)^{\text{th}}$ term from the $n^{\text{th}}$.

A simple example is

$$y_{n+1} = y_n + d \qquad y_1 = a$$

where $a$ and $d$ are constants.
It is straightforward to obtain an expression for $y_n$ in terms of $n$ as follows:

$$\begin{aligned} y_2 &= y_1 + d = a + d \\ y_3 &= y_2 + d = a + d + d = a + 2d \\ y_4 &= y_3 + d = a + 3d \\ &\vdots \\ y_n &= a + (n-1)d \end{aligned} \qquad (2)$$

This sequence characterised by a **constant difference** between successive terms

$$y_{n+1} - y_n = d \qquad n = 1, 2, 3, \ldots$$

is called an arithmetic sequence.

**Task**

Calculate the $n^{\text{th}}$ term of the arithmetic sequence defined by

$$y_{n+1} - y_n = 2 \qquad y_1 = 9.$$

Write out the first 4 terms of this sequence explicitly.

Suggest why an arithmetic sequence is also known as a linear sequence.

**Your solution**

**Answer**

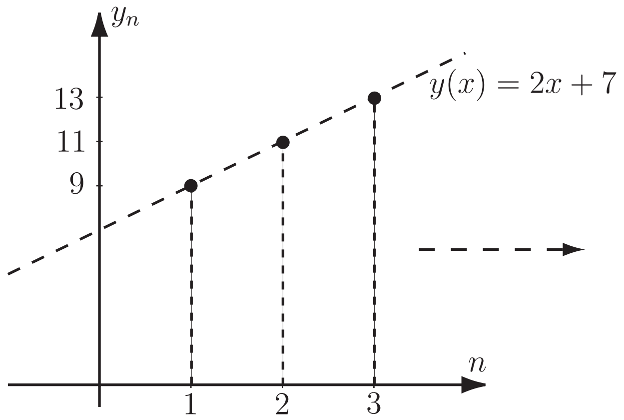
We have, using (2),

$y_n = 9 + (n-1)2$ or

$y_n = 2n + 7$

so $y_1 = 9$ (as given), $y_2 = 11,\ y_3 = 13,\ y_4 = 15, \ldots$

A graph of $y_n$ against $n$ would be just a set of points but all lie on the straight line $y = 2x + 7$, hence the term 'linear sequence'.




## Nomenclature

The equation

$$y_{n+1} - y_n = d \qquad (3)$$

is called a **difference equation** or **recurrence equation** or more specifically a first order, constant coefficient, linear, difference equation.

The sequence whose $n^{\text{th}}$ term is

$$y_n = a + (n-1)d \qquad (4)$$

is the **solution** of (3) for the **initial condition** $y_1 = a$.

The coefficients in (3) are the numbers preceding the terms $y_{n+1}$ and $y_n$ so are 1 and $-1$ respectively.

The classification **first** order for the difference equation (3) follows because the difference between the highest and lowest subscripts is $n + 1 - n = 1$.

Now consider again the sequence

$$\{y_n\} = \{2^n\}$$

Clearly

$$y_{n+1} - y_n = 2^{n+1} - 2^n = 2^n$$

so the difference here is dependent on $n$ i.e. is not constant. Hence the sequence $\{2^n\} = \{2, 4, 8, \ldots\}$ is not an arithmetic sequence.

**Task**

For the sequence $\{y_n\} = 2^n$ calculate $y_{n+1} - 2y_n$. Hence write down a difference equation and initial condition for which $\{2^n\}$ is the solution.

**Your solution**

**Answer**

$$y_{n+1} - 2y_n = 2^{n+1} - 2 \times 2^n = 2^{n+1} - 2^{n+1} = 0$$

Hence $y_n = 2^n$ is the solution of the **homogeneous** difference equation

$$y_{n+1} - 2y_n = 0 \qquad (5)$$

with initial condition $y_1 = 2$.

The term 'homogeneous' refers to the fact that the right-hand side of the difference equation (5) is zero.

