Parametric Curves 17.3

⚡ Introduction

In this Section we examine yet another way of defining curves - the parametric description. We shall see that this is, in some ways, far more useful than either the Cartesian description or the polar form. Although we shall only study planar curves (curves lying in a plane) the parametric description can be easily generalised to the description of spatial curves which twist and turn in three dimensional space.

⚡ Prerequisites

Before starting this Section you should . . .

- be familiar with Cartesian coordinates
- be familiar with trigonometric and hyperbolic functions and be able to manipulate them
- be able to differentiate simple functions
- be able to locate turning points and distinguish between maxima and minima.

⚡ Learning Outcomes

On completion you should be able to . . .

- sketch planar curves given in parametric form
- understand how the same curve can be described using different parameterisations
- recognise some conics given in parametric form

1. Parametric curves

Here we explore the use of a parameter t in the description of curves. We shall see that it has some advantages over the more usual Cartesian description. We start with a simple example.
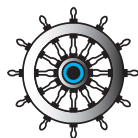
Example 6

Plot the curve $\qquad \underbrace{x = 2\cos t \qquad y = 3\sin t}_{\text{parametric equations of the curve}} \qquad \underbrace{0 \le t \le \dfrac{\pi}{2}}_{\text{parameter range}}$

Solution

The approach to sketching the curve is straightforward. We simply give the parameter t various values as it ranges through $0 \to \dfrac{\pi}{2}$ and, for each value of t, calculate corresponding values of (x, y) which are then plotted on a Cartesian xy plane. The value of t and the corresponding values of x, y are recorded in the following table:

t	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
x	2	1.98	1.90	1.78	1.62	1.41	1.18	0.91	0.62	0.31	0
y	0	0.47	0.93	1.36	1.76	2.12	2.43	2.67	2.85	2.96	3

Plotting the (x, y) coordinates gives the curve in Figure 16.
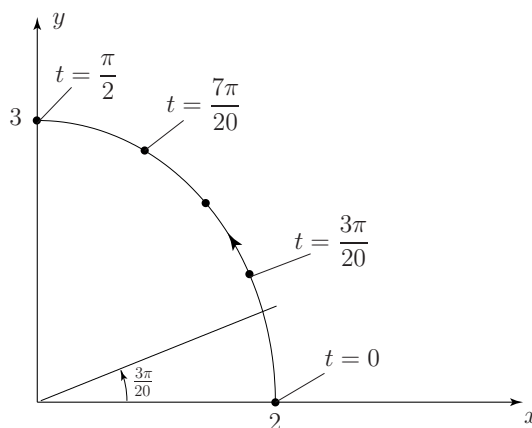

Figure 16

The curve in Figure 16 resembles part of an ellipse. This can be verified by eliminating t from the parametric equations to obtain an expression involving x, y only. If we divide the first parametric equation by 2 and the second by 3, square both and add we obtain

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t \equiv 1 \qquad \text{i.e.} \qquad \frac{x^2}{4} + \frac{y^2}{9} = 1$$

which we easily recognise as an ellipse whose major-axis is the y-axis. Also, as t ranges from $0 \to \dfrac{\pi}{2}$ $x = 2\cos t$ decreases from $2 \to 0$, and $y = 3\sin t$ increases from $0 \to 3$. We conclude that the

parametric equations $x = 2\cos t$, $y = 3\sin t$ together with the parametric range $0 \leq t \leq \dfrac{\pi}{2}$ describe that part of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ in the positive quadrant. On the curve in Figure 16 we have used an arrow to indicate the direction that we move along the curve as t **increases** from its initial value 0.

Task

Plot the curve $\qquad x = t + 1 \qquad y = 2t^2 - 3 \qquad 0 \leq t \leq 1$

Do you recognise this curve as a conic section?

First construct a table of (x, y) values as t ranges from $0 \rightarrow 1$:

Your solution

t	0	0.25	0.5	0.75	1
x					
y					

Answer

t	0	0.25	0.5	0.75	1
x	1	1.25	1.5	1.75	2
y	-3	-2.88	-2.5	-1.88	-1

Now plot the points on a Cartesian plane:

Your solution

Answer
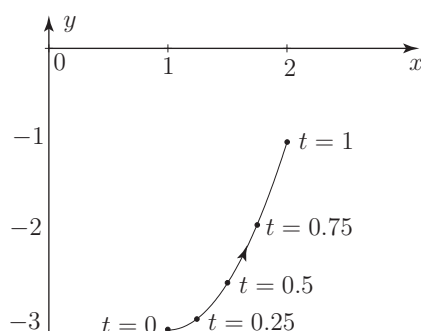
Now eliminate the t-variable from $x = t + 1$, $y = 2t^2 - 3$ to obtain the xy form of the curve:

Your solution

Answer

$y = 2x^2 - 4x - 1$ which is the equation of a parabola.

