# Applications of Differential Equations 19.4

## Introduction

Sections 19.2 and 19.3 have introduced several techniques for solving commonly occurring first-order and second-order ordinary differential equations. In this Section we solve a number of these equations which model engineering systems.

## Prerequisites

Before starting this Section you should ...

- understand what is meant by a differential equation
- be familiar with the terminology associated with differential equations: order, dependent variable and independent variable
- be able to integrate standard functions

## Learning Outcomes

On completion you should be able to ...

- recognise and solve first-order ordinary differential equations, modelling simple electrical circuits, projectile motion and Newton's law of cooling
- recognise and solve second-order ordinary differential equations with constant coefficients modelling free electrical and mechanical oscillations
- recognise and solve second-order ordinary differential equations with constant coefficients modelling forced electrical and mechanical oscillations

# 1. Modelling with first-order equations

## Applying Newton's law of cooling

In Section 19.1 we introduced Newton's law of cooling. The model equation is

$$\frac{d\theta}{dt} = -k(\theta - \theta_{\mathsf{s}}) \qquad \theta = \theta_0 \text{ at } t = 0. \tag{5}$$

where $\theta = \theta(t)$ is the temperature of the cooling object at time $t$, $\theta_{\mathsf{s}}$ the temperature of the environment (assumed constant) and $k$ is a thermal constant related to the object, $\theta_0$ is the initial temperature of the liquid.

**Task**

Solve this initial value problem:

$$\frac{d\theta}{dt} = -k(\theta - \theta_s), \qquad \theta = \theta_0 \quad \text{at} \quad t = 0$$

Separate the variables to obtain an equation connecting two integrals:

**Your solution**

**Answer**

$$\int \frac{d\theta}{\theta - \theta_{\mathsf{s}}} = -\int k \; dt$$

Now integrate both sides of this equation:

**Your solution**

**Answer**

$\ln(\theta - \theta_{\mathsf{s}}) = -kt + C$ where $C$ is constant

Apply the initial condition and take exponentials to obtain a formula for $\theta$:

**Your solution**

**Answer**

$\ln(\theta_0 - \theta_{\mathsf{s}}) = C$. Hence $\ln(\theta - \theta_{\mathsf{s}}) = -kt + \ln(\theta_0 - \theta_{\mathsf{s}})$ so that $\ln(\theta - \theta_{\mathsf{s}}) - \ln(\theta_0 - \theta_0) = -kt$

Thus, rearranging and inverting, we find:

$$\ln\left(\frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}}\right) = -kt \qquad \therefore \qquad \frac{\theta - \theta_{\mathsf{s}}}{\theta_0 - \theta_{\mathsf{s}}} = \mathsf{e}^{-kt} \quad \text{giving} \quad \theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathsf{e}^{-kt}.$$

The graph of $\theta$ against $t$ for $\theta = \theta_s + (\theta_0 - \theta_s)\mathrm{e}^{-kt}$ is shown in Figure 4 below.

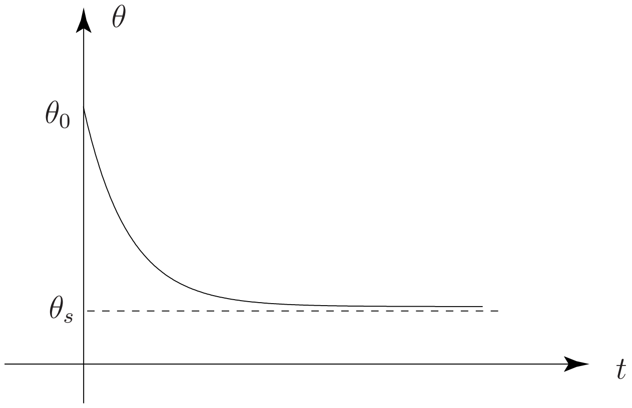



**Figure 4**

We see that as time increases ($t \to \infty$), then the temperature of the object cools down to that of the environment, that is: $\theta \to \theta_{\mathsf{s}}$.

We could have solved (5) by the integrating factor method, which you are now asked to do.

**Task**

We can write the equation for Newton's law of cooling (5) as

$$\frac{d\theta}{dt} + k\,\theta = k\theta_{\mathsf{s}}, \qquad \theta = \theta_0 \ \text{ at } \ t = 0 \qquad (6)$$

State the integrating factor for this equation:

**Your solution**

**Answer**

$\mathrm{e}^{\int k\,dt} = \mathrm{e}^{kt}$ is the integrating factor.

