Polar Coordinates 17.2

Introduction

In this Section we extend the use of polar coordinates. These were first introduced in HELM 2.8. They were also used in the discussion on complex numbers in HELM 10.2. We shall examine the application of polars to the description of curves, particularly conics. Some curves, spirals for example, which are very difficult to describe in terms of Cartesian coordinates (x, y) are relatively easily defined in polars $[r, \theta]$.

Prerequisites

Before starting this Section you should ...

- be familiar with Cartesian coordinates
- be familiar with trigonometric functions and how to manipulate then
- be able to simplify algebraic expressions and manipulate algebraic fractions

Learning Outcomes

On completion you should be able to ...

- understand how Cartesian coordinates and polar coordinates are related
- find the polar form of a curve given in Cartesian form
- recognise some conics given in polar form

1. Polar Coordinates

In this Section we consider the application of polar coordinates to the description of curves; in particular, to conics.

If the Cartesian coordinates of a point P are (x, y) then P can be located on a Cartesian plane as indicated in Figure 10.

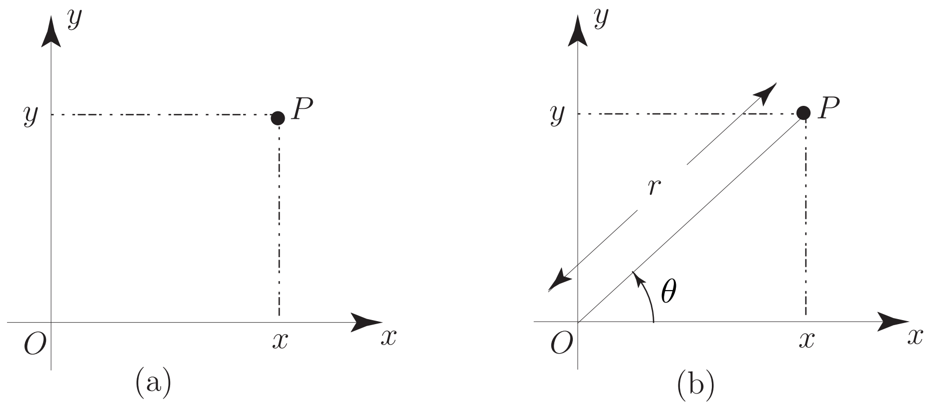


Figure 10

However, the same point P can be located by using polar coordinates r, θ where r is the distance of P from the origin and θ is the angle, measured anti-clockwise, that the line OP makes when measured from the positive x-direction. See Figure 10(b). In this Section we shall denote the polar coordinates of a point by using square brackets.

From Figure 10 it is clear that Cartesian and polar coordinates are directly related. The relations are noted in the following Key Point.

Key Point 5

If (x, y) are the Cartesian coordinates and $[r, \theta]$ the polar coordinates of a point P then

$$x = r\cos\theta \qquad y = r\sin\theta$$

and, equivalently,

$$r = +\sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

From these relations we see that it is a straightforward matter to calculate (x, y) given $[r, \theta]$. However, some care is needed (particularly with the determination of θ) if we want to calculate $[r, \theta]$ from (x, y).

Example 4

On a Cartesian plane locate points P, Q, R, S which have their locations specified by polar coordinates $[2, \frac{\pi}{2}]$, $[2, 3\frac{\pi}{2}]$, $[3, \frac{\pi}{6}]$, $[\sqrt{2}, \pi]$ respectively.

