

Stationary Points

18.3



Introduction

The calculation of the optimum value of a function of two variables is a common requirement in many areas of engineering, for example in thermodynamics. Unlike the case of a function of one variable we have to use more complicated criteria to distinguish between the various types of stationary point.



Prerequisites

Before starting this Section you should ...

- understand the idea of a function of two variables
- be able to work out partial derivatives



Learning Outcomes

On completion you should be able to ...

- identify local maximum points, local minimum points and saddle points on the surface $z = f(x, y)$
- use first partial derivatives to locate the stationary points of a function $f(x, y)$
- use second partial derivatives to determine the nature of a stationary point

1. The stationary points of a function of two variables

Figure 7 shows a computer generated picture of the surface defined by the function $z = x^3 + y^3 - 3x - 3y$, where both x and y take values in the interval $[-1.8, 1.8]$.

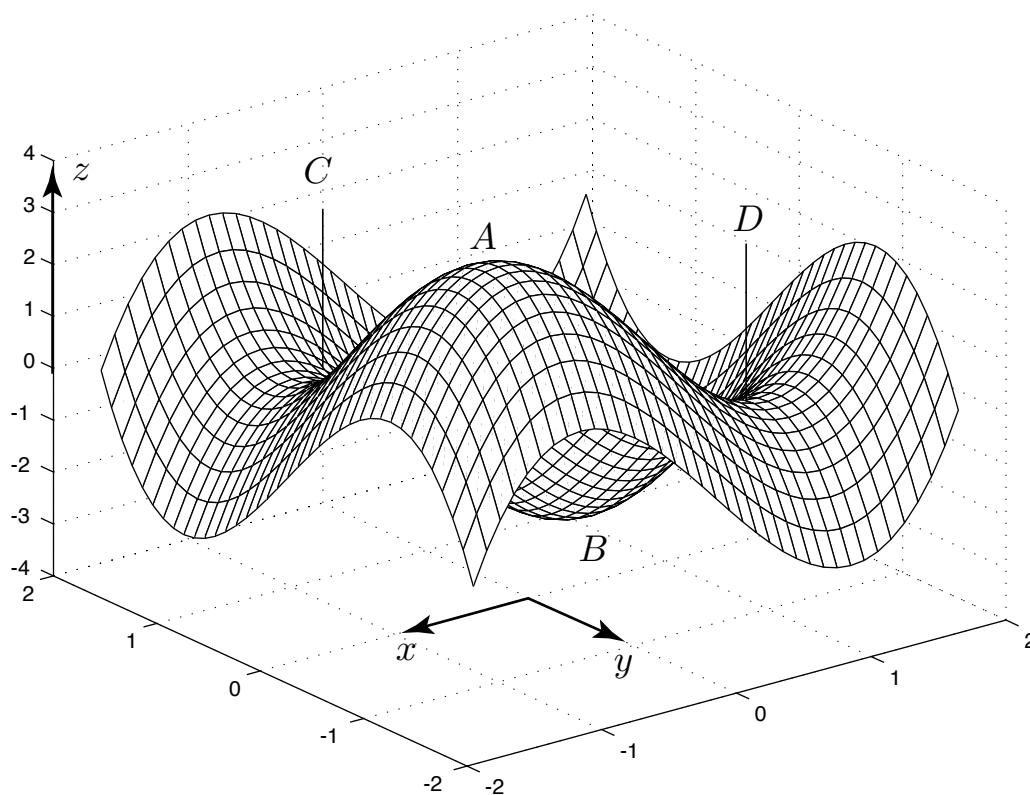


Figure 7

There are four features of particular interest on the surface. At point A there is a **local maximum**, at B there is a **local minimum**, and at C and D there are what are known as **saddle points**.

At A the surface is at its greatest height in the immediate neighbourhood. If we move on the surface from A we immediately lose height no matter in which direction we travel. At B the surface is at its least height in the neighbourhood. If we move on the surface from B we immediately gain height, no matter in which direction we travel.

The features at C and D are quite different. In some directions as we move away from these points along the surface we lose height whilst in others we gain height. The similarity in shape to a horse's saddle is evident.

At each point P of a *smooth* surface one can draw a unique plane which touches the surface there. This plane is called the **tangent plane** at P . (The tangent plane is a natural generalisation of the tangent line which can be drawn at each point of a smooth curve.) In Figure 7 at each of the points A, B, C, D the tangent plane to the surface is horizontal at the point of interest. Such points are thus known as **stationary points** of the function. In the next subsections we show how to locate stationary points and how to determine their nature using partial differentiation of the function $f(x, y)$,



In Figures 8 and 9 what are the features at A and B ?

