

# Transfer Functions

## 20.6

### Introduction

In this Section we introduce the concept of a transfer function and then use this to obtain a Laplace transform model of a linear engineering system. (A linear engineering system is one modelled by a constant coefficient ordinary differential equation.)

We shall also see how to obtain the impulse response of a linear system and hence to construct the general response by use of the convolution theorem.

### Prerequisites

Before starting this Section you should ...

- be able to use the convolution theorem
- be familiar with taking Laplace transforms and inverse Laplace transforms
- be familiar with the delta (impulse) function and its Laplace transform

### Learning Outcomes

On completion you should be able to ...

- find a transfer function of a linear system
- show how some linear systems may be combined together by combining appropriate transfer functions
- obtain the impulse response and the general response to a linear engineering system

# 1. Transfer functions and linear systems

Linear engineering systems are those that can be modelled by linear differential equations. We shall only consider those systems that can be modelled by constant coefficient ordinary differential equations.

Consider a system modelled by the second order differential equation.

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = f(t)$$

in which  $a, b, c$  are given constants and  $f(t)$  is a given function. In this context  $f(t)$  is often called the **input signal** or **forcing function** and the solution  $y(t)$  is often called the **output signal**.

We shall assume that the initial conditions are zero (in this case  $y(0) = 0, y'(0) = 0$ ).

Now, taking the Laplace transform of the differential equation, gives:

$$(as^2 + bs + c)Y(s) = F(s)$$

in which we have used  $y(0) = y'(0) = 0$  and where we have designated  $\mathcal{L}\{y(t)\} = Y(s)$  and  $\mathcal{L}\{f(t)\} = F(s)$ .

We define the **transfer function** of a system to be the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal with the initial conditions as zero. The transfer function (a function of  $s$ ), is denoted by  $H(s)$ . In this case

$$H(s) \equiv \frac{Y(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

Now, in the *special case* in which the input signal is the delta function,  $f(t) = \delta(t)$ , we have  $F(s) = 1$  and so,

$$H(s) = Y(s)$$

We call the solution to the differential equation *in this special case* the **unit impulse response function** and denote it by  $h(t)u(t)$  (we include the step function  $u(t)$  to emphasize its causality). So

$$h(t)u(t) = \mathcal{L}^{-1}\{H(s)\} \quad \text{when} \quad f(t) = \delta(t)$$

Now, keeping this in mind and returning to the general case in which the input signal  $f(t)$  is not necessarily the impulse function  $\delta(t)$ , we have:

$$Y(s) = H(s)F(s)$$

and so the solution for the output signal is, as usual, obtained by taking the inverse Laplace transform:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{H(s)F(s)\} \\ &= (h * f)(t) \end{aligned}$$

using the convolution theorem.



## Key Point 17

### Linear System Solution

The solution to a linear system, modelled by a constant coefficient ordinary differential equation, is given by the convolution of the unit impulse response function  $h(t)u(t)$  with the input function  $f(t)$ .

This approach provides yet another method of solving a linear system as Example 6 illustrates.



### Example 6

Find the impulse response function  $h(t)$  to a linear engineering system modelled by the differential equation:

$$\frac{d^2y}{dt^2} + 4y = e^{-t} \quad y(0) = 0 \quad y'(0) = 0$$

and hence solve the system.

#### Solution

Here

$$H(s) = \frac{1}{s^2 + 4} \quad \left( = \frac{1}{as^2 + bs + c} \quad \text{with } a = 1, b = 0, c = 4 \right)$$

This is obtained by replacing the forcing function  $e^{-t}$  by the impulse function  $\delta(t)$  and then taking the Laplace transform. Using this:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2} \sin 2t \cdot u(t)$$

Then the output  $y(t)$  corresponding to the input  $e^{-t}$  is given by the convolution of  $e^{-t}$  and  $h(t)$ . That is,

$$\begin{aligned} y(t) = (h * e^{-t})(t) &= \int_0^t \frac{1}{2} \sin 2(t-x) e^{-x} dx \\ &= \frac{1}{10} [\sin 2t - 2 \cos 2t + 2e^{-t}] \end{aligned}$$

(Note: the last integral can be determined by integrating by parts (twice), or by use of a computer algebra system such as Matlab.)



Use the transfer function approach to solve

$$\frac{dx}{dt} - 4x = \sin t \quad x(0) = 0.$$

Begin by finding the transfer function  $H(s)$ :

**Your solution**

**Answer**

You should find  $H(s) = 1/(s - 4)$  since the transfer function is the Laplace transform of the output  $X(s)$  when the input is a delta function  $\delta(t)$ .