Example 7

Sketch the curve $x = t^2 + 1$ $y = 2t^4 - 3$ $0 \leq t \leq 1$

Solution

This is very similar to the previous Task (except for t^4 replacing t^2 in the expression for y and t^2 replacing t in the expression for x). The corresponding table of values is

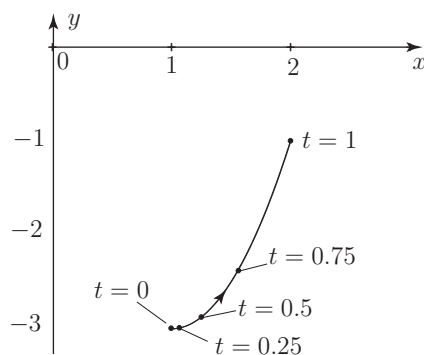| t	0	0.25	0.5	0.75	1
x	1	1.06	1.25	1.56	2
y	-3	-2.99	-2.88	-2.37	-1

Figure 17

We see that this is **identical** to the curve drawn previously. This is confirmed by eliminating the t-parameter from the expressions defining x, y. Here $t^2 = x - 1$ so $y = 2(x-1)^2 - 3$ which is the same as obtained in the last Task. The main difference is that particular values of t locate (in general) different (x, y) points on the curve for the two parametric representations.

We conclude that a given curve in the xy plane can have many (in fact infinitely many) parametric descriptions.

Task

Show that the two parametric representations below describe the same curve.

(a) $x = \cos t$ $y = \sin t$ $0 \leq t \leq \dfrac{\pi}{2}$

(b) $x = t$ $y = \sqrt{1 - t^2}$ $0 \leq t \leq 1$

Eliminate t from the parametric equations in (a):

Your solution

Answer

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

Eliminate t from the parametric equations in (b):

Your solution

Answer

$y = \sqrt{1 - x^2}$ $\therefore$ $y^2 = 1 - x^2$ or $x^2 + y^2 = 1$

What do you conclude?

Your solution

Answer

Both parametric descriptions represent (part of) a circle centred at the origin of radius 1.

2. General parametric form

We will assume that any curve in the xy plane may be written in parametric form:

$$\underbrace{x = g(t) \qquad y = h(t)}_{\text{parametric equations of the curve}} \qquad\qquad \underbrace{t_0 \leq t \leq t_1}_{\text{parameter range}}$$

in which $g(t)$, $h(t)$ are **given** functions of t and the parameter t ranges over the values $t_0 \to t_1$. As we give values to t within this range then corresponding values of x, y are calculated from $x = g(t)$, $y = h(t)$ which can then be plotted on an xy plane.

In HELM 12.3, we discovered how to obtain the derivative $\dfrac{dy}{dx}$ from a knowledge of the parametric derivatives $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$. We found

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} \right) \div \left(\frac{dx}{dt} \right)^3$$

Note that derivatives with respect to the parameter t are often denoted by a dot:

$$\frac{dx}{dt} \equiv \dot{x} \qquad \frac{dy}{dt} \equiv \dot{y} \qquad \frac{d^2x}{dt^2} \equiv \ddot{x} \quad \text{etc}$$

so that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Knowledge of the derivative is sometimes useful in curve sketching.

Example 8

Sketch the curve $\quad x = t^3 + 3t^2 + 2t \qquad y = 3 - 2t - t^2 \qquad -3 \le t \le 1.$

Solution

$$x = t^3 + 3t^2 + 2t = t(t+2)(t+1) \qquad y = 3 - 2t - t^2 = -(t+3)(t-1)$$

so that $x = 0$ when $t = 0, \ -1, \ -2$ and $y = 0$ when $t = -3, \ 1$. We calculate the values of x, y at various values of t:

t	-3	-2.50	-2	-1.50	-1	-0.50	0	0.50
x	-6	-1.88	0	0.38	0	-0.38	0	1.88
y	0	1.75	3	3.75	4	3.75	3	1.75