Multiplying (6) by this factor we find that

$$\mathrm{e}^{kt}\frac{d\theta}{dt} + k\mathrm{e}^{kt}\theta = k\theta_{\mathsf{s}}\mathrm{e}^{kt} \qquad \text{or, rearranging,} \qquad \frac{d}{dt}(\mathrm{e}^{kt}\theta) = k\theta_{\mathsf{s}}\mathrm{e}^{kt}$$

Now integrate this equation and apply the initial condition:

**Your solution**

**Answer**

Integration produces $\mathrm{e}^{kt}\theta = \theta_{\mathsf{s}}\mathrm{e}^{kt} + C$, where $C$ is an arbitrary constant. Then, applying the initial condition: when $t = 0$, $\theta_0 = \theta_{\mathsf{s}} + C$ so that $C = \theta_0 - \theta_{\mathsf{s}}$ gives the same result as before:

$$\theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathrm{e}^{-kt},$$

## Modelling electrical circuits

Another application of first-order differential equations arises in the modelling of electrical circuits. In Section 19.1 the differential equation for the RL circuit in Figure 5 below was shown to be

$$L\,\frac{di}{dt} + Ri = E$$

in which the initial condition is $i = 0$ at $t = 0$.

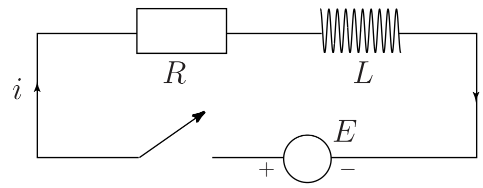



**Figure 5**

First we write this equation in standard form $\{\dfrac{dy}{dx} + P(x)y = Q(x)\}$ and obtain the integrating factor.

Dividing the differential equation through by $L$ gives

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}$$

which is now in standard form. The integrating factor is $\mathrm{e}^{\int \frac{R}{L}dt} = \mathrm{e}^{Rt/L}$.
Multiplying the equation in standard form by the integrating factor gives

$$\mathrm{e}^{Rt/L}\frac{di}{dt} + \mathrm{e}^{Rt/L}\frac{R}{L}i = \frac{E}{L}\mathrm{e}^{Rt/L}$$

or, rearranging,

$$\frac{d}{dt}(\mathrm{e}^{Rt/L}\ i) = \frac{E}{L}\mathrm{e}^{Rt/L}.$$

Now we integrate both sides and apply the initial condition to obtain the solution.

Integrating the differential equation gives:

$$\mathrm{e}^{Rt/L}\ i = \frac{E}{R}\ \mathrm{e}^{Rt/L} + C$$

where $C$ is a constant so

$$i = \frac{E}{R} + C\mathrm{e}^{-Rt/L}$$

Applying the initial condition $i = 0$ when $t = 0$ gives

$$0 = \frac{E}{R} + C$$

so that $C = -\dfrac{E}{R}$.

Finally, $i = \dfrac{E}{R}(1 - \mathrm{e}^{-Rt/L})$.

Note that as $t \to \infty$, $i \to \dfrac{E}{R}$ so as $t$ increases the effect of the inductor diminishes to zero.

**Task**

A spherical pill with volume $V$ and surface area $S$ is swallowed and slowly dissolves in the stomach, releasing an active component. In one model it is assumed that the capsule dissolves in the stomach acids such that the rate of change in volume, $\dfrac{dV}{dt}$, is directly proportional to the pill's surface area.

(a) Show that $\dfrac{dV}{dt} = -kV^{2/3}$ where $k$ is a positive real constant and solve this given that $V = V_0$ at $t = 0$.

(b) Experimental measurements indicate that for a 4 mm pill, half of the volume has dissolved after 3 hours. Find the rate constant $k$ (m s$^{-1}$).

(c) Estimate the time required for 95% of the pill to dissolve.

(a) First write down the formulae for volume of a sphere ($V$) and surface area of a sphere ($S$) and so express $S$ in terms of $V$ by eliminating $r$:

**Your solution**

**Answer**

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

From the $V$ equation $r = \left(\dfrac{3V}{4\pi}\right)^{1/3}$ so $S = (36\pi)^{1/3}V^{2/3} = kV^{2/3}$ for constant $k$.

Now write down the differential equation modelling the solution:

**Your solution**

**Answer**

$\dfrac{dV}{dt} = -kV^{2/3}$ (negative to represent a decrease with time)

Using the condition $V = V_0$ when $t = 0$, solve the differential equation:

**Your solution**

**Answer**
Solving by separation of variables gives

$$V = \left\{ \frac{1}{3}(C - kt) \right\}^{1/3}$$

and setting $V = V_0$ when $t = 0$ means

$V_0 = \left( \frac{1}{3} C \right)^3$ so $C = 3V_0^{1/3}$ and the solution is

$$V = \left\{ V_0^{1/3} - \frac{kt}{3} \right\}^3$$

(b) Impose the condition that half the volume has dissolved after 3 hours to find $k$:

**Your solution**

**Answer**

$$V = \left\{ V_0^{1/3} - \frac{kt}{3} \right\}^3$$

and when $t = 3$, $V = \dfrac{V_0}{2}$ so

$$\left( \frac{V_0}{2} \right)^{1/3} = V_0^{1/3} - k \qquad \text{and so} \qquad k = V_0^{1/3}(1 - (0.5)^{1/3})$$

(c) First write down the solution to the differential equation inserting the value of $k$ obtained in (b) and then use it to estimate the time to 95% dissolving:

**Your solution**

**Answer**

$$V = \left\{V_0^{1/3} - V_0^{1/3}(1-(0.5)^{1/3})\frac{t}{3}\right\}^3 \qquad \text{i.e.} \qquad V = V_0\left\{1-(1-(0.5)^{1/3})\frac{t}{3}\right\}^3$$

When 95% dissolved $V = 0.05V_0$ so

$$0.05V_0 = V_0\left\{1-(1-(0.5)^{1/3})\frac{t}{3}\right\}^3 \qquad \text{so} \qquad (0.05)^{1/3} = 1-(1-(0.5)^{1/3})\frac{t}{3}$$

so

$$t = 3\left\{\frac{1-(0.05)^{1/3}}{1-(0.5)^{1/3}}\right\} \approx 9.185 \approx 9 \text{ hr } 11 \text{ min}$$

# 2. Modelling free mechanical oscillations

Consider the following schematic diagram of a shock absorber:

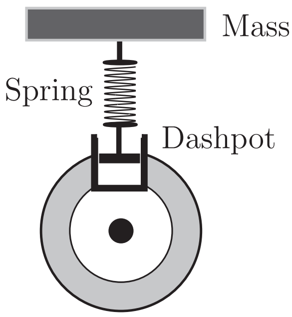



**Figure 6**

The equation of motion can be described in terms of the vertical displacement $x$ of the mass.

Let $m$ be the mass, $k\dfrac{dx}{dt}$ the damping force resulting from the dashpot and $nx$ the restoring force resulting from the spring. Here, $k$ and $n$ are constants.

Then the equation of motion is

$$m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - nx.$$

Suppose that the mass is displaced a distance $x_0$ initially and released from rest. Then at $t = 0$, $x = x_0$ and $\dfrac{dx}{dt} = 0$. Writing the differential equation in standard form gives

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = 0.$$

We shall see that the nature of the oscillations described by this differential equation depends crucially upon the relative values of the mechanical constants $m, k$ and $n$. This will be explored in subsequent Tasks.

**Task**

Find and solve the auxiliary equation of the differential equation

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = 0.$$

**Your solution**

**Answer**

Putting $x = \mathrm{e}^{\lambda t}$, the auxiliary equation is $\quad m\,\lambda^2 + k\,\lambda + n = 0.$

Hence $\lambda = \dfrac{-k \pm \sqrt{k^2 - 4m\,n}}{2m}$.

The value of $k$ controls the amount of damping in the system. We explore the solution for various values of $k$.

## Case 1: No damping

If $k = 0$ then there is no damping. We expect, in this case, that once motion has started it will continue for ever. The motion that ensues is called **simple harmonic motion**. In this case we have

$$\lambda = \frac{\pm\sqrt{-4m\,n}}{2m}, \quad \text{that is,} \quad \lambda = \pm\sqrt{\frac{n}{m}}\,\mathrm{i} \qquad \text{where} \quad \mathrm{i}^2 = -1.$$

and the solution for the displacement $x$ is:

$$x = A\cos\left(\sqrt{\frac{n}{m}}\,t\right) + B\sin\left(\sqrt{\frac{n}{m}}\,t\right) \quad \text{where } A, B \text{ are arbitrary constants.}$$

**Task**

Impose the initial conditions $x = x_0$ and $\dfrac{dx}{dt} = 0$ at $t = 0$ to find the unique solution to the ODE:

**Your solution**

**Answer**

$$\frac{dx}{dt} = -\sqrt{\frac{n}{m}}\, A\sin\left(\sqrt{\frac{n}{m}}\, t\right) + \sqrt{\frac{n}{m}} B\cos\left(\sqrt{\frac{n}{m}}\, t\right)$$

When $t = 0$, $\dfrac{dx}{dt} = 0$ so that $\quad \sqrt{\dfrac{n}{m}}\, B = 0 \quad$ so that $\quad B = 0.$

Therefore $\quad x = A\cos\left(\sqrt{\dfrac{n}{m}}\, t\right).$

Imposing the remaining initial condition: when $t = 0,\ x = x_0$ so that $x_0 = A$ and finally:

$$x = x_0\cos\left(\sqrt{\frac{n}{m}}\, t\right).$$

## Case 2: Light damping

If $k^2 - 4mn < 0$, i.e. $k^2 < 4mn$ then the roots of the auxiliary equation are complex:

$$\lambda_1 = \frac{-k + \mathrm{i}\sqrt{4mn - k^2}}{2m} \qquad \lambda_2 = \frac{-k - \mathrm{i}\sqrt{4mn - k^2}}{2m}$$

Then, after some rearrangement:

$$x = \mathrm{e}^{-kt/2m}\left[A\cos pt + B\sin pt\right] \quad \text{in which} \quad p = \sqrt{4mn - k^2}/2m.$$

**Task**

If $m = 1,\ n = 1$ and $k = 1$ find $\lambda_1$ and $\lambda_2$ and then find the solution for the displacement $x$.