More generally it follows that

$$y_{n+1} - Ay_n = 0 \qquad y_1 = A$$

has solution sequence $\{y_n\}$ with general term

$$y_n = A^n$$

## A second order difference equation

Second order difference equations are characterised, as you would expect, by a difference of 2 between the highest and lowest subscripts. A famous example of a constant coefficient second order difference equation is

$$y_{n+2} = y_{n+1} + y_n \quad \text{or} \quad y_{n+2} - y_{n+1} - y_n = 0 \qquad (6)$$

The solution $\{y_n\}$ of (6) is a sequence where any term is the sum of the two preceding ones.

**Task**

What additional information is needed if (6) is to be solved?

**Your solution**

**Answer**

**Two** initial conditions, the values of $y_1$ and $y_2$ must be specified so we can calculate

$$\begin{aligned} y_3 &= y_2 + y_1 \\ y_4 &= y_3 + y_2 \end{aligned}$$

and so on.

**Task**

Find the first 6 terms of the solution sequence of (6) for each of the following sets of initial conditions

(a) $y_1 = 1 \qquad y_2 = 3$

(b) $y_1 = 1 \qquad y_2 = 1$

**Your solution**

**Answer**

(a) $\{1, 3, 4, 7, 11, 18 \ldots\}$

(b) $\{1, 1, 2, 3, 5, 8, \ldots\}$ (7)

The sequence (7) is a very famous one; it is known as the **Fibonacci Sequence**. It follows that the solution sequence of the difference equation (6)

$$y_{n+1} = y_{n+1} + y_n$$

with initial conditions $y_1 = y_2 = 1$ is the Fibonacci sequence. What is not so obvious is what is the general term $y_n$ of this sequence.

One way of obtaining $y_n$ in this case, and for many other linear constant coefficient difference equations, is via a technique involving $Z-$transforms which we shall introduce shortly.

## Shifting of sequences

### Right Shift

Recall the sequence $\{y_n\} = \{n^2\}$ or, writing out the first few terms explicitly,

$$\{y_n\} = \{1, 4, 9, 16, 25, \ldots\}$$

The sequence $\{v_n\} = \{0, 1, 4, 9, 16, 25, \ldots\}$ contains the same numbers as $y_n$ but they are all shifted one place to the right. The general term of this shifted sequence is

$$v_n = (n-1)^2 \qquad n = 1, 2, 3, \ldots$$

Similarly the sequence

$$\{w_n\} = \{0, 0, 1, 4, 9, 16, 25, \ldots\}$$

has general term

$$w_n = \begin{cases} (n-2)^2 & n = 2, 3, \ldots \\ 0 & n = 1 \end{cases}$$

**Task**

For the sequence $\{y_n\} = \{2^n\} = \{2, 4, 8, 16, \ldots\}$ write out explicitly the first 6 terms and the general terms of the sequences $v_n$ and $w_n$ obtained respectively by shifting the terms of $\{y_n\}$

(a) one place to the right (b) three places the the right.

**Your solution**

**Answer**

(a)

$$\{v_n\} \;=\; \{0, 2, 4, 8, 16, 32 \ldots\} \qquad v_n = \begin{cases} 2^{n-1} & n = 2, 3, 4, \ldots \\ 0 & n = 1 \end{cases}$$

(b)

$$\{w_n\} \;=\; \{0, 0, 0, 2, 4, 8 \ldots\} \qquad w_n = \begin{cases} 2^{n-3} & n = 4, 5, 6, \ldots \\ 0 & n = 1, 2, 3 \end{cases}$$

The operation of shifting the terms of a sequence is an important one in digital signal processing and digital control. We shall have more to say about this later. For the moment we just note that in a digital system a right shift can be produced by **delay unit** denoted symbolically as follows:

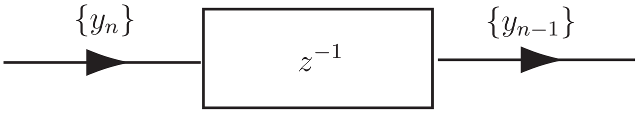



**Figure 2**

A shift of 2 units to the right could be produced by 2 such delay units in series:

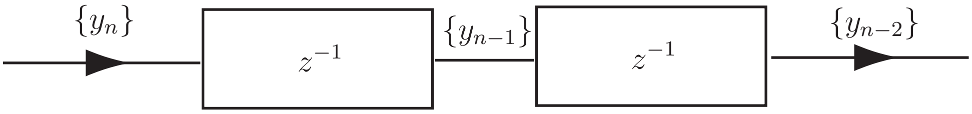



**Figure 3**

(The significance of writing $z^{-1}$ will emerge later when we have studied $z-$transforms.)