We see that $t = -2$ and $t = 0$ give rise to the **same** coordinate values for (x, y). This represents a **double-point** in the curve which is one where the curve crosses itself. Now

$$\frac{dx}{dt} = 3t^2 + 6t + 2, \quad \frac{dy}{dt} = -2 - 2t \qquad \therefore \qquad \frac{dy}{dx} = \frac{-2(1+t)}{3t^2 + 6t + 2}$$

so there is a turning point when $t = -1$. The reader is urged to calculate $\dfrac{d^2y}{dx^2}$ and to show that this is negative when $t = -1$ (i.e. at $x = 0, \ y = 4$) indicating a maximum when. (The reader should check that vertical tangents occur at $t = -0.43$ and $t = -1.47$, to 2 d.p.)
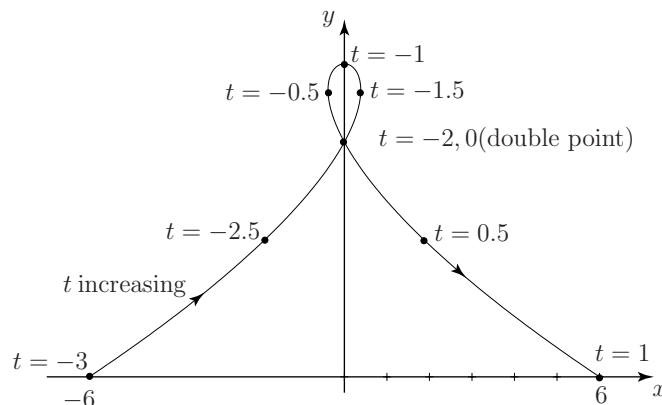
We can now make a reasonable sketch of the curve:

Figure 18

3. Standard forms of conic sections in parametric form

We have seen above that, given a curve in the xy plane, there is no unique way of representing it in parametric form. However, for some commonly occurring curves, particularly the conics, there are accepted standard parametric equations.

The parabola

The standard parametric equations for a parabola are: $x = at^2 \qquad y = 2at$

Clearly, we have $t = \dfrac{y}{2a}$ and by eliminating t we get $x = a\left(\dfrac{y^2}{4a^2}\right)$ or $y^2 = 4ax$ which we recognise as the standard Cartesian description of a parabola. As an illustration, Figure 19 shows the curve with $a = 2$ and $-1 \le t \le 2.3$
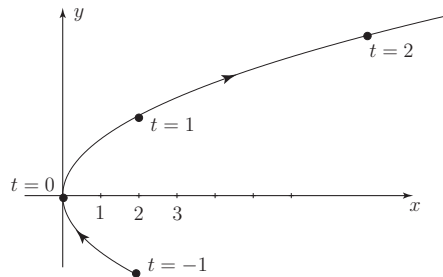

Figure 19

The ellipse

Here, the standard equations are $x = a\cos t \qquad y = b\sin t$

Again, eliminating t (dividing the first equation by a, the second by b, squaring and adding) we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t \equiv 1 \quad \text{or, in more familiar form:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we choose the range for t as $0 \le t \le \dfrac{7\pi}{4}$ the following segment of the ellipse is obtained.
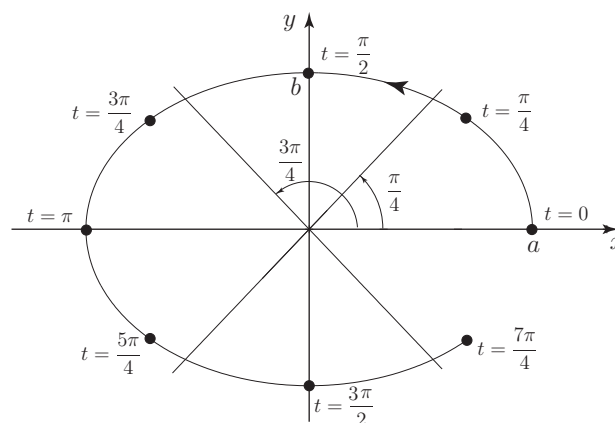

Figure 20

Here we note that (except in the special case when $a = b$, giving a circle) the parameter t is **not** the angle that the radial line makes with the the positive x-axis.

