Contents 19

Differential Equations

Learning outcomes

In this Workbook you will learn what a differential equation is and how to recognise some of the basic different types. You will learn how to apply some common techniques used to obtain general solutions of differential equations and how to fit initial or boundary conditions to obtain a unique solution. You will appreciate how differential equations arise in applications and you will gain some experience in applying your knowledge to model a number of engineering problems using differential equations.

Modelling with Differential Equations 19.1

Introduction

Many models of engineering systems involve the **rate of change** of a quantity. There is thus a need to incorporate derivatives into the mathematical model. These mathematical models are examples of **differential equations**.

Accompanying the differential equation will be one or more conditions that let us obtain a unique solution to a particular problem. Often we solve the differential equation first to obtain a general solution; then we apply the conditions to obtain the unique solution. It is important to know which conditions must be specified in order to obtain a unique solution.

Prerequisites

Before starting this Section you should ...

- be able to differentiate; (HELM 11)
- be able to integrate; (HELM 13)

Learning Outcomes

On completion you should be able to ...

- understand the use of differential equations in modelling engineering systems
- identify the order and type of a differential equation
- recognise the nature of a general solution
- determine the nature of the appropriate additional conditions which will give a unique solution to the equation

1. Case study: Newton's law of cooling

When a hot liquid is placed in a cooler environment, experimental observation shows that its temperature decreases to approximately that of its surroundings. A typical graph of the temperature of the liquid plotted against time is shown in Figure 1.

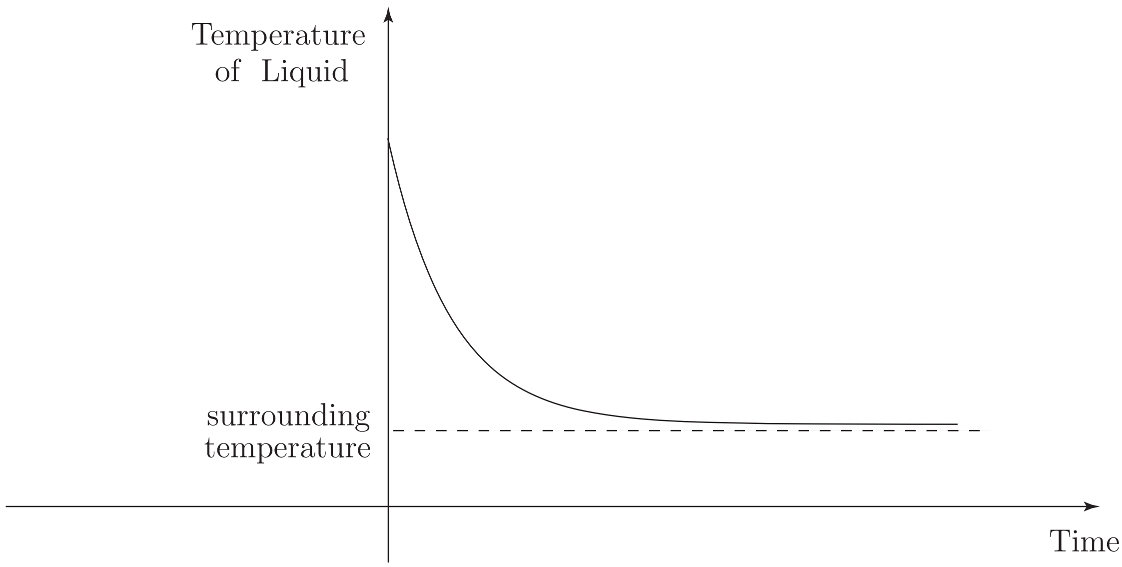


Figure 1

After an initially rapid decrease the temperature changes progressively less rapidly and eventually the curve appears to 'flatten out'.

Newton's law of cooling states that the rate of cooling of liquid is proportional to the difference between its temperature and the temperature of its environment (the ambient temperature). To convert this into mathematics, let t be the time elapsed (in seconds, s), θ the temperature of the liquid (°C),and θ_0 the temperature of the liquid at the start $(t = 0)$. The temperature of the surroundings is denoted by θ_{s}.

Task

Write down the mathematical equation which is equivalent to Newton's law of cooling and state the accompanying condition.

First, find an expression for the rate of cooling, and an expression for the difference between the liquid's temperature and that of the environment:

Your solution

Answer

The rate of cooling is the rate of change of temperature with time: $\dfrac{d\theta}{dt}$.

The temperature difference is $\theta - \theta_{\mathsf{s}}$.

