Errors and Percentage Change

18.4

Introduction

When one variable is related to several others by a functional relationship it is possible to estimate the percentage change in that variable caused by given percentage changes in the other variables. For example, if the values of the input variables of a function are measured and the measurements are in error, due to limits on the precision of measurement, then we can use partial differentiation to estimate the effect that these errors have on the forecast value of the output.

Prerequisites

Before starting this Section you should ...

- understand the definition of partial derivatives and be able to find them

Learning Outcomes

On completion you should be able to ...

- calculate small errors in a function of more than one variable
- calculate approximate values for absolute error, relative error and percentage relative error

1. Approximations using partial derivatives

Functions of two variables

We saw in HELM 16.5 how to expand a function of a single variable $f(x)$ in a Taylor series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

This can be written in the following alternative form (by replacing $x - x_0$ by h so that $x = x_0 + h$):

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots$$

This expansion can be generalised to functions of two or more variables:

$$f(x_0 + h,\ y_0 + k) \simeq f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0)$$

where, assuming h and k to be small, we have ignored higher-order terms involving powers of h and k. We define δf to be the change in $f(x, y)$ resulting from small changes to x_0 and y_0, denoted by h and k respectively. Thus:

$$\delta f = f(x_0 + h,\ y_0 + k) - f(x_0, y_0)$$

and so $\delta f \simeq hf_x(x_0, y_0) + kf_y(x_0, y_0)$. Using the notation δx and δy instead of h and k for small increments in x and y respectively we may write

$$\delta f \simeq \delta x . f_x(x_0, y_0) + \delta y . f_y(x_0, y_0)$$

Finally, using the more common notation for partial derivatives, we write

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$

Informally, the term δf is referred to as the **absolute error** in $f(x, y)$ resulting from errors $\delta x, \delta y$ in the variables x and y respectively. Other measures of error are used. For example, the **relative error** in a variable f is defined as $\dfrac{\delta f}{f}$ and the **percentage relative error** is $\dfrac{\delta f}{f} \times 100$.

Key Point 5

Measures of Error

If δf is the change in f at (x_0, y_0) resulting from small changes h, k to x_0 and y_0 respectively, then $\delta f = f(x_0 + h,\ y_0 + k) - f(x_0, y_0)$, and

The absolute error in f is δf.

The relative error in f is $\dfrac{\delta f}{f}$.

The percentage relative error in f is $\dfrac{\delta f}{f} \times 100$.

Note that to determine the error numerically we need to know not only the actual values of δx and δy but also the values of x and y at the point of interest.

Example 12

Estimate the absolute error for the function $f(x, y) = x^2 + x^3 y$

Solution

$f_x = 2x + 3x^2 y;\ \ f_y = x^3.$
Then $\delta f \simeq (2x + 3x^2 y)\delta x + x^3 \delta y$

Task

Estimate the absolute error for $f(x, y) = x^2 y^2 + x + y$ at the point $(-1, 2)$ if $\delta x = 0.1$ and $\delta y = 0.025$. Compare the estimate with the exact value of the error.

First find f_x and f_y:

Your solution

$f_x =$ $\qquad\qquad$ $f_y =$

Answer

$f_x = 2xy^2 + 1, \quad f_y = 2x^2 y + 1$

Now obtain an expression for the absolute error:

Your solution

Answer

$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2 y + 1)\delta y$

Now obtain the estimated value of the absolute error at the point of interest:

Your solution

Answer

$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2 y + 1)\delta y = (-7)(0.1) + (5)(0.025) = -0.575.$

Finally compare the estimate with the exact value:

Your solution

Answer
The actual error is calculated from

$$\delta f = f(x_0 + \delta x,\ y_0 + \delta y) - f(x_0, y_0) = f(-0.9, 2.025) - f(-1, 2) = -0.5534937.$$

We see that there is a reasonably close correspondence between the two values.

Functions of three or more variables

If f is a function of several variables $x, y, u, v, \ldots$ the error induced in f as a result of making small errors $\delta x, \delta y, \delta u, \delta v \ldots$ in $x, y, u, v, \ldots$ is found by a simple generalisation of the expression for two variables given above:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v + \ldots$$

Example 13

Suppose that the area of triangle ABC is to be calculated by measuring two sides and the included angle. Call the sides b and c and the angle A.
Then the area S of the triangle is given by $S = \frac{1}{2}bc\sin A.$

Now suppose that the side b is measured as 4.00 m, c as 3.00 m and A as 30^o. Suppose also that the measurements of the sides could be in error by as much as $\pm\ 0.005$ m and of the angle by $\pm\ 0.01^o$. Calculate the likely maximum error induced in S as a result of the errors in the sides and angle.