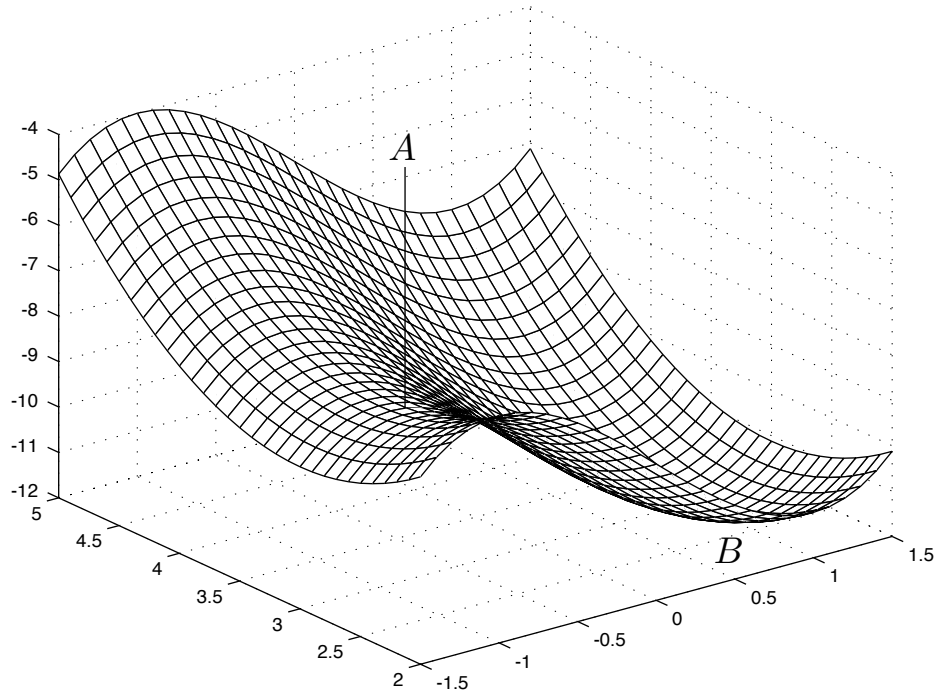


Figure 8

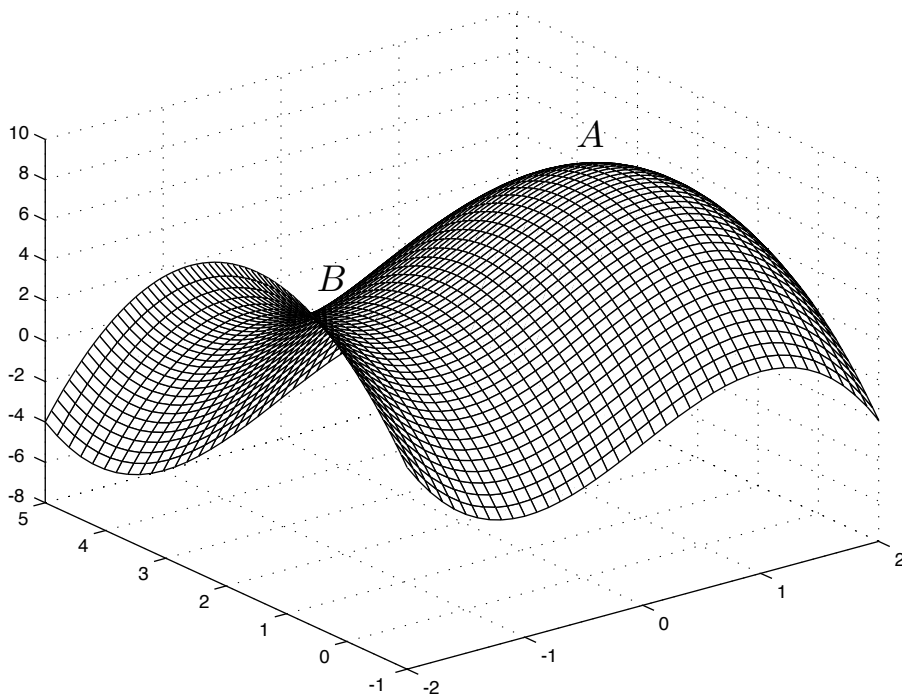


Figure 9

Your solution

AnswerFigure 8 A is a saddle point, B is a local minimum.Figure 9 A is a local maximum, B is a saddle point.

