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Functions of Several Variables

Learning outcomes

In this Workbook you will learn about functions of two or more variables. You will learn that a function of two variables can be interpreted as a surface. You will learn how to sketch simple surfaces. You will learn what a partial derivative is and how the partial derivative of any order may be found. As an important application of partial differentiation you will learn how to locate the turning points of functions of several variables. In particular, for functions of two variables, you will learn how to distinguish between maxima and minima points. Finally you will apply your knowledge to the topic of error analysis.

Functions of Several Variables

18.1

Introduction

A function of a single variable $y = f(x)$ is interpreted graphically as a planar curve. In this Section we generalise the concept to functions of more than one variable. We shall see that a function of two variables $z = f(x, y)$ can be interpreted as a surface. Functions of two or more variables often arise in engineering and in science and it is important to be able to deal with such functions with confidence and skill. We see in this Section how to sketch simple surfaces. In later Sections we shall examine how to determine the rate of change of $f(x, y)$ with respect to x and y and also how to obtain the optimum values of functions of several variables.

Prerequisites

Before starting this Section you should . . .

- understand the Cartesian coordinates (x, y, z) of three-dimensional space.
- be able to sketch simple 2D curves

Learning Outcomes

On completion you should be able to . . .

- understand the mathematical description of a surface
- sketch simple surfaces
- use the notation for a function of several variables

1. Functions of several variables

We know that $f(x)$ is used to represent a function of one variable: the input variable is x and the output is the value $f(x)$. Here x is the **independent** variable and $y = f(x)$ is the **dependent variable**.

Suppose we consider a function with **two independent** input variables x and y, for example

$$f(x, y) = x + 2y + 3.$$

If we specify values for x and y then we have a **single** value $f(x, y)$. For example, if $x = 3$ and $y = 1$ then $f(x, y) = 3 + 2 + 3 = 8$. We write $f(3, 1) = 8$.

Task

Find the values of $f(2, 1)$, $f(-1, -3)$ and $f(0, 0)$ for the following functions.

(a) $f(x, y) = x^2 + y^2 + 1$ (b) $f(x, y) = 2x + xy + y^3$

Your solution

Answer

(a) $f(2, 1) = 2^2 + 1^2 + 1 = 6$; $f(-1, -3) = (-1)^2 + (-3)^2 + 1 = 11$; $f(0, 0) = 1$

(b) $f(2, 1) = 4 + 2 + 1 = 7$; $f(-1, -3) = -2 + 3 - 27 = -26$; $f(0, 0) = 0$

In a similar way we can define a function of three independent variables. Let these variables be x, y and u and the function $f(x, y, u)$.

Example 1

Given $f(x, y, u) = x^2 + yu + 2$, find $f(0, 1, 0)$, $f(-1, -1, 2)$.

Solution

$$f(0, 1, 0) = 0^2 + 1 \times 0 + 2 = 2; \quad f(-1, -1, 2) = 1 - 2 + 2 = 1$$

Task

(a) Find $f(2, -1, 1)$ for $f(x, y, u) = xy + yu + ux$.

(b) Evaluate $f(x, y, u, t) = x^2 - y^2 - u^2 - 2t$ when $x = 1$, $y = -2$, $u = 3$, $t = 1$.

Your solution

Answer

(a) $f(2,-1,1) = 2 \times (-1) + (-1) \times 1 + 1 \times 2 = -1$

(b) $f(1,-2,3,1) = 1^2 - (-2)^2 - 3^2 - 2 \times 1 = -14$ (this is a function of 4 independent variables).

2. Functions of two variables

The aim of this Section is to enable the reader to gain confidence in dealing with functions of several variables. In order to do this we often concentrate on functions of just two variables. The latter have an easy geometrical interpretation and we can therefore use our geometrical intuition to help understand the meaning of much of the mathematics associated with such functions. We begin by reminding the reader of the Cartesian coordinate system used to locate points in three dimensions. A point P is located by specifying its Cartesian coordinates (a, b, c) defined in Figure 1.