Solution

Here S is a function of three variables b, c, A. We calculate $\quad S = \frac{1}{2} \times 4 \times 3 \times \frac{1}{2} = 3$ m^2.

Now $\frac{\partial S}{\partial b} = \frac{1}{2}c\sin A$, $\quad \frac{\partial S}{\partial c} = \frac{1}{2}b\sin A$ and $\frac{\partial S}{\partial A} = \frac{1}{2}bc\cos A$, so

$$\delta S \simeq \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c + \frac{\partial S}{\partial A}\delta A = \frac{1}{2}c\sin A\ \delta b + \frac{1}{2}b\sin A\ \delta c + \frac{1}{2}bc\cos A\ \delta A.$$

Here $|\delta b|_{\max} = |\delta c|_{\max} = 0.005$ and $|\delta A|_{\max} = \frac{\pi}{180} \times 0.01$ (A must be measured in radians). Substituting these values we see that the maximum error in the calculated value of S is given by the approximation

$$\begin{aligned} |\delta S|_{\max} &\simeq \left(\frac{1}{2} \times 3 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times 3 \times \frac{\sqrt{3}}{2}\right)\frac{\pi}{180} \times 0.01 \\ &\simeq 0.0097 \text{ m}^2 \end{aligned}$$

Hence the estimated value of S is in error by up to about $\pm\ 0.01$ m^2.

Engineering Example 2

Measuring the height of a building

The height h of a building is estimated from (i) the known horizontal distance x between the point of observation M and the foot of the building and (ii) the elevation angle θ between the horizontal and the line joining the point of observation to the top of the building (see Figure 10). If the measured horizontal distance is $x = 150$ m and the elevation angle is $\theta = 40^\circ$, estimate the error in measured building height due to an error of 0.1° degree in the measurement of the angle of elevation.

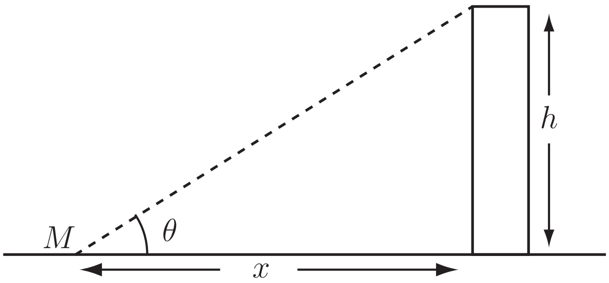

Figure 10: Geometry of the measurement

The variables $x,\ \theta$, and h are related by

$$\tan\theta = h/x.$$

or

$$x\tan\theta = h. \qquad (1)$$

The error in h resulting from a measurement error in θ can be deduced by differentiating (1):

$$\frac{d(x\tan\theta)}{d\theta} = \frac{dh}{d\theta} \qquad \Rightarrow \qquad \tan\theta\,\frac{dx}{d\theta} + x\,\frac{d(\tan\theta)}{d\theta} = \frac{dh}{d\theta}.$$

This can be written

$$\tan\theta\,\frac{dx}{d\theta} + x\sec^2\theta = \frac{dh}{d\theta}. \qquad (2)$$

Equation (2) gives the relationship among the small variations in variables x, h and θ. Since x is assumed to be without error and independent of θ, $\dfrac{dx}{d\theta} = 0$ and equation (2) becomes

$$x\sec^2\theta = \frac{dh}{d\theta}. \qquad (3)$$

Equation (3) can be considered to relate the error in building height δh to the error in angle $\delta\theta$:

$$\frac{\delta h}{\delta\theta} \simeq x\sec^2\theta.$$

It is given that $x = 150$ m.

The incidence angle $\theta = 40^\circ$ can be converted to radians i.e. $\theta = 40\pi/180$ rad $= 2\pi/9$ rad.

Then the error in angle $\delta\theta = 0.1^\circ$ needs to be expressed in radians for consistency of the units in (3).

So $\delta\theta = 0.1\pi/180$ rad $= \pi/1800$ rad. Hence, from Equation (3)

$$\delta h = 150\frac{\pi}{1800 \times \cos^2(2\pi/9)} \approx 0.45 \text{ m.}$$

So the error in building height resulting from an error in elevation angle of $0.1°$ is about 0.45 m.