2. Location of stationary points

As we said in the previous subsection, the tangent plane to the surface $z = f(x, y)$ is horizontal at a stationary point. A condition which guarantees that the function $f(x, y)$ will have a stationary point at a point (x_0, y_0) is that, at that point both $f_x = 0$ and $f_y = 0$ simultaneously.



Verify that $(0, 2)$ is a stationary point of the function $f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$ and find the stationary value $f(0, 2)$.

First, find f_x and f_y :

Your solution**Answer**

$$f_x = 16x \quad ; \quad f_y = 12y - 6y^2$$

Now find the values of these partial derivatives at $x = 0$, $y = 2$:

Your solution**Answer**

$$f_x = 0 \quad , \quad f_y = 24 - 24 = 0$$

Hence $(0, 2)$ is a stationary point.

The **stationary value** is $f(0, 2) = 0 + 24 - 16 + 5 = 13$

**Example 9**

Find a second stationary point of $f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$.

Solution

$f_x = 16x$ and $f_y \equiv 6y(2 - y)$. From this we note that $f_x = 0$ when $x = 0$, and $f_y = 0$ and when $y = 0$, so $x = 0$, $y = 0$ i.e. $(0, 0)$ is a second stationary point of the function.

It is important when solving the simultaneous equations $f_x = 0$ and $f_y = 0$ to find stationary points not to miss any solutions. A useful tip is to factorise the left-hand sides and consider systematically all the possibilities.

**Example 10**

Locate the stationary points of

$$f(x, y) = x^4 + y^4 - 36xy$$

SolutionFirst we write down the partial derivatives of $f(x, y)$

$$\frac{\partial f}{\partial x} = 4x^3 - 36y = 4(x^3 - 9y) \quad \frac{\partial f}{\partial y} = 4y^3 - 36x = 4(y^3 - 9x)$$

Now we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$:

$$x^3 - 9y = 0 \quad \text{(i)}$$

$$y^3 - 9x = 0 \quad \text{(ii)}$$

$$\text{From (ii) we obtain: } x = \frac{y^3}{9} \quad \text{(iii)}$$

Now substitute from (iii) into (i)

$$\begin{aligned} & \frac{y^9}{9^3} - 9y = 0 \\ \Rightarrow & y^9 - 9^4y = 0 \\ \Rightarrow & y(y^8 - 3^8) = 0 \quad \text{(removing the common factor)} \\ \Rightarrow & y(y^4 - 3^4)(y^4 + 3^4) = 0 \quad \text{(using the difference of two squares)} \end{aligned}$$

We therefore obtain, as the only solutions:

$$y = 0 \text{ or } y^4 - 3^4 = 0 \quad \text{(since } y^4 + 3^4 \text{ is never zero)}$$

The last equation implies:

$$\begin{aligned} (y^2 - 9)(y^2 + 9) &= 0 \quad \text{(using the difference of two squares)} \\ \therefore y^2 &= 9 \text{ and } y = \pm 3. \end{aligned}$$

Now, using (iii): when $y = 0$, $x = 0$, when $y = 3$, $x = 3$, and when $y = -3$, $x = -3$.The stationary points are $(0, 0)$, $(-3, -3)$ and $(3, 3)$.

Locate the stationary points of

$$f(x, y) = x^3 + y^2 - 3x - 6y - 1.$$

First find the partial derivatives of $f(x, y)$:**Your solution**

Answer

$$\frac{\partial f}{\partial x} = 3x^2 - 3, \quad \frac{\partial f}{\partial y} = 2y - 6$$

Now solve simultaneously the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$:

Your solution**Answer**

$$3x^2 - 3 = 0 \text{ and } 2y - 6 = 0.$$

Hence $x^2 = 1$ and $y = 3$, giving stationary points at $(1, 3)$ and $(-1, 3)$.

3. The nature of a stationary point

We state, without proof, a relatively simple test to determine the nature of a stationary point, once located. If the surface is very flat near the stationary point then the test will not be sensitive enough to determine the nature of the point. The test is dependent upon the values of the second order derivatives: f_{xx} , f_{yy} , f_{xy} and also upon a combination of second order derivatives denoted by D where

$$D \equiv \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2, \text{ which is also expressible as } D \equiv f_{xx}f_{yy} - (f_{xy})^2$$

The test is as follows:



Key Point 4

Test to Determine the Nature of Stationary Points

1. At each stationary point work out the three second order partial derivatives.
2. Calculate the value of $D = f_{xx}f_{yy} - (f_{xy})^2$ at each stationary point.
Then, test each stationary point in turn:
3. If $D < 0$ the stationary point is a **saddle point**.