In the study of the orbits of planets and satellites it is often preferable to use plane polar coordinates (r, θ) to treat the problem. In these coordinates an ellipse has an equation of the form $\dfrac{1}{r} = A + B\cos\theta$, with A and B positive numbers such that $B < A$. Not only is there a difference in the equations on passing from Cartesian to polar coordinates; there is also a change in the origin of coordinates. The polar coordinate equation is using a focus of the ellipse as the origin. In the Cartesian description the foci are two points at $+e$ along the x-axis, where e obeys the equation $e = a - b$, if we assume that $a < b$ i.e. we choose the long axis of the ellipse as the x-axis. This problem gives some practice at algebraic manipulation and also indicates some shortcuts which can be made once the mathematics of the ellipse has been understood.

Example 9

An ellipse is described in plane polar coordinates by the equation

$$\frac{1}{r} = 2 + \cos\theta$$

Convert the equation to Cartesian form. [Hint: remember that $x = r\cos\theta$.]

Solution

Multiplying the given equation by r and then using $x = r\cos\theta$ gives the results

$$1 = 2r + x \qquad \text{so that} \qquad 2r = 1 - x$$

We now square the second equation, remembering that $r^2 = x^2 + y^2$. We now have

$$4(x^2 + y^2) = (1 - x)^2 = 1 + x^2 - 2x \quad \text{so that} \quad 3x^2 + 2x + 4y^2 = 1$$

We now recall the method of completing the square, which allows us to set

$$3x^2 + 2x = 3(x^2 + \frac{2x}{3})^2 - \frac{1}{9})$$

Putting this result into the equation and collecting terms leads to the final result

$$\frac{(x + \frac{1}{3})^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with} \quad a = \frac{2}{3} \text{ and } b = \sqrt{\frac{1}{3}}.$$

This is the standard Cartesian form for the equation of an ellipse **but** we must remember that we started from a polar equation with a focus of the ellipse as origin. The presence of the term $x + \dfrac{1}{3}$ in the equation above actually tells us that the focus being used as origin was a distance of $\dfrac{1}{3}$ to the right of the centre of the ellipse at $x = 0$.

The preceding piece of algebra was necessary in order to convince us that the original equation in plane polar coordinates **does** represent an ellipse. However, now that we are convinced of this we can go back and try to extract information in a more speedy way from the equation in its original (r, θ) form.

Solution (contd.)

Try setting $\theta = 0$ and $\theta = \pi$ in the equation

$$\frac{1}{r} = 2 + \cos\theta$$

We find that at $\theta = 0$ we have $r = \frac{1}{3}$ while at $\theta = \pi$ we have $r = 1$. These r values correspond to the two ends of the ellipse, so the long axis has a total length $1 + \frac{1}{3} = \frac{4}{3}$. This tells us that $a = \frac{2}{3}$, exactly as found by our longer algebraic derivation. We can further deduce that the focus acting as origin must be at a distance of $\frac{1}{3}$ from the centre of the ellipse in order to lead to the two r values at $\theta = 0$ and $\theta = \pi$. If we now use the equation $e = a - b$ mentioned earlier then we find that $\frac{1}{9} = \frac{4}{9} - b^2$, so that $b = \sqrt{\frac{1}{3}}$, as obtained by our lengthy algebra.

The hyperbola

The standard equations are $\quad x = a\cosh t \qquad y = b\sinh t$.

In this case, to eliminate t we use the identity $\cosh^2 t - \sinh^2 t \equiv 1$ giving rise to the equation of the hyperbola in Cartesian form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

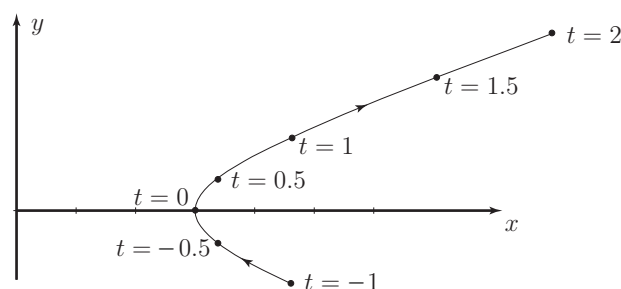In Figure 21 we have chosen a parameter range $-1 \leq t \leq 2$.

Figure 21

To obtain the complete curve the parameter range $-\infty < t < \infty$ must be used. These parametric equations only give the right-hand branch of the hyperbola. To obtain the left-hand branch we would use $\quad x = -a\cosh t \qquad y = b\sinh t$

Exercises

1. In the following sketch the given parametric curves. Also, eliminate the parameter to give the Cartesian equation in x and y.