Solution

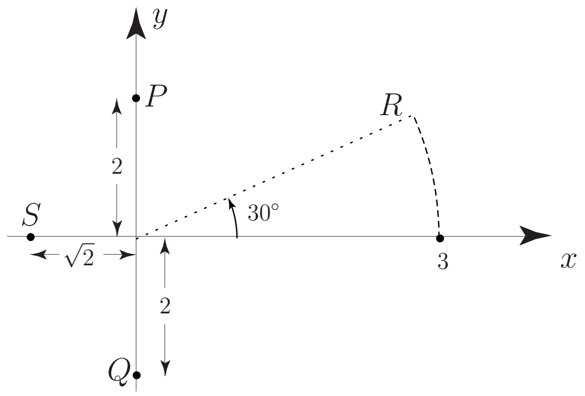


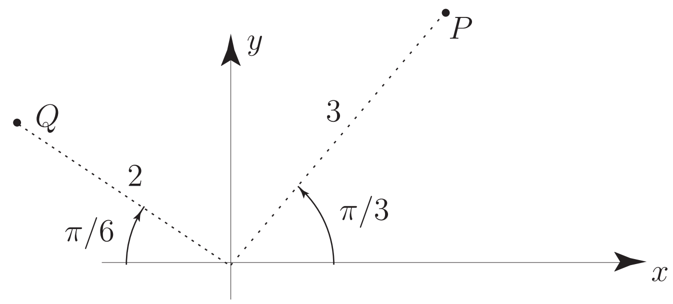
Figure 11

Task

Two points P, Q have polar coordinates $[3, \frac{\pi}{3}]$ and $[2, \frac{5\pi}{6}]$ respectively. By locating these points on a Cartesian plane find their equivalent Cartesian coordinates.

Your solution

Answer



$$P: \ (3\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3}) \equiv (\frac{3}{2}, \frac{3\sqrt{3}}{2})$$

$$Q: \ (-2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}) \equiv (\frac{-2\sqrt{3}}{2}, 1)$$

The polar coordinates of a point are not unique. So, the polar coordinates $[a, \theta]$ and $[a, \phi]$ **represent the same point** in the Cartesian plane provided θ and ϕ differ by an integer multiple of 2π. See Figure 12.

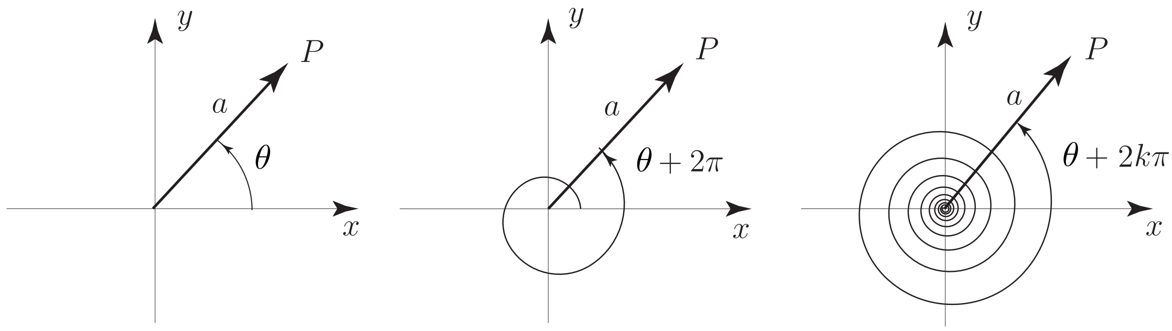


Figure 12

For example, the polar coordinates $[2, \frac{\pi}{3}]$, $[2, \frac{7\pi}{3}]$, $[2, -\frac{5\pi}{3}]$ all represent the same point in the Cartesian plane.

Key Point 6

By convention, we measure the positive angle θ in an **anti-clockwise direction**.

The angle $-\phi$ is interpreted as the angle ϕ measured in a clockwise direction.