**Your solution**

**Answer**

$\lambda = \dfrac{-1 + \mathrm{i}\sqrt{4-1}}{2} = -1/2 \pm \mathrm{i}\sqrt{3}/2$. Hence $x = \mathrm{e}^{-t/2}\left[A\cos\dfrac{\sqrt{3}}{2}t + B\sin\dfrac{\sqrt{3}}{2}t\right]$.

Impose the initial conditions $x = x_0,\ \dfrac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants and hence find the solution to the ODE:

**Your solution**

**Answer**

Differentiating, we obtain

$$\frac{dx}{dt} = -\frac{1}{2}\mathrm{e}^{-t/2}\left[A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right] + \mathrm{e}^{-t/2}\left[-\frac{\sqrt{3}}{2}A\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}B\cos\frac{\sqrt{3}}{2}t\right]$$

At $t = 0$,

$$x = x_0 = A \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B \qquad \text{(ii)}$$

Solving (i) and (ii) we obtain

$$A = x_0 \qquad B = \frac{\sqrt{3}}{3}x_0 \quad \text{then} \quad x = x_0\mathrm{e}^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right].$$

The graph of $x$ against $t$ is shown in Figure 7. This is the case of light damping. As the damping in the system decreases (i.e. $k \to 0$ ) the number of oscillations (in a given time interval) will increase. In many mechanical systems these oscillations are usually unwanted and the designer would choose a value of $k$ to either reduce them or to eliminate them altogether. For the choice $k^2 = 4mn$, known as the **critical damping** case, all the oscillations are absent.

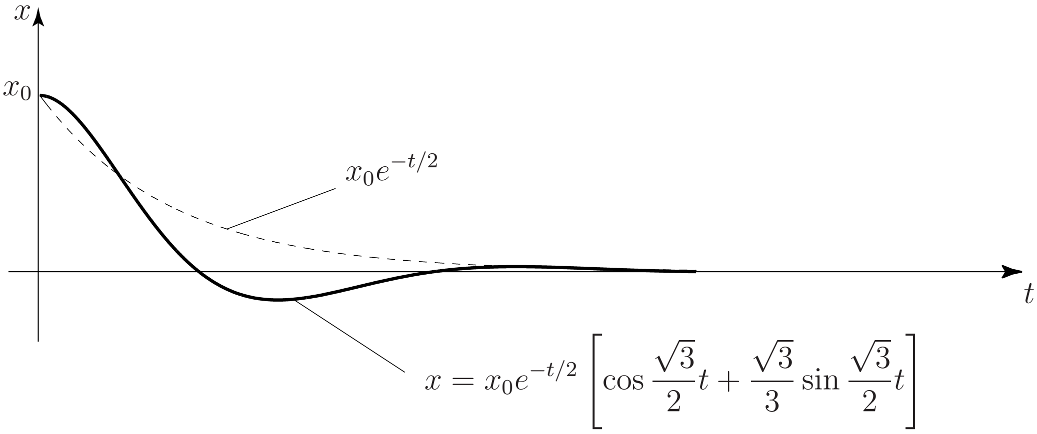



**Figure 7**

## Case 3: Heavy damping

If $k^2 - 4mn > 0$, i.e. $k^2 > 4mn$, then there are two real roots of the auxiliary equation, $\lambda_1$ and $\lambda_2$:

$$\lambda_1 = \frac{-k + \sqrt{k^2 - 4mn}}{2m} \qquad \lambda_2 = \frac{-k - \sqrt{k^2 - 4mn}}{2m}$$

Then

$$x = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}.$$

**Task**

If $m = 1,\ n = 1$ and $k = 2.5$ find $\lambda_1$ and $\lambda_2$ and then find the solution for the displacement $x$.

**Your solution**

**Answer**

$$\lambda = \frac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -1.25 \pm 0.75$$

Hence $\lambda_1,\ \lambda_2 = -0.5, -2$ and so $x = A\mathrm{e}^{-0.5t} + B\mathrm{e}^{-2t}$

Impose the initial conditions $x = x_0,\ \dfrac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants and hence find the solution to the ODE.

**Your solution**

**Answer**

Differentiating, we obtain

$$\frac{dx}{dt} = -0.5A\mathrm{e}^{-0.5t} - 2B\mathrm{e}^{-2t}$$

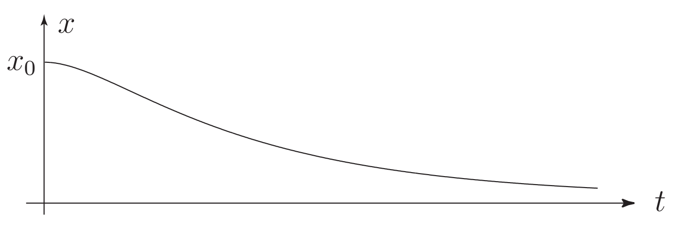
At $t = 0$,

$$x = x_0 = A + B \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -0.5A - 2B \qquad \text{(ii)}$$

Solving (i) and (ii) we obtain $\quad A = \dfrac{4}{3}x_0 \qquad B = -\dfrac{1}{3}x_0$ then $\quad x = \dfrac{1}{3}x_0(4\mathrm{e}^{-0.5t} - \mathrm{e}^{-2t})$.