Now formulate Newton's law of cooling:

Your solution

Answer

You should obtain $\dfrac{d\theta}{dt} \propto (\theta - \theta_{\mathsf{s}})$ or, equivalently: $\dfrac{d\theta}{dt} = -k(\theta - \theta_{\mathsf{s}})$. k is a positive constant of proportion and the negative sign is present because $(\theta - \theta_{\mathsf{s}})$ is positive, whereas $\dfrac{d\theta}{dt}$ must be negative, since θ decreases with time. The units of k are s^{-1}. The accompanying condition is $\theta = \theta_0$ at $t = 0$ which simply states the temperature of the liquid when the cooling begins.

In the above Task we call t the independent variable and θ the dependent variable. Since the condition is given at $t = 0$ we refer to it as an initial condition. For future reference, the solution of the above differential equation which satisfies the initial condition is $\theta = \theta_{\mathsf{s}} + (\theta_0 - \theta_{\mathsf{s}})\mathsf{e}^{-kt}$.

2. The general solution of a differential equation

Consider the equation $y = A\mathsf{e}^{2x}$ where A is an arbitrary constant. If we differentiate it we obtain

$$\frac{dy}{dx} = 2A\mathsf{e}^{2x}$$

and so, since $y = A\mathsf{e}^{2x}$ we obtain

$$\frac{dy}{dx} = 2y.$$

Thus a differential equation satisfied by y is

$$\frac{dy}{dx} = 2y.$$

Note that we have eliminated the arbitrary constant.
Now consider the equation

$$y = A\cos 3x + B\sin 3x$$

where A and B are arbitrary constants. Differentiating, we obtain

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x.$$

Differentiating a second time gives

$$\frac{d^2y}{dx^2} = -9\,A\cos 3x - 9\,B\sin 3x.$$

The right-hand side is simply (-9) times the expression for y. Hence y satisfies the differential equation

$$\frac{d^2y}{dx^2} = -9y.$$

Task

Find a differential equation satisfied by $y = A\cosh 2x + B\sinh 2x$ where A and B are arbitrary constants.

Your solution

Answer

Differentiating once we obtain $\dfrac{dy}{dx} = 2A\sinh 2x + 2B\cosh 2x$

Differentiating a second time we obtain $\dfrac{d^2y}{dx^2} = 4A\cosh 2x + 4B\sinh 2x$

Hence $\dfrac{d^2y}{dx^2} = 4y$

We have seen that an expression including one arbitrary constant required one differentiation to obtain a differential equation which eliminated the arbitrary constant. Where two constants were present, two differentiations were required. Is the converse true? For example, would a differential equation involving $\dfrac{dy}{dx}$ as the only derivative have a general solution with one arbitrary constant and would a differential equation which had $\dfrac{d^2y}{dx^2}$ as the highest derivative produce a general solution with two arbitrary constants? The answer is, usually, yes.

Task

Integrate twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{w}{2}(\ell x - x^2),$$

where w and ℓ are constants, to find a general solution for y.

Your solution

Answer

Integrating once: $\dfrac{dy}{dx} = \dfrac{w}{2}\left(\dfrac{\ell x^2}{2} - \dfrac{x^3}{3}\right) + A$ where A is an arbitrary constant (of integration).

Integrating again: $y = \dfrac{w}{2}\left(\dfrac{\ell x^3}{6} - \dfrac{x^4}{12}\right) + A\,x + B$ where B is a second arbitrary constant.

Consider the simple differential equation

$$\frac{dy}{dx} = 2x.$$

On integrating, we obtain the general solution

$$y = x^2 + C$$

where C is an arbitrary constant. As C varies we get different solutions, each of which belongs to the family of solutions. Figure 2 shows some examples.