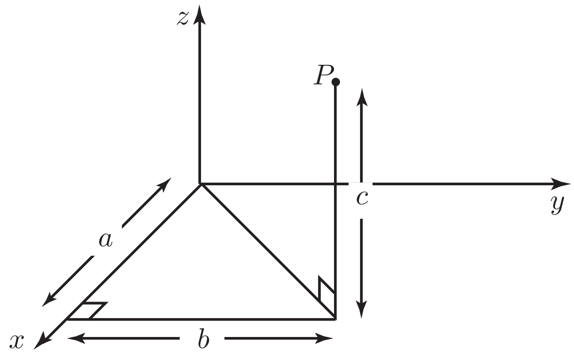


Figure 1

Within this 3-dimensional space we can consider simple surfaces. Perhaps the simplest is the **plane**. From HELM 9.6 on vectors we recall the general equation of a plane:

$$Ax + By + Cz = D$$

where $A,\ B,\ C,\ D$ are constants. This plane intersects the $x-$axis (where $y = z = 0$) at the point $\left(\dfrac{D}{A},\ 0,\ 0\right)$, intersects the $y-$axis (where $x = z = 0$) at the point $\left(0,\ \dfrac{D}{B}, 0\right)$ and the $z-$axis (where $x = y = 0$) at the point $\left(0,\ 0,\ \dfrac{D}{C}\right)$. See Figure 2 where the dotted lines are hidden from view behind the plane which passes through three points marked on the axes.

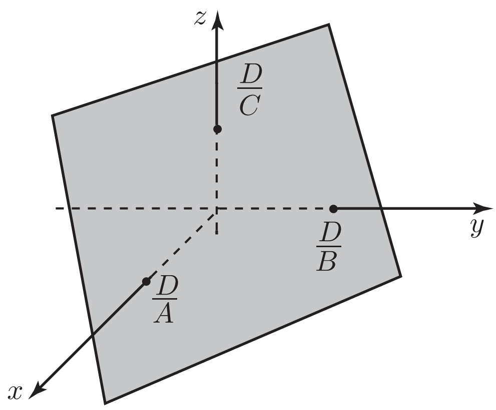


Figure 2

There are some special cases of note.

- $B = C = 0 \quad A \neq 0.$

  Here the plane is $x = D/A$. This plane (for any given values of D and A) is parallel to the zy plane a distance D/A units from it. See Figure 3a.

- $A = 0,\ C = 0 \quad B \neq 0$

  Here the plane is $y = D/B$ and is parallel to the zx plane at a distance D/B units from it. See Figure 3b.

- $A = 0,\ B = 0 \quad C \neq 0$

  Here the plane is $z = D/C$ which is parallel to the xy plane a distance D/C units from it. See Figure 3c.

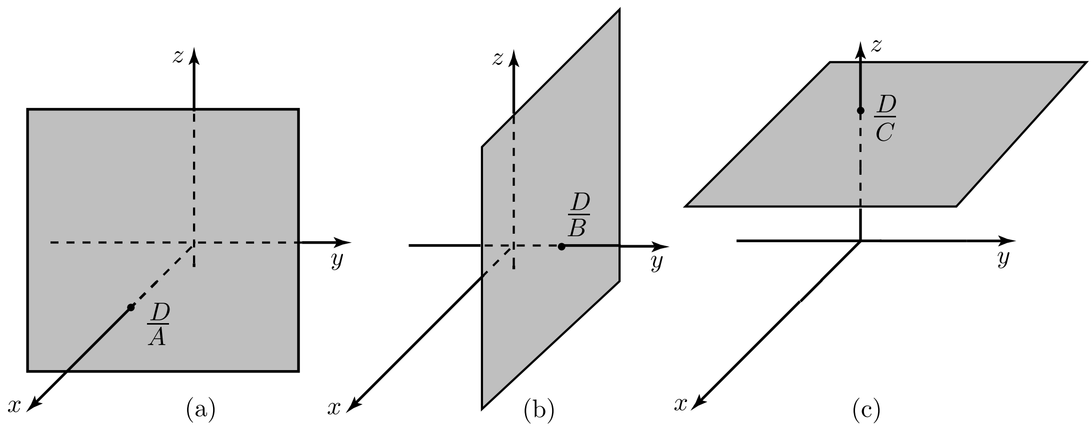


Figure 3

Planes are particularly simple examples of **surfaces**. Generally, a surface is described by a relation connecting the three variables x, y, z. In the case of the plane this relation is linear $Ax+By+Cz = D$. In some cases, as we have seen, one or two variables may be absent from the relation. In three dimensions such a relation **still** defines a surface, for example $z = 0$ defines the plane of the x- and y-axes.