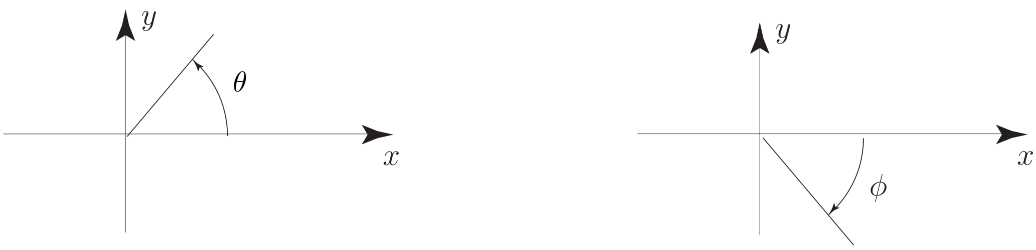


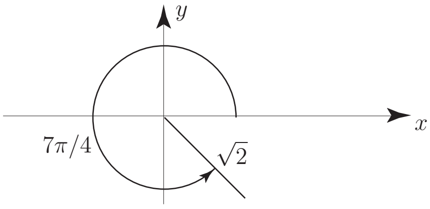
Figure 13

Exercises

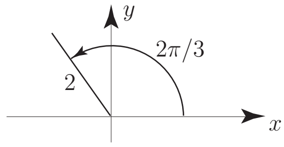
1. The Cartesian coordinates of P, Q are $(1, -1)$ and $(-1, \sqrt{3})$. What are their equivalent polar coordinates?
2. Locate the points P, Q, R with polar coordinates $[1, \frac{\pi}{3}]$, $[2, \frac{7\pi}{3}]$, $[2, \frac{10\pi}{3}]$. What do you notice?

Answer

1.



$(1,-1) \rightarrow [\sqrt{2},\ 7\pi/4]$



$(-1,\sqrt{3}) \rightarrow [2,\ 2\pi/3]$

2. All these points lie on a straight line through the origin.

2. Simple curves in polar coordinates

We are used to describing the equations of curves in Cartesian variables x, y. Thus $x^2 + y^2 = 1$ represents a circle, centre the origin, and of radius 1, and $y = 2x^2$ is the equation of a parabola whose axis is the y-axis and with vertex located at the origin. (In colloquial terms the vertex is the 'sharp end' of a conic.) We can convert these equations into polar form by using the relations $x = r\cos\theta$, $y = r\sin\theta$.

Example 5

Find the polar coordinate form of

(a) the circle $x^2 + y^2 = 1$ (b) the parabola $y = 2x^2$.

Solution

(a) Using $x = r\cos\theta$, $y = r\sin\theta$ in the expression $x^2 + y^2 = 1$ we have

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 1 \quad \text{or} \quad r^2(\cos^2\theta + \sin^2\theta) = 1$$

giving $r^2 = 1$. We simplify this to $r = 1$ (since $r = -1$ is invalid being a negative distance). Of course we might have guessed this answer since the relation $r = 1$ states that every point on the curve is a constant distance 1 away from the origin.
(b) Repeating the approach used in (a) for $y = 2x^2$ we obtain:

$$r\sin\theta = 2(r\cos\theta)^2 \qquad \text{i.e.} \qquad r\sin\theta - 2r^2\cos^2\theta = 0$$

Therefore $r(\sin\theta - 2r\cos^2\theta) = 0$. Either $r = 0$ (which is a single point, the origin, and is clearly not a parabola) or

$$\sin\theta - 2r\cos^2\theta = 0 \qquad \text{giving, finally} \qquad r = \frac{1}{2}\tan\theta\sec\theta.$$

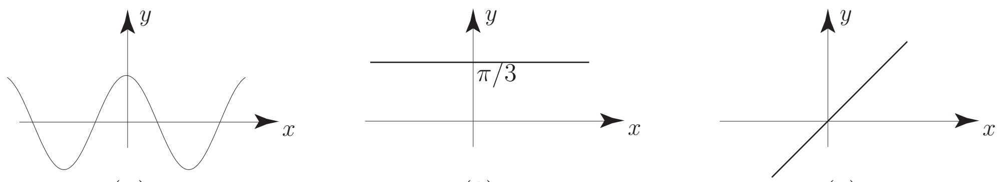
This is the polar equation of this particular parabola, $y = 2x^2$.

Task

Sketch the curves
(a) $y = \cos x$ (b) $y = \dfrac{\pi}{3}$ (c) $y = x$

Your solution

Answer



(a) (b) (c)

Task

Sketch the curve $r = \cos\theta$.