**Left Shift**

Suppose we again consider the sequence of squares

$$\{y_n\} = \{1, 4, 9, 16, 25, \ldots\}$$

with $y_n = n^2$.

Shifting all the numbers one place to the **left** (or **advancing** the sequence) means that the sequence $\{v_n\}$ generated has terms

$$v_0 = y_1 = 1 \qquad v_1 = y_2 = 4 \qquad v_2 = y_3 = 9 \ldots$$

and so has general term

$$v_n = (n+1)^2 \qquad n = 0, 1, 2, \ldots$$
$$= y_{n+1}$$

Notice here the appearance of the zero subscript for the first time.

Shifting the terms of $\{v_n\}$ one place to the left or equivalently the terms of $\{y_n\}$ two places to the left generates a sequence $\{w_n\}$ where

$$w_{-1} = v_0 = y_1 = 1 \qquad w_0 = v_1 = y_2 = 4$$

and so on.

The general term is

$$w_n = (n+2)^2 \qquad n = -1, 0, 1, 2, \ldots$$
$$= y_{n+2}$$

If $\{y_n\} = \{1, 1, 2, 3, 5, \ldots\}$ $n = 1, 2, 3, \ldots$ is the Fibonacci sequence, write out the terms of the sequences $\{y_{n+1}\}$, $\{y_{n+2}\}$.

**Your solution**

**Answer**

$$y_{n+1} = \{1, \underset{\uparrow}{1}, 2, 3, 5, \ldots\}$$ where $y_0 = 1$ (arrowed), $y_1 = 1$, $y_2 = 2, \ldots$

$$y_{n+2} = \{1, 1, \underset{\uparrow}{2}, 3, 5, \ldots\}$$ where $y_{-1} = 1$, $y_0 = 1$ (arrowed), $y_1 = 2$, $y_2 = 3, \ldots$

It should be clear from this discussion of left shifted sequences that the simpler idea of a sequence 'beginning' at $n = 1$ and containing only terms $y_1, y_2, \ldots$ has to be modified.

We should instead think of a sequence as **two-sided** i.e. $\{y_n\}$ defined for **all** integer values of $n$ and zero. In writing out the 'middle' terms of a two sided sequence it is convenient to show by an arrow the term $y_0$.

For example the sequence $\{y_n\} = \{n^2\}$ $n = 0, \pm 1, \pm 2, \ldots$ could be written

$$\{\ldots 9, 4, 1, \underset{\uparrow}{0}, 1, 4, 9, \ldots\}$$

A sequence which is zero for negative integers $n$ is sometimes called a **causal** sequence.
For example the sequence, denoted by $\{u_n\}$,

$$u_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ 1 & n = 0, 1, 2, 3, \ldots \end{cases}$$

is causal. Figure 4 makes it clear why $\{u_n\}$ is called the **unit step sequence**.

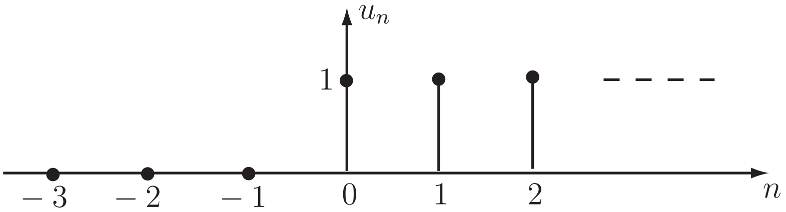



**Figure 4**

The 'curly bracket' notation for the unit step sequence with the $n = 0$ term arrowed is

$$\{u_n\} = \{\ldots, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, \ldots\}$$


Draw graphs of the sequences $\{u_{n-1}\}, \{u_{n-2}\}, \{u_{n+1}\}$ where $\{u_n\}$ is the unit step sequence.

**Your solution**

**Answer**