Although **any** relation connecting x, y, z defines a surface, by convention, one of the variables (usually z) is chosen as the dependent variable and the other two therefore are independent variables. For the case of a plane $Ax + By + Cz = D$ (and $C \neq 0$) we would write, for example,

$$z = \frac{1}{C}(D - Ax - By)$$

Generally a surface is defined by a relation of the form

$$z = f(x, y)$$

where the expression on the right is any relation involving two variables x, y.

Sketching surfaces

A plane is relatively easy to sketch since it is flat all we need to know about it is where it intersects the three coordinate axes. For more general surfaces what we do is to sketch curves (like contours) which lie on the surface. If we draw enough of these curves our 'eye' will naturally interpret the shape of the surface.

Let us see, for example, how we sketch $z = x^2 + y^2$.
Firstly we confirm that $z = x^2+y^2$ **is a surface** since this is a relation connecting the three coordinate variables x, y, z. In the standard notation our function of two variables is $f(x, y) = x^2+y^2$. To sketch the surface we fix one of the variables at a constant value.

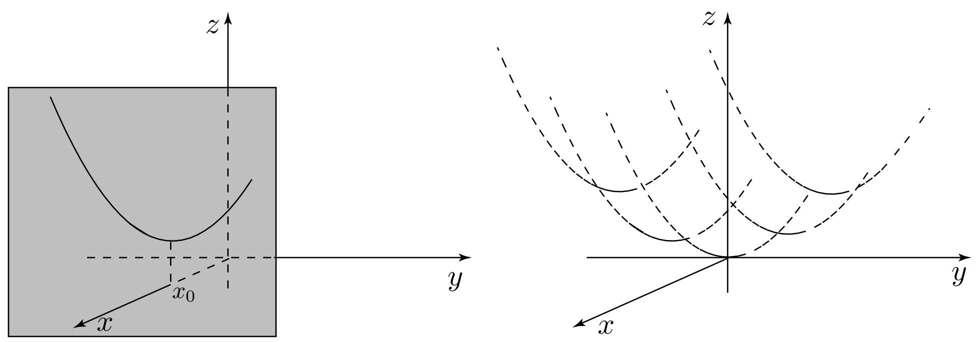
- Fix x at value x_0.

From our discussion above we remember that $x = x_0$ is the equation of a plane parallel to the zy plane. In this case our relation becomes:

$$z = x_0^2 + y^2$$

Since z is now a function of a **single** variable y, with x_0^2 held constant, this relation: $z = x_0^2 + y^2$ defines a **curve** which **lies in the plane** $x = x_0$.

In Figure 4(a) we have drawn this curve (a **parabola**). Now by changing the value chosen for x_0 we will obtain a sequence of curves, each a parabola, lying in a different plane, and each being a part of the surface we are trying to sketch. In Figure 4(b) we have drawn some of the curves of this sequence.



(a) (b)

Figure 4

What we have done is to **slice** the surface by planes parallel to the zy plane. Each slice intersects the surface in a curve. In this case we have not yet plotted enough curves to accurately visualise the surface so we need to draw other surface curves.

- Fix y at value y_0

Here $y = y_0$ (the equation of a plane parallel to the zx plane.) In this case the surface becomes

$$z = x^2 + y_0^2$$

Again z is a function of single variable (since y_0 is fixed) and describes a curve: again the curve is a **parabola**, but this time residing on the plane $y = y_0$. For each different y_0 we choose a different parabola is obtained: each lying on the surface $z = x^2+y^2$. Some of these curves have been sketched in Figure 5(a). These have then combined with the curves of Figure 4(b) to produce Figure 5(b).