The graph of $x$ against $t$ is shown below. This is the case of heavy damping.




Other cases are dealt with in the Exercises at the end of the Section.

# 3. Modelling forced mechanical oscillations

Suppose now that the mass is subject to a force $f(t)$ after the initial disturbance. Then the equation of motion is

$$m\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + nx = f(t)$$

Consider the case $f(t) = F\cos\omega t$, that is, an oscillatory force of magnitude $F$ and angular frequency $\omega$. Choosing specific values for the constants in the model: $m = n = 1$, $k = 0$, and $\omega = 2$ we find

$$\frac{d^2x}{dt^2} + x = F\cos 2t$$

**Task**

Find the complementary function for the differential equation

$$\frac{d^2x}{dt^2} + x = F\cos 2t$$

**Your solution**

**Answer**

The homogeneous equation is

$$\frac{d^2x}{dt^2} + x = 0$$

with auxiliary equation $\lambda^2 + 1 = 0$. Hence the complementary function is

$$x_{\text{cf}} = A\cos t + B\sin t.$$

Now find a particular integral for the differential equation:

**Your solution**

**Answer**

Try $x_{\mathsf{p}} = C\cos 2t + D\sin 2t$ so that $\dfrac{d^2x_{\mathsf{p}}}{dt^2} = -4C\cos 2t - 4D\sin 2t$. Substituting into the differential equation gives

$$(-4C + C)\cos 2t + (-4D + D\ \sin 2t) \equiv F\cos 2t.$$

Comparing coefficients gives $-3C = F$ and $-3D = 0$ so that $D = 0,\quad C = -\dfrac{1}{3}F$ and $x_{\mathsf{p}} = -\dfrac{1}{3}F\cos 2t$. The general solution of the differential equation is therefore

$$x = x_{\mathsf{p}} + x_{\mathsf{cf}} = -\frac{1}{3}F\cos 2t + A\cos t + B\sin t.$$

Finally, apply the initial conditions to find the solution for the displacement $x$:

**Your solution**

**Answer**

We need to determine the derivative and apply the initial conditions:

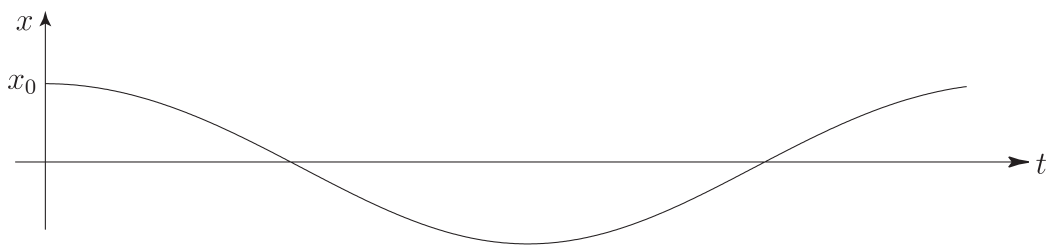
$$\frac{dx}{dt} = \frac{2}{3}F\sin 2t - A\sin t + B\cos t.$$

At $t = 0$ $\qquad x = x_0 = -\dfrac{1}{3}F + A$ and $\dfrac{dx}{dt} = 0 = B$

Hence $\qquad B = 0$ and $A = x_0 + \dfrac{1}{3}F.$

Then $\qquad x = -\dfrac{1}{3}F\cos 2t + \left(x_0 + \dfrac{1}{3}F\right)\cos t.$

The graph of $x$ against $t$ is shown below.




If the angular frequency $\omega$ of the applied force is *nearly equal to* that of the free oscillation the phenomenon of **beats** occurs. If the angular frequencies are *equal* we get the phenomenon of **resonance**. Note that we can eliminate resonance by introducing damping into the system.

# 4. Modelling forces on beams

## Engineering Example 3

### Shear force and bending moment of a beam

**Introduction**

The beam is a fundamental part of most structures we see around us. It may be used in many ways depending as how its ends are fixed. One end may be rigidly fixed and the other free (called **cantilevered**) or both ends may be resting on supports (called **simply supported**). Other combinations are possible. There are three basic quantities of interest in the deformation of beams, the deflection, the shear force and the bending moment.

For a beam which is supporting a load of $w$ (measured in N m$^{-1}$ and which may represent the self-weight of the beam or may be an external load), the shear force is denoted by $S$ and measured in N m$^{-1}$ and the bending moment is denoted by $M$ and measured in N m$^{-1}$.

The quantities $M$, $S$ and $w$ are related by

$$\frac{dM}{dz} = S \qquad (1)$$

and

$$\frac{dS}{dz} = -w \qquad (2)$$

where $z$ measures the position along the beam. If one of the quantities is known, the others can be calculated from the Equations (1) and (2). In words, the shear force is the negative of the derivative (with respect to position) of the bending moment and the load is the derivative of the shear force. Alternatively, the shear force is the negative of the integral (with respect to position) of the load and the bending moment is the integral of the shear force. The negative sign in Equation (2) reflects the fact that the load is normally measured positively in the downward direction while a positive shear force refers to an upward force.