Task

Estimate the maximum error in $f(x, y) = x^2 + y^2 + xy$ at the point $x = 2,\ y = 3$ if maximum errors $\pm\ 0.01$ and $\pm\ 0.02$ are made in x and y respectively.

First find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial y} =$

Answer

$\dfrac{\partial f}{\partial x} = 2x + y; \quad \dfrac{\partial f}{\partial y} = 2y + x.$

For $x = 2$ and $y = 3$ calculate the value of $f(x, y)$:

Your solution

Answer

$f(2, 3) = 2^2 + 3^2 + 2 \times 3 = 19.$

Now since the error in the measured value of x is $\pm\ 0.01$ and in y is $\pm\ 0.02$ we have $|\delta x|_{\max} = 0.01, \quad |\delta y|_{\max} = 0.02$. Write down an expression to approximate to $|\delta f|_{\max}$:

Your solution

Answer

$|\delta f|_{\max} \simeq |(2x + y)|\ |\delta x|_{\max} + |(2y + x)|\ |\delta y|_{\max}$

Calculate $|\delta f|_{\max}$ at the point $x = 2,\ y = 3$ and give bounds for $f(2, 3)$:

Your solution

Answer

$$\begin{aligned} |\delta f|_{\max} &\simeq (2 \times 2 + 3) \times 0.01 + (2 \times 3 + 2) \times 0.02 \\ &= 0.07 + 0.16 = 0.23. \end{aligned}$$

Hence we quote $f(2, 3) = 19 \pm\ 0.23$, which can be expressed as $18.77 \leq f(2, 3) \leq 19.23$

2. Relative error and percentage relative error

Two other measures of error can be obtained from a knowledge of the expression for the absolute error. As mentioned earlier, the relative error in f is $\dfrac{\delta f}{f}$ and the percentage relative error is $\left(\dfrac{\delta f}{f} \times 100\right)\%$. Suppose that $f(x,y) = x^2 + y^2 + xy$ then

$$\begin{aligned} \delta f &\simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \\ &= (2x+y)\delta x + (2y+x)\delta y \end{aligned}$$

The relative error is

$$\begin{aligned} \frac{\delta f}{f} &\simeq \frac{1}{f}\frac{\partial f}{\partial x}\delta x + \frac{1}{f}\frac{\partial f}{\partial y}\delta y \\ &= \frac{(2x+y)}{x^2+y^2+xy}\delta x + \frac{(2y+x)}{x^2+y^2+xy}\delta y \end{aligned}$$

The actual value of the relative error can be obtained if the actual errors of the independent variables are known and the values of x and y at the point of interest. In the special case where the function is a combination of powers of the input variables then there is a short cut to finding the relative error in the value of the function. For example, if $f(x,y,u) = \dfrac{x^2y^4}{u^3}$ then

$$\frac{\partial f}{\partial x} = \frac{2xy^4}{u^3}, \qquad \frac{\partial f}{\partial y} = \frac{4x^2y^3}{u^3}, \qquad \frac{\partial f}{\partial u} = -\frac{3x^2y^4}{u^4}$$

Hence

$$\delta f \simeq \frac{2xy^4}{u^3}\delta x + \frac{4x^2y^3}{u^3}\delta y - \frac{3x^2y^4}{u^4}\delta u$$

Finally,

$$\frac{\delta f}{f} \simeq \frac{2xy^4}{u^3} \times \frac{u^3}{x^2y^4}\delta x + \frac{4x^2y^3}{u^3} \times \frac{u^3}{x^2y^4}\delta y - \frac{3x^2y^4}{u^4} \times \frac{u^3}{x^2y^4}\delta u$$

Cancelling down the fractions,

$$\frac{\delta f}{f} \simeq 2\frac{\delta x}{x} + 4\frac{\delta y}{y} - 3\frac{\delta u}{u} \qquad (1)$$

so that

rel. error in $f \simeq 2\times$ (rel. error in x) $+ 4\times$ (rel. error in y) $- 3\times$ (rel. error in u).

Note that if we write

$$f(x,y,u) = x^2y^4u^{-3}$$

we see that the coefficients of the relative errors on the right-hand side of (1) are the powers of the appropriate variable.

To find the percentage relative error we simply multiply the relative error by 100.