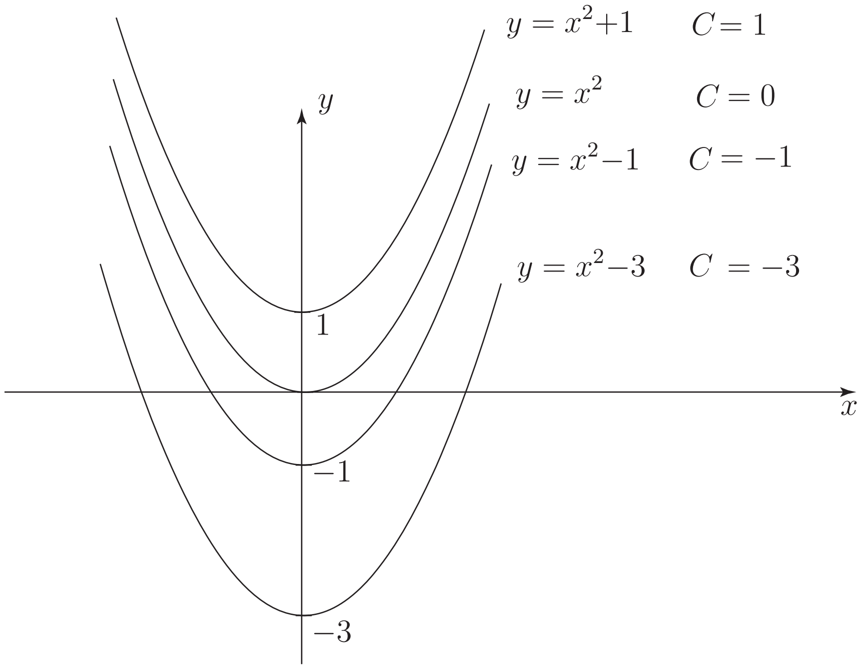


Figure 2

It can be shown that no two members of this family of graphs ever meet and that through each point in the x-y plane passes one, and only one, of these graphs. Hence if we specify the boundary condition $y = 2$ when $x = 0$, written $y(0) = 2$, then using $y = x^2 + c$:

$$2 = 0 + C \quad \text{so that} \quad C = 2$$

and $y = x^2 + 2$ is the unique solution.

Find the unique solution of the differential equation $\dfrac{dy}{dx} = 3x^2$ which satisfies the condition $y(1) = 4$.

Your solution

Answer

You should obtain $y = x^3 + 3$ since, by a single integration we have $y = x^3 + C$, where C is an arbitrary constant. Now when $x = 1,\ y = 4$ so that $4 = 1 + C$. Hence $C = 3$ and the unique solution is $y = x^3 + 3$.

Example 1

Solve the differential equation $\dfrac{d^2y}{dx^2} = 6x$ subject to the conditions

(a) $y(0) = 2$ and $y(1) = 3$

(b) $y(0) = 2$ and $y(1) = 5$

(c) $y(0) = 2$ and $\dfrac{dy}{dx} = 1$ at $x = 0$.

Solution

(a) Integrating the differential equation once produces $\dfrac{dy}{dx} = 3x^2 + A$. The general solution is found by integrating a second time to give $y = x^3 + Ax + B$, where A and B are arbitrary constants.

Imposing the conditions $y(0) = 2$ and $y(1) = 3$: at $x = 0$ we have $y = 2 = 0 + 0 + B = B$ so that $B = 2$, and at $x = 1$ we have $y = 3 = 1 + A + B = 1 + A + 2$. Therefore $A = 0$ and the solution is

$$y = x^3 + 2.$$

(b) Here the second condition is $y(1) = 5$ so at $x = 1$

$$y = 5 = 1 + A + 2 \qquad \text{so that} \quad A = 2$$

and the solution in this case is

$$y = x^3 + 2x + 2.$$

(c) Here the second condition is

$$\frac{dy}{dx} = 1 \text{ at } x = 0 \qquad \text{i.e.} \qquad y'(0) = 1$$

then since $\dfrac{dy}{dx} = 3x^2 + A$, putting $x = 0$ we get:

$$\frac{dy}{dx} = 1 = 0 + A$$

so that $A = 1$ and the solution in this case is $y = x^3 + x + 2$.

3. Classifying differential equations

When solving differential equations (either analytically or numerically) it is important to be able to recognise the various kinds that can arise. We therefore need to introduce some terminology which will help us to distinguish one kind of differential equation from another.

- An **ordinary differential equation** (ODE) is any relation between a function of a **single variable** and its derivatives. (All differential equations studied in this workbook are ordinary.)
- The **order** of a differential equation is the order of the highest derivative in the equation.
- A differential equation is **linear** if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable or its derivatives.

Example 2

Classify the differential equations specifying the order and type (linear/non-linear)

(a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = x^2$

(b) $\dfrac{d^2x}{dt^2} = \left(\dfrac{dx}{dt}\right)^3 + 3x$

(c) $\dfrac{dx}{dt} - x = t^2$

(d) $\dfrac{dy}{dt} + \cos y = 0$

(e) $\dfrac{dy}{dt} + y^2 = 4$

Solution

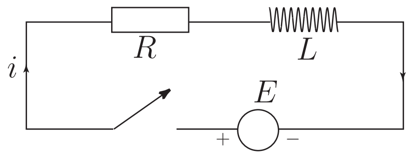
(a) Second order, linear.
(b) Second order, non-linear (because of the cubic term).
(c) First order, linear.
(d) First order, non-linear (because of the $\cos y$ term).
(e) First order, non-linear (because of the y^2 term).

Note that in (a) the independent variable is x whereas in the other cases it is t.

In (a), (d) and (e) the dependent variable is y and in (b) and (c) it is x.

Exercises

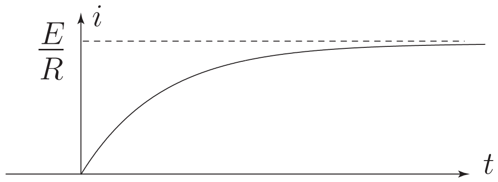
1. In this RL circuit the switch is closed at $t = 0$ and a constant voltage E is applied.