First complete the table of values. Enter values to 2 d.p. and work in radians:

Your solution

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
r							

Answer

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
r	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00

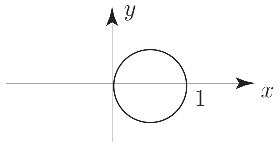
You will see that the values of θ for $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$ give rise to negative values of r (and hence invalid).

Now sketch the curve:

Your solution

Answer

circle: centre $\left(\frac{1}{2}, 0\right)$, radius $\frac{1}{2}$.



Task

Sketch the curve $\theta = \pi/3$.

Your solution

Answer

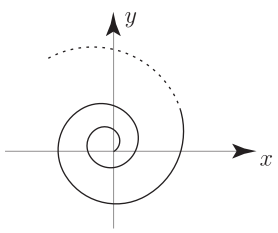
Radial line passing through the origin at angle $\dfrac{\pi}{3}$ to the positive x-axis.

Task

Sketch the curve $r = \theta$.

Your solution

Answer



3. Standard conics in polar coordinates

In the previous Section we merely stated the standard equations of the conics using Cartesian coordinates. Here we consider an alternative definition of a conic and use this different approach to obtain the equations of the standard conics in polar form. Consider a straight line $x = -d$ (this will be the directrix of the conic) and let e be the eccentricity of the conic (e is a positive real number). It can be shown that the set of points P in the (x, y) plane which satisfy the condition

$$\frac{\text{distance of } P \text{ from origin}}{\text{perpendicular distance from } P \text{ to the line}} = e$$

is a conic with eccentricity e. In particular, it is an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$. See Figure 14.

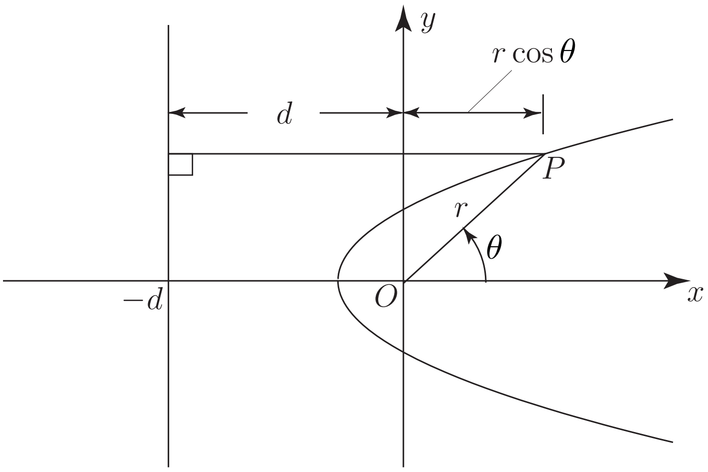


Figure 14

We can obtain the polar coordinate form of this conic in a straightforward manner. If P has polar coordinates $[r, \theta]$ then the relation above gives

$$\frac{r}{d + r\cos\theta} = e \qquad \text{or} \qquad r = e(d + r\cos\theta)$$

Thus, solving for r: $\qquad r = \dfrac{ed}{1 - e\cos\theta}$

This is the equation of the conic.
In all of these conics it can be shown that one of the foci is located at the origin. See Figure 15 in which the pertinent details of the conics are highlighted.