If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ the stationary point is a **local minimum**.

If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ the stationary point is a **local maximum**.

If $D = 0$ then the test is inconclusive (we need an alternative test).

**Example 11**

The function: $f(x, y) = x^4 + y^4 - 36xy$ has stationary points at $(0, 0)$, $(-3, -3)$, $(3, 3)$. Use Key Point 4 to determine the nature of each stationary point.

Solution

We have $\frac{\partial f}{\partial x} = f_x = 4x^3 - 36y$ and $\frac{\partial f}{\partial y} = f_y = 4y^3 - 36x$.

Then $\frac{\partial^2 f}{\partial x^2} = f_{xx} = 12x^2$, $\frac{\partial^2 f}{\partial y^2} = f_{yy} = 12y^2$, $\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = -36$.

A tabular presentation is useful for calculating $D = f_{xx}f_{yy} - (f_{xy})^2$:

	Point	Point	Point
Derivatives	$(0, 0)$	$(-3, -3)$	$(3, 3)$
f_{xx}	0	108	108
f_{yy}	0	108	108
f_{xy}	-36	-36	-36
D	< 0	> 0	> 0

$(0, 0)$ is a saddle point; $(-3, -3)$ and $(3, 3)$ are both local minima.



Determine the nature of the stationary points of $f(x, y) = x^3 + y^2 - 3x - 6y - 1$, which are $(1, 3)$ and $(1, -3)$.

Write down the three second partial derivatives:

Your solution

Answer

$$f_{xx} = 6x, \quad f_{yy} = 2, \quad f_{xy} = 0.$$

Now complete the table below and determine the nature of the stationary points:

Your solution

	Point	Point
Derivatives	(1, 3)	(-1, 3)
f_{xx}		
f_{yy}		
f_{xy}		
D		

Answer

	Point	Point
Derivatives	(1, 3)	(-1, 3)
f_{xx}	6	-6
f_{yy}	2	2
f_{xy}	0	0
D	> 0	< 0

State the nature of each stationary point:

Your solution

Answer

(1, 3) is a local minimum; (-1, 3) is a saddle point.

For most functions the procedures described above enable us to distinguish between the various types of stationary point. However, note the following example, in which these procedures fail.

Given $f(x, y) = x^4 + y^4 + 2x^2y^2$.

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy^2, \quad \frac{\partial f}{\partial y} = 4y^3 + 4x^2y,$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 4y^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2 + 4x^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 8xy$$

Location: The stationary points are located where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, that is, where

$4x^3 + 4xy^2 = 0$ and $4y^3 + 4x^2y = 0$. A simple factorisation implies $4x(x^2 + y^2) = 0$ and $4y(y^2 + x^2) = 0$. The only solution which satisfies both equations is $x = y = 0$ and therefore the only stationary point is $(0, 0)$.

Nature: Unfortunately, all the second partial derivatives are zero at $(0, 0)$ and therefore $D = 0$, so the test, as described in Key Point 4, fails to give us the necessary information.

However, in this example it is easy to see that the stationary point is in fact a local minimum. This could be confirmed by using a computer generated graph of the surface near the point $(0, 0)$. Alternatively, we observe $x^4 + y^4 + 2x^2y^2 \equiv (x^2 + y^2)^2$ so $f(x, y) \geq 0$, the only point where $f(x, y) = 0$ being the stationary point. This is therefore a local (and global) minimum.

Exercises

Determine the nature of the stationary points of the function in each case:

- $f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$
- $f(x, y) = x^3 + 15x^2 - 20y^2 + 10$
- $f(x, y) = 4 - x^2 - xy - y^2$
- $f(x, y) = 2x^2 + y^2 + 3xy - 3y - 5x + 8$
- $f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2) + 1$
- $f(x, y) = x^4 + y^4 + 2x^2y^2 + 2x^2 + 2y^2 + 1$

Answers

- $(0, 0)$ local minimum, $(0, 2)$ saddle point.
- $(0, 0)$ saddle point, $(-10, 0)$ local maximum.
- $(0, 0)$ local maximum.
- $(-1, 3)$ saddle point.
- $(0, 0)$ saddle point, $(1, 0)$ local minimum, $(-1, 0)$ local minimum.
- $f(x, y) \equiv (x^2 + y^2 + 1)^2$, local minimum at $(0, 0)$.