Task

If $f = \dfrac{x^3 y}{u^2}$ and x, y, u are subject to percentage relative errors of $1\%,\ -1\%$ and 2% respectively find the approximate percentage relative error in f.

First find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial u}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial y} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial u} =$

Answer

$\dfrac{\partial f}{\partial x} = \dfrac{3x^2 y}{u^2}, \quad \dfrac{\partial f}{\partial y} = \dfrac{x^3}{u^2}, \quad \dfrac{\partial f}{\partial u} = -\dfrac{2x^3 y}{u^3}.$

Now write down an expression for δf

Your solution

$\delta f \simeq$

Answer

$\delta f \simeq \dfrac{3x^2 y}{u^2}\delta x + \dfrac{x^3}{u^2}\delta y - \dfrac{2x^3 y}{u^3}\delta u$

Hence write down an expression for the percentage relative error in f:

Your solution

Answer

$\dfrac{\delta f}{f} \times 100 \simeq \dfrac{3x^2 y}{u^2} \times \dfrac{u^2}{x^3 y}\delta x \times 100 + \dfrac{x^3}{u^2} \times \dfrac{u^2}{x^3 y}\delta y \times 100 - \dfrac{2x^3 y}{u^3} \times \dfrac{u^2}{x^3 y}\delta u \times 100$

Finally, calculate the value of the percentage relative error:

Your solution

Answer

$$\begin{aligned} \frac{\delta f}{f} \times 100 &\simeq 3\frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 - 2\frac{\delta u}{u} \times 100 \\ &= 3(1) - 1 - 2(2) = -2\% \end{aligned}$$

Note that $f = x^3 y u^{-2}$.

Engineering Example 3

Error in power to a load resistance

Introduction

The power required by an electrical circuit depends upon its components. However, the specification of the rating of the individual components is subject to some uncertainity. This Example concerns the calculation of the error in the power required by a circuit shown in Figure 11 given a formula for the power, the values of the individual components and the percentage errors in them.

Problem in words

The power delivered to the load resistance R_L for the circuit shown in Figure 11 is given by

$$P = \frac{25R_L}{(R+R_L)^2}$$

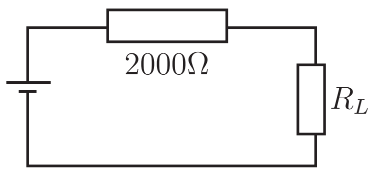

Figure 11: Circuit with a load resistance

If $R = 2000\ \Omega$ and $R_L = 1000\ \Omega$ with a maximum possible error of 5% in either, find P and estimate the maximum error in P.

Mathematical statement of the problem

We can calculate P by substituting $R = 2000$ and $R_L = 1000$ into $P = \dfrac{25R_L}{(R+R_L)^2}$.

We need to calculate the absolute errors in R and R_L and use these in the approximation $\delta P \approx \dfrac{P}{R}\delta R + \dfrac{P}{R_L}\delta R_L$ to calculate the error in P.

Mathematical analysis

At $R = 2000$ and $R_L = 1000$

$$P = \frac{25 \times 1000}{(1000+2000)^2} = \frac{25}{9000} = \frac{25}{9} \times 10^{-3} \approx 2.77 \times 10^{-3} \text{ watts.}$$

A 5% error in R gives $|\delta R|_{\max} = \dfrac{5}{100} \times 2000 = 100$ and $|\delta R_L|_{\max} = \dfrac{5}{100} \times 1000 = 50$

$$|\delta P|_{\max} \approx \frac{P}{R}\ |\delta R|_{\max} + \frac{P}{R_L}\ |\delta R_L|_{\max}$$

We need to calculate the values of the partial derivatives at $R = 2000$ and $R_L = 1000$.

$$P = \frac{25R_L}{(R+R_L)^2} = 25R_L(R+R_L)^{-2}$$

$$\frac{P}{R} = -50R_L(R+R_L)^{-3}$$

$$\frac{P}{R_L} = 25(R+R_L)^{-2} - 50R_L(R+R_L)^{-3}$$

So $\dfrac{P}{R}(2000, 1000) = -50(1000)(3000)^{-3} = \dfrac{-50}{1000^2 \times 27} = -\dfrac{50}{27} \times 10^{-6}$

$$\frac{P}{R_L}(2000, 1000) = 25(3000)^{-2} - 50(1000)(3000)^{-3} = \left(\frac{25}{9} - \frac{50}{27}\right) \times 10^{-6}$$

$$= \left(\frac{75-50}{27}\right) \times 10^{-6} = \frac{25}{27} \times 10^{-6}$$