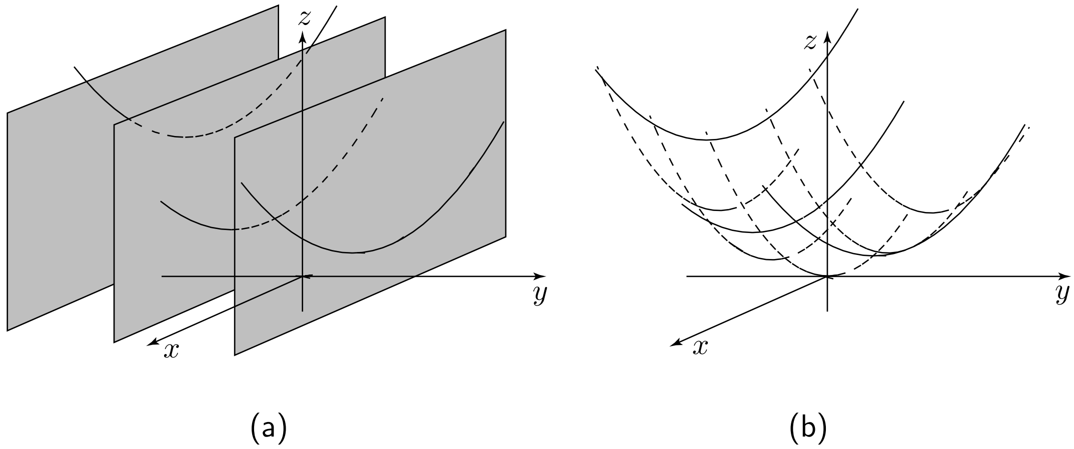


Figure 5

We now have an idea of what the surface defined by $z = x^2 + y^2$ looks like but to complete the picture we draw a final sequence of curves.

- Fix z at value z_0.

We have $z = z_0$ (the equation of a plane parallel to the xy plane.) In this case the surface becomes

$$z_0 = x^2 + y^2$$

But this is the equation of a **circle** centred on $x = 0$, $y = 0$ of radius $\sqrt{z_0}$. (Clearly we **must** choose $z_0 \geq 0$ because $x^2 + y^2$ cannot be negative.) As we vary z_0 we obtain different circles, each lying on a different plane $z = z_0$. In Figure 6 we have combined the circles with the curves of Figure 5(b) to obtain a good visualisation of the surface $z = x^2 + y^2$.

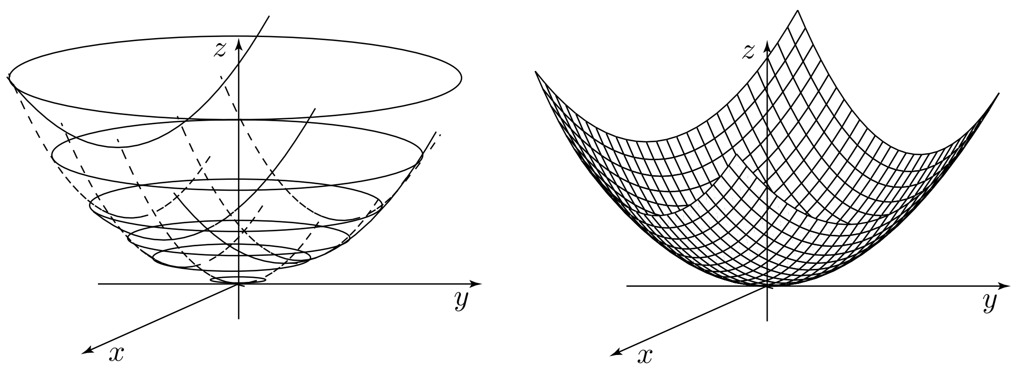


Figure 6

(Technically the surface is called a **paraboloid**, obtained by **rotating** a parabola about the $z-$axis.)

With the wide availability of sophisticated graphics packages the need to be able to sketch a surface is not as important as once it was. However, we urge the reader to attempt simple surface sketching in the initial stages of this study as it will enhance understanding of functions of two variables.