   (a) $x = t, \quad y = 2 - t \quad 0 \leq t \leq 1$ (b) $x = 2 - t, \quad y = t + 1 \quad 0 \leq t \leq \infty$

   (c) $x = \dfrac{2}{t} \quad y = t - 2 \quad 0 < t < 3$ (d) $x = 3\sin\dfrac{\pi t}{2} \quad y = 4\cos\dfrac{\pi t}{2} \quad -1 \leq t \leq 0.5$

2. Find the tangent line to the parametric curve $x = t^2 - t \ y - t^2 + t$ at the point where $t = 1$.

3. For each of the following curves expressed in parametric form obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and use this information to help make a sketch.
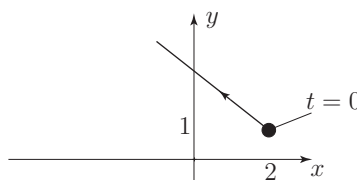
   (a) $x = t^2 - 2t, \quad y = t^2 - 4t$

   (b) $x = t^3 - 3t - 2, \quad y = t^2 - t - 2$
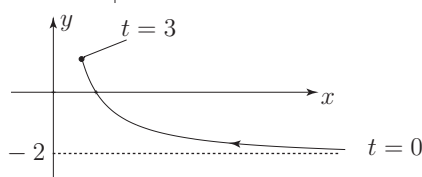
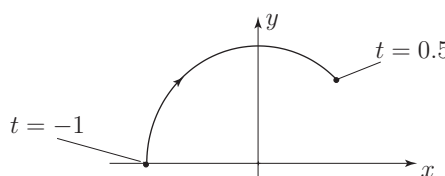## Answers

1.   (a) $y = 2 - x$
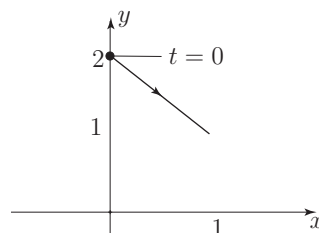
    (b) $y = 3 - x$

    (c) $y = \dfrac{2}{x} - 2 \qquad \therefore \qquad x(y + 2) = 2$
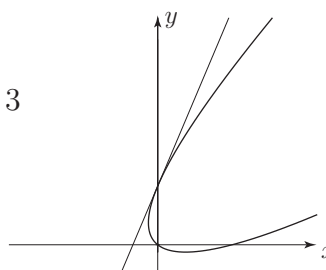

    (d) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

2. $\dfrac{dy}{dt} = 2t + 1 \qquad \dfrac{dx}{dt} = 2t - 1$

    $\therefore \qquad \dfrac{dy}{dx} = \dfrac{2t + 1}{2t - 1} \qquad$ when $t = 1$ then $\dfrac{dy}{dx} = 3$

   when $t = 1 \quad x = 0, \quad y = 2$
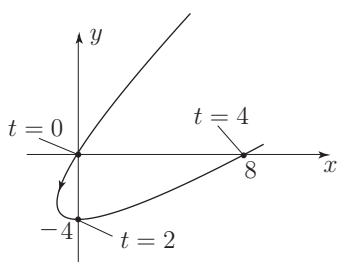
    $\therefore \qquad$ tangent line is $y = 3x + 2$

Answer

3. (a) $\dfrac{dy}{dt} = 2t - 4 \qquad \dfrac{dx}{dt} = 2t - 2$

$\dfrac{d^2y}{dt^2} = 2 \qquad \dfrac{d^2x}{dt^2} = 2$

$\dfrac{dy}{dx} = \dfrac{2t-4}{2t-2} = \dfrac{t-2}{t-1} \qquad \dfrac{d^2y}{dx^2} = \dfrac{[(2t-2)-(2t-4)]2}{8(t-1)^3} = \dfrac{1}{2(t-1)^3}$

(b) $x = (t-2)(t^2 + 2t + 1) = (t-2)(t+1)^2$

$y = (t+1)(t-2)$

$\dfrac{dy}{dt} = 2t - 1 \qquad \dfrac{dx}{dt} = 3t^2 - 3$

$\dfrac{d^2y}{dt^2} = 2 \qquad \dfrac{d^2x}{dt^2} = 6t$

$\dfrac{d^2y}{dx^2} = \dfrac{[2(3t^2-3)-(2t-1)6t]}{(3t^2-3)^3} = \dfrac{-6t^2+6t-6}{27(t^2-1)^3}$