Now obtain an expression for the solution  $x(t)$  in terms of the convolution:

**Your solution**

**Answer**

You should obtain  $x(t) = (\sin t * h)(t)$  where

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s - 4}\right\} = e^{4t}u(t) \quad \text{and} \quad x(t) = \int_0^t (\sin x)e^{4(t-x)}u(t-x) dx$$

Now complete the evaluation of this integral:

**Your solution**

**Answer**

If  $t > 0$  then  $u(t - x) = 1$  and so

$$\begin{aligned} x(t) &= \int_0^t \sin x e^{4(t-x)} dx = e^{4t} \left\{ \left[ -\frac{\sin x}{4} e^{-4x} \right]_0^t - \int_0^t -\frac{\cos x}{4} e^{-4x} dx \right\} \\ &= e^{4t} \left\{ -\frac{\sin t}{4} e^{-4t} + \frac{1}{4} \left( \left[ -\frac{\cos x}{4} e^{-4x} \right]_0^t - \int_0^t \frac{\sin x}{4} e^{-4x} dx \right) \right\} \end{aligned}$$

Therefore  $x(t) = -\frac{1}{4} \sin t - \frac{1}{16} \cos t + \frac{1}{16} e^{4t} - \frac{1}{16} x(t)$

Hence  $x(t) = \frac{1}{17} (-4 \sin t - \cos t + e^{4t})$

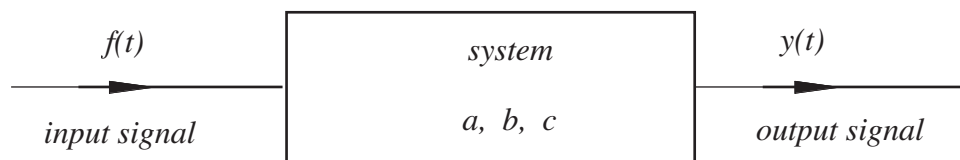
## 2. Modelling linear systems by transfer functions

We have seen previously that an engineering system can be modelled by one or more differential equations. However, with the introduction of the transfer function we have an alternative model which we examine in this Section.

It will be helpful to develop a pictorial approach to system modelling. To begin, we can imagine a differential equation:

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = f(t)$$

as being a model of the engineering system which transforms the input signal  $f(t)$  into an output signal  $y(t)$  (the solution of the differential equation). The **system** is characterised by the values of the coefficients  $a, b, c$ . A different engineering system will be characterised by a different set of coefficients. These coefficients are *independent* of the input signal. Changing the input signal does not change the system. It is the system that changes the input signal into the output signal. This is easy to describe pictorially (Figure 21).



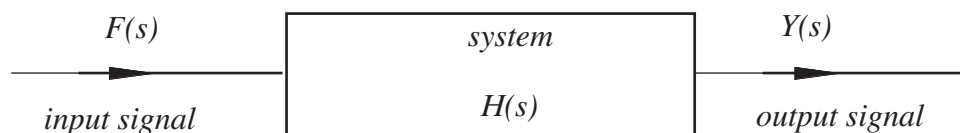
**Figure 21:** Block diagram describing the system in the  $t$ -domain

In a block diagram the system is represented by a rectangular box and the input and output signals represented by lines with an arrow to indicate the 'flow'.

After the Laplace transform of the differential equation is taken, the differential equation is transformed into

$$Y(s) \equiv H(s)F(s) \quad H(s) \equiv \frac{1}{as^2 + bs + c}$$

in which  $H(s)$  is the transfer function. The latter characterises (in Laplace transform terms) the engineering system from which it was derived. The relation, connecting the Laplace transform of the output  $Y(s)$  to the Laplace transform of the input  $F(s)$ , can also be described schematically (Figure 22).

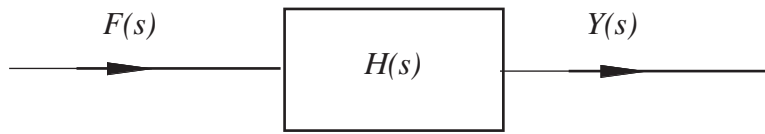


**Figure 22:** Block diagram describing the system in the  $s$ -domain

We can begin to model an engineering system directly in terms of transfer functions. In order to do this effectively we need to know how transfer functions are to be combined together. Before we do this we first extend our block diagrams to allow for 'interactions'.

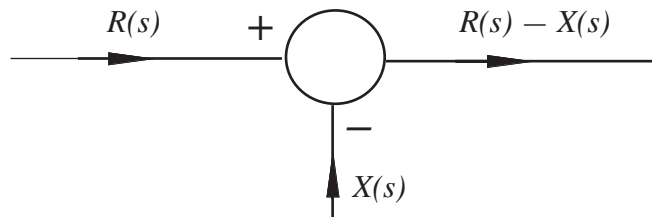
There are three basic components occurring in block diagrams which we now describe.