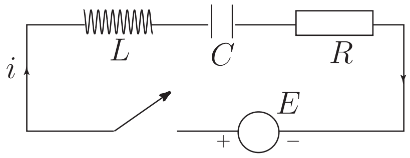
The voltage across the resistor is iR where i is the current flowing in the circuit and R is the (constant) resistance. The voltage across the inductance is $L\dfrac{di}{dt}$ where L is the constant inductance.

Kirchhoff's law of voltages states that the applied voltage is the sum of the other voltages in the circuit. Write down a differential equation for the current i and state the initial condition.

2. The diagram below shows the graph of i against t (from Exercise 1). What information does this graph convey?



3. In the LCR circuit below the voltage across the capacitor is q/C where q is the charge on the capacitor, and C is the capacitance. Note that $\dfrac{dq}{dt} = i$. Find a differential equation for i and write down the initial conditions if the initial charge is zero and the switch is closed at $t = 0$.



4. Find differential equations satisfied by

   (a) $y = A\cos 4x + B\sin 4x$

   (b) $x = A\,\mathrm{e}^{-2t}$

   (c) $y = A\sin x + B\sinh x + C\cos x + D\cosh x$ (harder)

5. Find the family of solutions of the differential equation $\dfrac{dy}{dx} = -2x$. Sketch the curves of some members of the family on the same axes. What is the solution if $y(1) = 3$?

6. (a) Find the general solution of the differential equation $y'' = 12x^2$.

   (b) Find the solution which satisfies $y(0) = 2,\ y(1) = 8$

   (c) Find the solution which satisfies $y(0) = 1,\ y'(0) = -2$.

7. Classify the differential equations

   (a) $\dfrac{d^2x}{dt^2}+3\,\dfrac{dx}{dt} = x$ (b) $\dfrac{d^3y}{dx^3} = \left(\dfrac{dy}{dx}\right)^2+\dfrac{dy}{dx}$ (c) $\dfrac{dy}{dx}+y = \sin x$ (d) $\dfrac{d^2y}{dx^2}+y\dfrac{dy}{dx} = 2.$

Answers

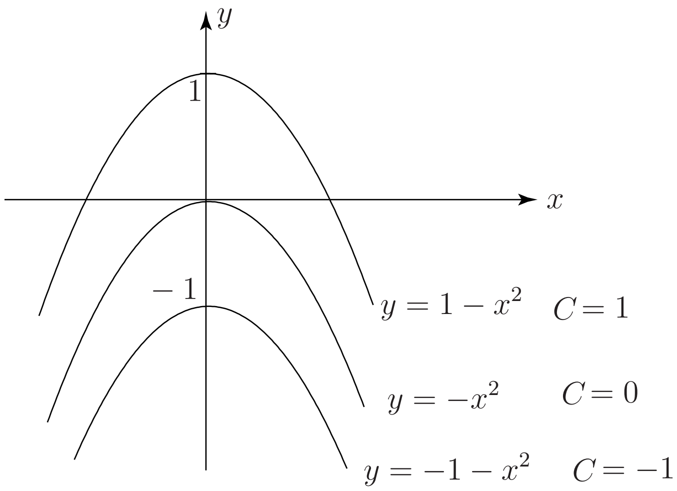
1. $L\,\dfrac{di}{dt}+R\,i=E$; $i=0$ at $t=0$.

2. Current increases rapidly at first, then less rapidly and tends to the value $\dfrac{E}{R}$ which is what it would be in the absence of L.

3. $L\,\dfrac{d^2q}{dt^2}+R\,\dfrac{dq}{dt}+\dfrac{q}{C}=E;\qquad q=0$ and $i=\dfrac{dq}{dt}=0$ at $t=0$.

4. (a) $\dfrac{d^2y}{dx^2}=-16y$ (b) $\dfrac{dx}{dt}=-2x$ (c) $\dfrac{d^4y}{dx^4}=y$

5. $y=-x^2+C$



If $3=-1+C$ then $C=4$ and $y=-x^2+4$.

6.

(a) $y=x^4+A\,x+B$

(b) When $x=0,\ y=2=B$; hence $B=2$. When $x=1,\ y=8=1+A+B=3+A$ hence $A=5$ and $y=x^4+5x+2$.

(c) When $x=0\ y=1=B$. Hence $B=1$; $\dfrac{dy}{dx}=y'=4x^3+A$, so at $x=0$, $y'=-2=A$.

Therefore $y=x^4-2x+1$

7. (a) Second order, linear

(b) Third order, non-linear *(squared term)*

(c) First order, linear

(d) Second order, non-linear *(product term)*