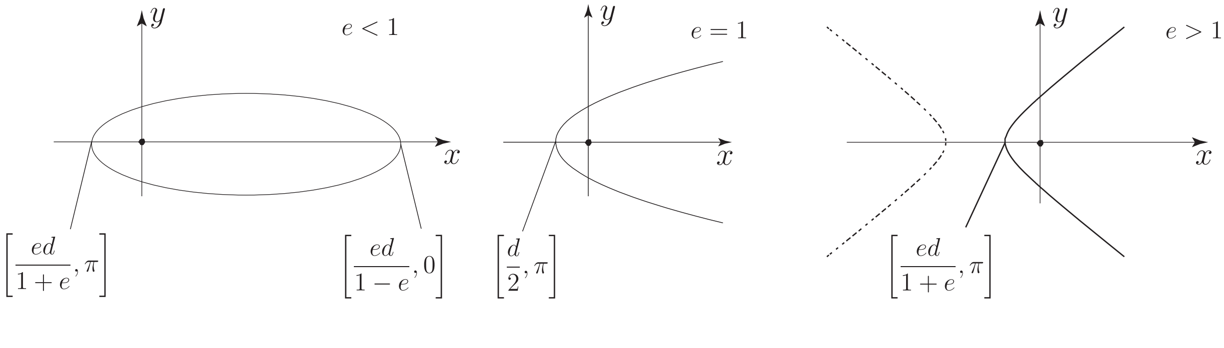


Figure 15

Task

Sketch the ellipse $r = \dfrac{4}{2 - \cos\theta}$ and locate the coordinates of its vertices.

Your solution

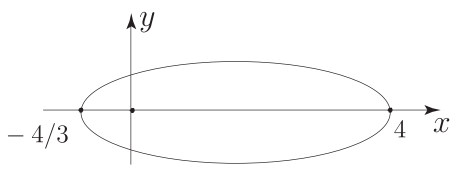
Answer

Here

$$r = \frac{4}{2 - \cos\theta} = \frac{2}{1 - \frac{1}{2}\cos\theta} \qquad \text{so} \qquad e = \frac{1}{2}$$

Then

$$de = 2 \qquad \frac{de}{1+e} = \frac{2}{\frac{3}{2}} = \frac{4}{3} \qquad \text{and} \qquad \frac{de}{1-e} = \frac{2}{\frac{1}{2}} = 4$$



Exercises

1. Sketch the polar curves (a) $r = \dfrac{1}{1-\cos\theta}$ (b) $r = e^{-\theta}$ (c) $r = \dfrac{6}{3-\cos\theta}$.

2. Find the polar form of the following curves given in Cartesian form:

   (a) $y^2 = 1 + 2x$ (b) $2xy = 1$

3. Find the Cartesian form of the following curves given in polar form

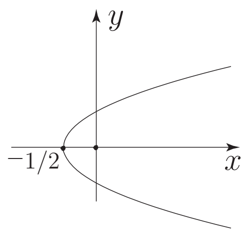
   (a) $r = \dfrac{2}{\sin\theta + 2\cos\theta}$ (b) $r = 3\cos\theta$

   Do you recognise these equations?

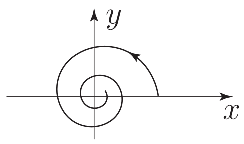
Answers

1.

(a) parabola $e = 1,\ d = 1$

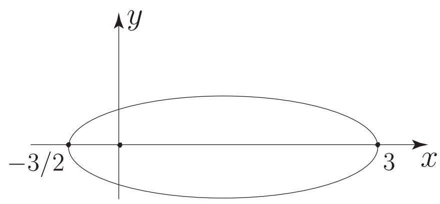


(b) decreasing spiral



(c) $r = \dfrac{2}{1 - \frac{1}{3}\cos\theta}$

ellipse since $e = \frac{1}{3} < 1$. Also $de = 2$



2.

(a) $r^2\sin^2\theta = 1 + 2r\cos\theta \quad \therefore \quad r = \dfrac{\cos\theta + 1}{1-\cos^2\theta} = \dfrac{1}{1-\cos\theta}$

(b) $2r^2\cos\theta\sin\theta = 1 \quad \therefore \quad r^2 = \text{cosec}\, 2\theta$

3.

(a) $r(\sin\theta + 2\cos\theta) = 2 \quad \therefore \quad y + 2x = 2$ which is a straight line

(b) $r = 3\cos\theta \quad \therefore \quad \sqrt{x^2+y^2} = \dfrac{3x}{\sqrt{x^2+y^2}} \quad \therefore \quad x^2 + y^2 = 3x$

in standard form: $\left(x - \dfrac{3}{2}\right)^2 + y^2 = \dfrac{9}{4}$. i.e. a circle, centre $\left(\dfrac{3}{2},\ 0\right)$ with radius $\dfrac{3}{2}$