The **first component**, we have already met: the block relating the **input to the output** (Figure 23):



**Figure 23:** Input to Output

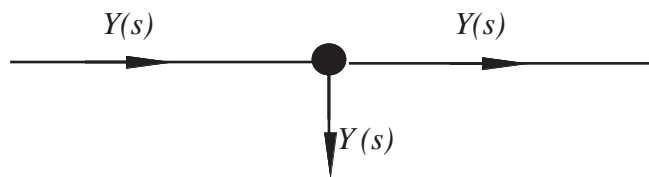
The **second component** is called a **summing point** (Figure 24):



**Figure 24:** Summing point

Here we have shown two incoming signals  $R(s), X(s)$  (but at a general summing point there may be *many* incoming signals) and one outgoing signal (there should never be more than *one* outgoing signal). The sign attached to the incoming signal defines whether the signal is adding to (+) or subtracting from (-) the summing point. The outgoing signal is then calculated in an obvious way, taking these signs into account.

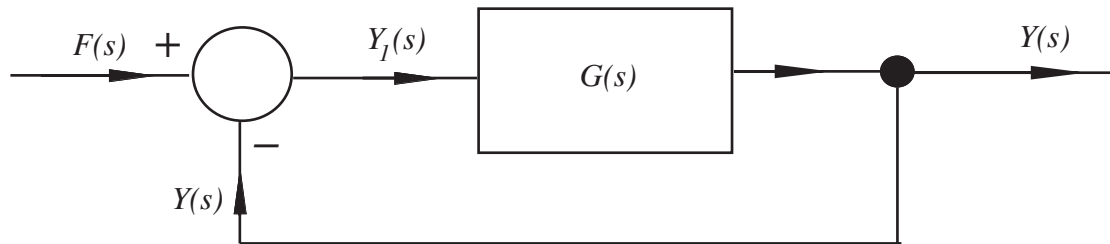
The **third component** is a **take-off point** (Figure 25):



**Figure 25:** Take-off point

Here the value of the signal  $Y(s)$  is found in such a way as not to affect the signal that is being transmitted. (This situation can never be precisely realised in practice, but using sensitive measuring devices it can be well approximated. As a simple example consider the problem of measuring the temperature of a certain volume of liquid. The act of putting a thermometer in the liquid will usually slightly affect the temperature we are trying to observe.)

An example of a block diagram is the so-called negative feedback loop, shown in Figure 26 (we are using  $G(s)$  to denote the transfer function):



**Figure 26:** Negative feedback loop

Here, the output signal is tapped and subtracted from the input signal. Hence

$$Y(s) = G(s)Y_1(s)$$

because  $Y_1(s)$  is the input signal to the system characterised by transfer function  $G(s)$ . However, at the summing point  $Y_1(s) = F(s) - Y(s)$  and so

$$Y(s) = G(s)(F(s) - Y(s))$$

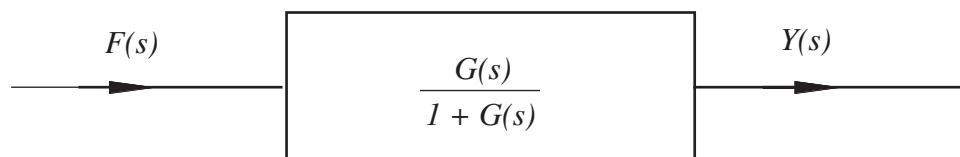
from which we easily obtain:

$$Y(s) = \left[ \frac{G(s)}{1 + G(s)} \right] F(s)$$

so that, in terms of input and output signals, the feedback loop is characterised by a transfer function

$$\frac{G(s)}{1 + G(s)}.$$

In some feedback loops the tapped signal  $Y(s)$  may be modified in some way before feedback. Using the overall transfer function we can now picture the feedback loop in a simpler way (Figure 27):



**Figure 27:** Feedback loop transfer function

Another type of block diagram occurs when the output from one system becomes the input to another system. For example consider the system of coupled differential equations:

$$\begin{aligned} \frac{dx}{dt} + x &= f(t) \\ 3\frac{dy}{dt} - y &= x(t) \\ x(0) = 0 \quad y(0) &= 0 \end{aligned}$$

in which  $f(t)$  is a given input signal.