**Problem posed in words**

A beam is fixed rigidly at one end and free to move at the other end (like a diving board). It only has to support its own weight. Find the shear force and the bending moment along its length.

**Mathematical statement of problem**

A uniform beam of length $L$, supports its own weight $w_o$ (a constant). At one end $(z = 0)$, the beam is fixed rigidly while the other end $(z = L)$ is free to move. Find the shear force $S$ and the bending moment $M$ as functions of $z$.

**Mathematical analysis**

As $w$ is a constant, Equation (2) gives

$$S = -\int w dz = -\int w_o dz = -w_o z + C.$$

At the free end $(z = L)$, the shear force $S = 0$ so $C = w_oL$ giving

$$S = w_o (L - z)$$

This expression can be substituted into Equation (1) to give

$$M = \int S dz = \int w_o (L - z)\, dz = \int (w_o L - w_o z)\, dz = w_o L z - \frac{w_o}{2} z^2 + K.$$

Once again, $M = 0$ at the free end $z = L$ so $K$ is given by $K = -\frac{w_o}{2}L^2$. Thus

$$M = w_o L z - \frac{w_o}{2} z^2 - \frac{w_o}{2} L^2$$

The diagrams in Figure 8 show the load $w$ (Figure 8a), the shear force $S$ (Figure 8b) and the bending moment $M$ (Figure 8c) as functions of position $z$.

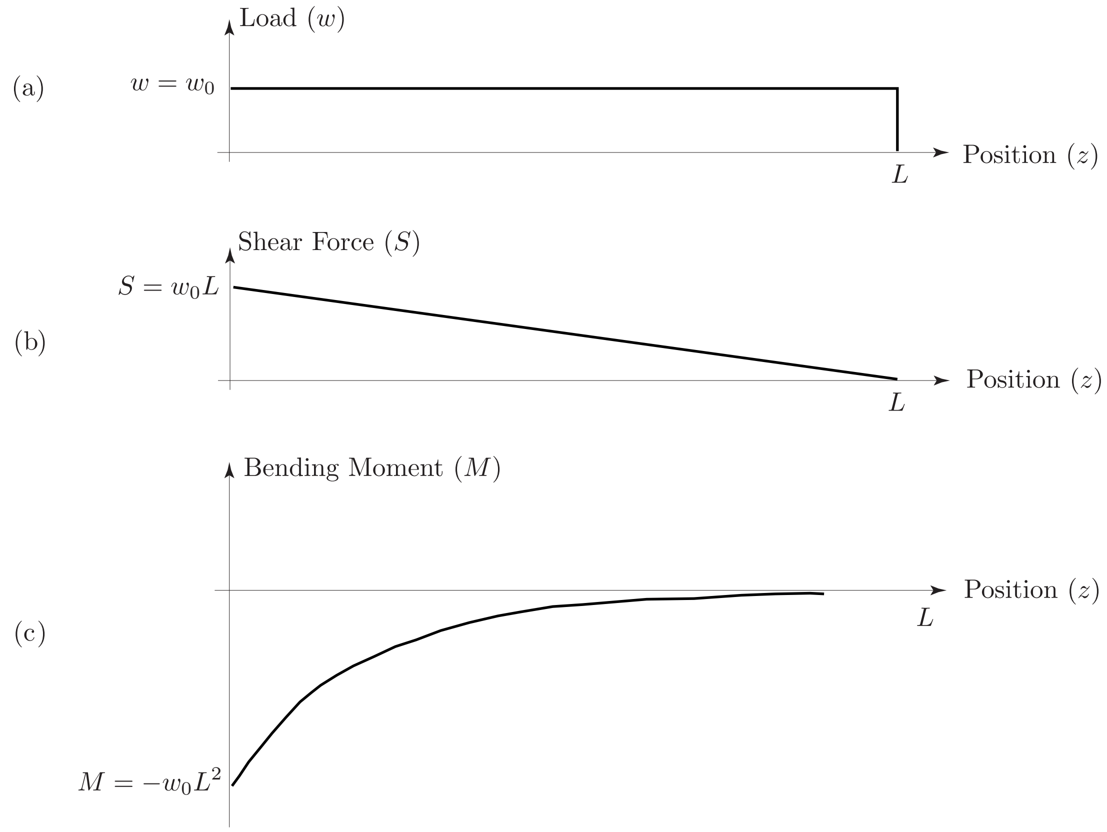



**Figure 8**: The loading (a), shear force (b) and bending moment (c) as functions of position z

**Interpretation**

The beam deforms (as we might have expected) with the shear force and bending moments having maximum values at the fixed end and minimum (zero) values at the free end. You can easily experience this for yourselves: simply hold a wooden plank (not too heavy) at one end with both hands so that it is horizontal. As you try this with planks of increasing length (and hence weight) you will find it increasingly difficult to support the weight of the plank (this is the shear force) and increasingly difficult to keep the plank horizontal (this is the bending moment).