Substituting these values into $|\delta P|_{\max} \approx \dfrac{P}{R}\ |\delta R|_{\max} + \dfrac{P}{R_L}\,|\delta R_L|_{\max}$ we get:

$$|\delta P|_{\max} = \frac{50}{27} \times 10^{-6} \times 100 + \frac{25}{27} \times 10^{-6} \times 50 = \left(\frac{5000}{27} + \frac{25 \times 50}{27}\right) \times 10^{-6} \approx 2.315 \times 10^{-4}$$

Interpretation

At $R = 2000$ and $R_L = 1000$, P will be 2.77×10^{-3} W and, assuming 5$\%$ errors in the values of the resistors, then the error in $P \approx \pm 2.315 \times 10^{-4}$ W. This represents about 8.4% error. So the error in the power is greater than that in the individual components.

Exercises

1. The sides of a right-angled triangle enclosing the right-angle are measured as 6 m and 8 m. The maximum errors in each measurement are $\pm$ 0.1m. Find the maximum error in the calculated area.

2. In Exercise 1, the angle opposite the 8 m side is calculated from $\tan\theta = 8/6$ as $\theta = 53°8'$. Calculate the approximate maximum error in that angle.

3. If $v = \sqrt{\dfrac{3x}{y}}$ find the maximum percentage error in v due to errors of 1% in x and 3% in y.

4. If $n = \dfrac{1}{2L}\sqrt{\dfrac{E}{d}}$ and L, E and d can be measured correct to within 1%, how accurate is the calculated value of n?

5. The area of a segment of a circle which subtends an angle θ is given by $A = \dfrac{1}{2}r^2(\theta - \sin\theta)$. The radius r is measured with a percentage error of $+0.2\%$ and θ is measured as 45^0 with an error of $= +0.1°$. Find the percentage error in the calculated area.

Answers

1. $A = \frac{1}{2}xy \qquad \delta A \approx \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y \qquad \delta A \approx \frac{y}{2}\delta x + \frac{x}{2}\delta y$

   Maximum error $= |y\delta x| + |x\delta y| = 0.7$ m^2.

2. $\theta = \tan^{-1}\frac{y}{x}$ so $\delta\theta = \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y = -\frac{y}{x^2+y^2}\delta x + \frac{x}{x^2+y^2}\delta y$

   Maximum error in θ is $\left|\frac{-8}{6^2+8^2}(0.1)\right| + \left|\frac{6}{6^2+8^2}(0.1)\right| = 0.014$ rad. This is 0.8^0.

3. Take logarithms of both sides: $\ln v = \frac{1}{2}\ln 3 + \frac{1}{2}\ln x - \frac{1}{2}\ln y$ so $\frac{\delta v}{v} \approx \frac{\delta x}{2x} - \frac{\delta y}{2y}$

   Maximum percentage error in $v = \left|\frac{\delta x}{2x}\right| + \left|-\frac{\delta y}{2y}\right| = \frac{1}{2}\% + \frac{3}{2}\% = 2\%$.

4. Take logarithms of both sides:

   $$\ln n = -\ln 2 - \ln L + \frac{1}{2}\ln E - \frac{1}{2}\ln d \qquad \text{so} \qquad \frac{\delta n}{n} = -\frac{\delta L}{L} + \frac{\delta E}{2E} - \frac{\delta d}{2d}$$

   Maximum percentage error in $n = \left|-\frac{\delta L}{L}\right| + \left|\frac{\delta E}{2E}\right| + \left|-\frac{\delta d}{2d}\right| = 1\% + \frac{1}{2}\% + \frac{1}{2}\% = 2\%$.

5. $A = \frac{1}{2}r^2(\theta - \sin\theta)$ so $\frac{\delta A}{A} = \frac{2\delta r}{r} + \frac{1-\cos\theta}{\theta - \sin\theta}\delta\theta$

   $$\frac{\delta A}{A} = 2(0.2)\% + \left\{\frac{1 - \frac{1}{\sqrt{2}}}{\frac{\pi}{4} - \frac{1}{\sqrt{2}}}\right\}\frac{\pi}{1800} \times 100\% = (0.4 + 0.65)\% = 1.05\%$$