In terms of Laplace transforms we have, as usual

$$sX(s) + X(s) = F(s) \quad 3sY(s) - Y(s) = X(s)$$

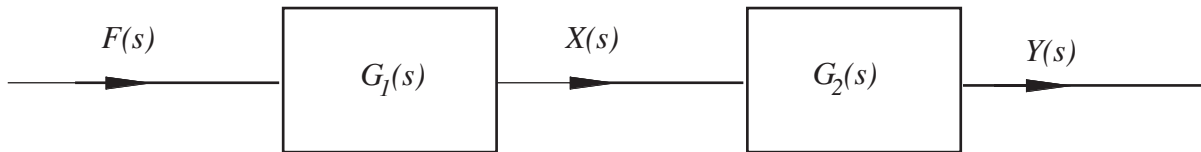
so the transfer function for the first equation ( $G_1(s)$  say) satisfies

$$G_1(s) \equiv \frac{X(s)}{F(s)} = \frac{1}{s+1}$$

whilst the transfer function for the second equation  $G_2(s)$  satisfies

$$G_2(s) \equiv \frac{Y(s)}{X(s)} = \frac{1}{3s-1}$$

In pictorial terms this is shown in Figure 28:



**Figure 28**

So we have two transfer functions 'in series'. To find how they combine we simply find an expression connecting the final output  $Y(s)$  to the initial input  $F(s)$ . Clearly

$$X(s) = G_1(s)F(s) \quad \text{and so} \quad Y(s) = G_2(s)X(s) = [G_2(s)G_1(s)] F(s)$$

So transfer functions in series are simply multiplied together. In this case the overall transfer function  $H(s)$  is:

$$H(s) = G_1(s)G_2(s) = \frac{1}{(s+1)(3s-1)}$$

Note that this result could be found directly from the differential equations used to model this system. If we differentiate the second differential equation of the original pair we get:

$$3\frac{d^2y}{dt^2} - \frac{dy}{dt} = \frac{dx}{dt}$$

Rearranging the first equation gives  $\frac{dx}{dt} = f(t) - x$

Substituting gives:  $3\frac{d^2y}{dt^2} - \frac{dy}{dt} = f(t) - x = f(t) - \left[3\frac{dy}{dt} - y\right]$

or

$$3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - y = f(t)$$

which, on taking Laplace transforms, gives the  $s$ -relation  $(3s^2 + 2s - 1)Y(s) = F(s)$  implying a transfer function:

$$H(s) = \frac{1}{3s^2 + 2s - 1} = \frac{1}{(s+1)(3s-1)}$$

as obtained above.





## Engineering Example 3

### System response

An engineering system is modelled by the block diagram in Figure 29:

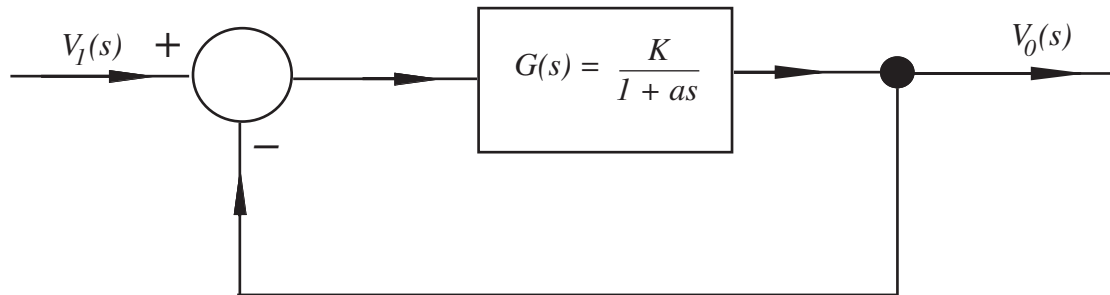


Figure 29

Determine the system response  $v_0(t)$  when the input function is a unit step function when  $K = 2.5$  and  $a = 0$ .

#### Solution

If the system has an overall transfer function  $H(s)$  then  $V_0(s) = H(s)V_1(s)$ . But this particular system is the negative feedback loop described earlier and so

$$H(s) = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{1 + as}}{1 + \frac{K}{1 + as}} = \frac{K}{K + 1 + as}$$

In this particular case

$$H(s) = \frac{2.5}{3.5 + 0.5s} = \frac{5}{7 + s}$$

Thus the impulse response  $h(t)$  is

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{(7 + s)}\right\} = 5e^{-7t}u(t)$$

and so the response to a step input  $u(t)$  is given by the convolution of  $h(t)$  with  $u(t)$

$$\begin{aligned} v_0(t) &= \int_0^t u(t-x)5e^{-7x}u(t) dx \\ &= \int_0^t 5e^{-7x} dx \quad t > 0 \\ &= \left[-\frac{5}{7}e^{-7x}\right]_0^t = -\frac{5}{7}[e^{-7t} - 1] \end{aligned}$$

**Table of Laplace Transforms**

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\frac{1}{s}$
3	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
4	$e^{-at} u(t)$	$\frac{1}{s+a}$
5	$\sin at \cdot u(t)$	$\frac{a}{s^2 + a^2}$
6	$\cos at \cdot u(t)$	$\frac{s}{s^2 + a^2}$
7	$e^{-at} \sin bt \cdot u(t)$	$\frac{b}{(s+a)^2 + b^2}$
8	$e^{-at} \cos bt u(t)$	$\frac{s+a}{(s+a)^2 + b^2}$