This mathematical model is an excellent description of real beams.

## Engineering Example 4

# Deflection of a uniformly loaded beam

### Introduction

A uniformly loaded beam of length $L$ is supported at both ends as shown in Figure 9. The deflection $y(x)$ is a function of horizontal position $x$ and obeys the ordinary differential equation (ODE)

$$\frac{d^4y}{dx^4}(x) = \frac{1}{EI}q(x) \qquad (1)$$

where $E$ is Young's modulus, $I$ is the moment of inertia and $q(x)$ is the load per unit length at point $x$. We assume in this problem that $q(x) = q$ a constant. The boundary conditions are (i) no deflection at $x = 0$ and $x = L$ (ii) no curvature of the beam at $x = 0$ and $x = L$.

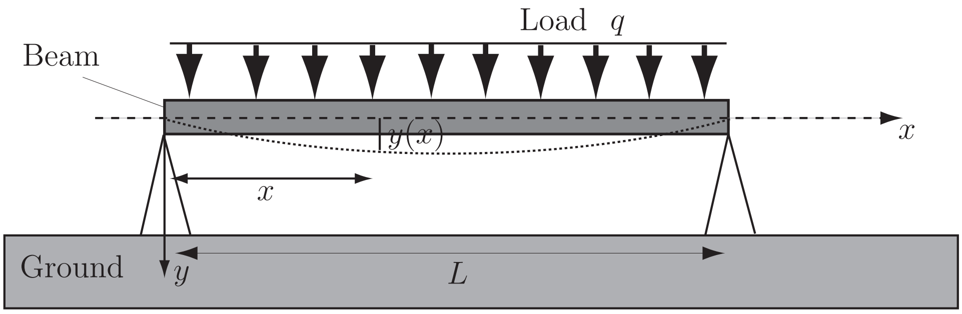



**Figure 9**: The bending beam, parameters involved in the mathematical formulation

### Problem in words

Find the deflection of a beam, supported so that that there is no deflection and no curvature of the beam at its ends, subject to a uniformly distributed load, as a function of position along the beam.

### Mathematical statement of problem

Find the equation of the curve $y(x)$ assumed by the bending beam that satisfies the ODE (1). Use the coordinate system shown in Figure 9 where the origin is at the left extremity of the beam. In this coordinate system, the boundary conditions, which require that there is no deflection at $x = 0$ and $x = L$, and that there is no curvature of the beam at $x = 0$ and $x = L$, are

(a) $y(0) = 0$

(b) $y(L) = 0$

(c) $\frac{d^2y}{dx^2}\Big|_{x=0} = 0$

(d) $\frac{d^2y}{dx^2}\Big|_{x=L} = 0$

(e) $\frac{d^2y}{dx^2}\Big|_{x=L} = 0$

Note that $\dfrac{dy(x)}{dx}$ and $\dfrac{d^2y(x)}{dx^2}$ are respectively the slope and the radius of curvature of the curve at point $(x, y)$.

**Mathematical analysis**

Integrating Equation (1) leads to:

$$EI\frac{d^3y}{dx^3}(x) = qx + A \tag{2}$$

Integrating a second time:

$$EI\frac{d^2y}{dx^2}(x) = qx^2/2 + Ax + B \tag{3}$$

Integrating a third time:

$$EI\frac{dy}{dx}(x) = qx^3/6 + Ax^2/2 + Bx + C \tag{4}$$

Integrating a fourth time:

$$EIy(x) = qx^4/24 + Ax^3/6 + Bx^2/2 + Cx + D. \tag{5}$$

The boundary conditions (a) and (b) enable determination of the constants of integration $A, B, C, D$. Indeed, the boundary condition (a), $y(0) = 0,$ and Equation (5) give

$$EIy(0) = q \times (0)^4/24 + A \times (0)^3/6 + B \times (0)^2/2 + C \times (0) + D = 0$$

which yields $D = 0.$
The boundary condition (b), $y(L) = 0,$ and Equation (5) give

$$EIy(L) = qL^4/24 + AL^3/6 + BL^2/2 + CL + D.$$

Using the newly found value for $D$ one writes

$$qL^4/24 + AL^3/6 + BL^2/2 + CL = 0 \tag{6}$$

The boundary condition (c) obtained from the definition of the radius of curvature, $\dfrac{d^2y}{dx^2}(0) = 0,$ and Equation (3) give

$$I\frac{d^2y}{dx^2}(0) = q \times (0)^2/2 + A \times (0) + B$$

which yields $B = 0$ . The boundary condition (d), $\dfrac{d^2y}{dx^2}(L) = 0,$ and Equation (3) give

$$EI\frac{d^2y}{dx^2}(L) = qL^2/2 + AL = 0$$

which yields $A = -qL/2$ . The expressions for $A, B, D$ are introduced in Equation (6) to find the last unknown constant $C$. This leads to $qL^4/24 - qL^4/12 + CL = 0$ or $C = qL^3/24$. Finally, Equation (5) and the values of constants lead to the solution

$$y(x) = [qx^4/24 - qLx^3/12 + qL^3x/24]/EI. \tag{7}$$

**Interpretation**

The predicted deflection is zero at both ends as required, and you may check that it is symmetrical about the centre of the beam by switching to the coordinate system $(X, Y)$ with $L/2 - x = X$ and $y = Y$ and verifying that the deflection $Y(X)$ is symmetrical about the vertical axis, i.e. $Y(X) = Y(-X)$.

## Exercises

1. In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant $E$. Given that $\dfrac{dq}{dt} = i$ where $q$ is the charge in the capacitor, $i$ the current in the circuit, $R$ the resistance and $C$ the capacitance the equation for the circuit is

   $$Ri + \frac{q}{C} = E.$$

   If the initial charge is zero find the charge subsequently.

2. If the voltage in the RC circuit is $E = E_0 \cos \omega t$ find the charge and the current at time $t$.

3. An object is projected from the Earth's surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance.

   The equation of motion is

   $$m\,v\frac{dv}{dx} = -m\,g\frac{R^2}{x^2}$$

   where the mass of the object is $m$, its distance from the centre of the Earth is $x$ and the radius of the Earth is $R$.

4. The radial stress $p$ at distance $r$ from the axis of a thick cylinder subjected to internal pressure is given by $p + r\dfrac{dp}{dr} = A - p$ where $A$ is a constant. If $p = p_0$ at the inner wall $(r = r_1)$ and is negligible $(p = 0)$ at the outer wall $(r = r_2)$ find an expression for $p$.

5. The equation for an LCR circuit with applied voltage $E$ is

   $$L\frac{di}{dt} + Ri + \frac{1}{C}q = E.$$

   By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 10^{-4}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

6. Consider the free vibration problem in Section 19.4 subsection 2 (page 57) when $m = 1,\ n = 1$ and $k = 2$ (critical damping).

   Find the solution for $x(t)$.

7. Repeat Exercise 6 for the case $m = 1,\ n = 1$ and $k = 1.5$ (light damping)

8. Consider the forced vibration problem in Section 19.4 subsection 2 with $m = 1,\ n = 25,\ k = 8,\ E = \sin 3t,\ x_0 = 0$ with an initial velocity of 3.

9. This refers to the Task on page 55 concerning modelling the dissolving of a pill in the stomach.

   An alternative model supposes that the pill is very rapidly permeated by stomach acids and the small granules contained in the capsule dissolve individually. In this case, the rate of change of volume is assumed to be directly proportional to the volume. Using the experimental data given in the Task, estimate the time for 95% of the pill to dissolve, based on this alternative model, and compare results.

**Answers**

1. Use the equation in the form $R\dfrac{dq}{dt} + \dfrac{q}{C} = E$ or $\dfrac{dq}{dt} + \dfrac{1}{RC}q = \dfrac{E}{R}$.

   The integrating factor is $\mathrm{e}^{t/RC}$ and the general solution is

   $q = EC(1 - \mathrm{e}^{-t/RC})$ and as $t \to \infty \;\; q \to EC.$

2. $q = \dfrac{E_0 C}{1 + \omega^2 R^2 C^2} \left[\cos \omega t - \mathrm{e}^{-t/RC} + \omega RC \sin \omega t\right]$

   $$i = \frac{dq}{dt} = \frac{E_0 C}{1 + \omega^2 R^2 C^2} \left[-\omega \sin \omega t + \frac{1}{RC}\,\mathrm{e}^{-t/RC} + \omega^2 RC \cos \omega t\right].$$

3. $v_{\text{min}} = \sqrt{2gR}$. If $R = 6378$ km and $g = 9.81$ m s$^{-2}$ then $v_{\text{min}} = 11.2$ km s$^{-1}$.

4. $p = \dfrac{p_0 r_1^2}{r_1^2 - r_2^2}\left(1 - \dfrac{r_2^2}{r^2}\right)$

5. $q = 0.1 - \dfrac{1}{10\sqrt{3}}\mathrm{e}^{-50t}(\sin 50\sqrt{3}t + \sqrt{3}\cos 50\sqrt{3}t) \qquad i = \dfrac{20}{\sqrt{3}}\mathrm{e}^{-50t}\sin 50\sqrt{3}t.$

6. $x = x_0(1 + t)e^{-t}$

7. $x = x_0 e^{-0.75t}(\cos \dfrac{\sqrt{7}}{4}t + \dfrac{3}{\sqrt{7}}\sin \dfrac{\sqrt{7}}{4}t)$

8. $x = \dfrac{1}{104}\left[\mathrm{e}^{-4t}\,(3\cos 3t + 106 \sin 3t) - 3\cos 3t + 2\sin 3t)\right]$

9. This leads to $\dfrac{dV}{dt} = -kV$ and $V = V_0\mathrm{e}^{-kt}$ where $k = \dfrac{1}{3}\ln 2$. The time taken is about 4 hr 19 min. This is much less than the other model, as should be expected.