The Mean Value and the Root-Mean-Square Value

14.2

Introduction

Currents and voltages often vary with time and engineers may wish to know the mean value of such a current or voltage over some particular time interval. The mean value of a time-varying function is defined in terms of an integral. An associated quantity is the **root-mean-square** (r.m.s). For example, the r.m.s. value of a current is used in the calculation of the power dissipated by a resistor.

Prerequisites

Before starting this Section you should . . .

- be able to calculate definite integrals
- be familiar with a table of trigonometric identities

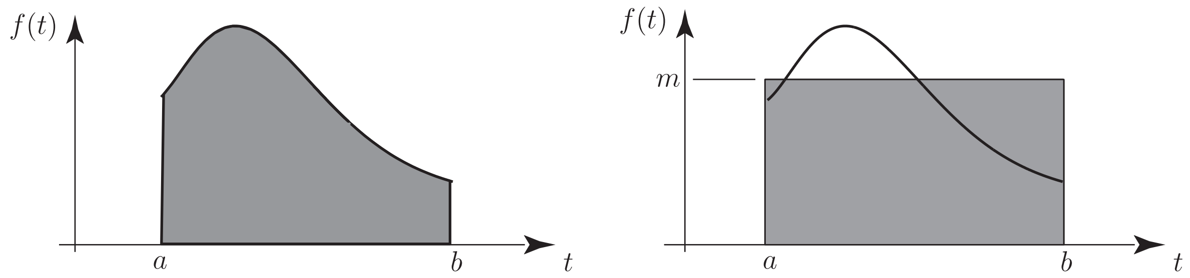
Learning Outcomes

On completion you should be able to . . .

- calculate the mean value of a function
- calculate the root-mean-square value of a function

1. Average value of a function

Suppose a time-varying function $f(t)$ is defined on the interval $a \leq t \leq b$. The area, A, under the graph of $f(t)$ is given by the integral $A = \int_a^b f(t)\,dt$. This is illustrated in Figure 5.



(a) the area under the curve from $t = a$ to $t = b$

(b) the area under the curve and the area of the rectangle are equal

Figure 5

On Figure 3 we have also drawn a rectangle with base spanning the interval $a \leq t \leq b$ and which has the same area as that under the curve. Suppose the height of the rectangle is m. Then

$$\text{area of rectangle} = \text{area under curve} \quad \Rightarrow \quad m(b-a) = \int_a^b f(t)\,dt \quad \Rightarrow \quad m = \frac{1}{b-a}\int_a^b f(t)\,dt$$

The value of m is the **mean value** of the function across the interval $a \leq t \leq b$.

Key Point 2

The **mean value** of a function $f(t)$ in the interval $a \leq t \leq b$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,dt$

The mean value depends upon the interval chosen. If the values of a or b are changed, then the mean value of the function across the interval from a to b will in general change as well.

Example 2

Find the mean value of $f(t) = t^2$ over the interval $1 \leq t \leq 3$.

Solution

Using Key Point 2 with $a = 1$ and $b = 3$ and $f(t) = t^2$

$$\text{mean value} = \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{3-1}\int_1^3 t^2\,dt = \frac{1}{2}\left[\frac{t^3}{3}\right]_1^3 = \frac{13}{3}$$

Find the mean value of $f(t) = t^2$ over the interval $2 \leq t \leq 5$.

Use Key Point 2 with $a = 2$ and $b = 5$ to write down the required integral:

Your solution

mean value =

Answer

$$\frac{1}{5-2}\int_2^5 t^2\,dt$$

Now evaluate the integral:

Your solution

mean value =

Answer

$$\frac{1}{5-2}\int_2^5 t^2\,dt = \frac{1}{3}\left[\frac{t^3}{3}\right]_2^5 = \frac{1}{3}\left[\frac{125}{3} - \frac{8}{3}\right] = \frac{117}{9} = 13$$

Engineering Example 2

Sonic boom

Introduction

Impulsive signals are described by their peak amplitudes and their duration. Another quantity of interest is the total energy of the impulse. The effect of a blast wave from an explosion on structures, for example, is related to its total energy. This Example looks at the calculation of the energy on a sonic boom. Sonic booms are caused when an aircraft travels faster than the speed of sound in air. An idealized sonic-boom pressure waveform is shown in Figure 6 where the instantaneous sound pressure $p(t)$ is plotted versus time t. This wave type is often called an N-wave because it resembles the shape of the letter N. The energy in a sound wave is proportional to the square of the sound pressure.

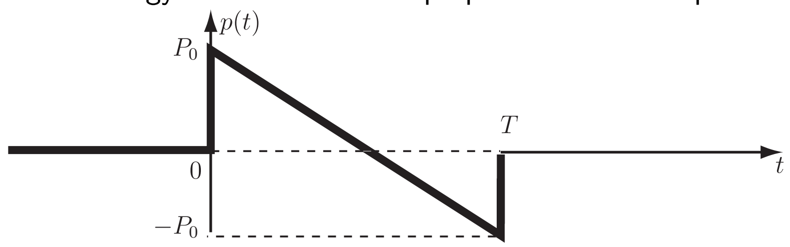


Figure 6: An idealized sonic-boom pressure waveform

Problem in words

Calculate the energy in an ideal N-wave sonic boom in terms of its peak pressure, its duration and the density and sound speed in air.

Mathematical statement of problem

Represent the positive peak pressure by P_0 and the duration by T. The total acoustic energy E carried across unit area normal to the sonic-boom wave front during time T is defined by

$$E = <p(t)^2> T/\rho c \qquad (1)$$

where ρ is the air density, c the speed of sound and the time average of $[p(t)]^2$ is

$$<p(t)^2> = \frac{1}{T}\int_0^T p(t)^2\, dt \qquad (2)$$

(a) Find an appropriate expression for $p(t)$.

(b) Hence show that E can be expressed in terms of $P_0,\ T,\ \rho$ and c as $\quad E = \dfrac{TP_0^2}{3\rho c}$.

Mathematical analysis

(a) The interval of integration needed to compute (2) is $[0, T]$. Therefore it is necessary to find an expression for $p(t)$ only in this interval. Figure 6 shows that, in this interval, the dependence of the sound pressure p on the variable t is linear, i.e. $p(t) = at + b$.

From Figure 6 also $p(0) = P_0$ and $p(T) = -P_0$. The constants a and b are determined from these conditions.

At $t = 0,\ a \times 0 + b = P_0$ implies that $b = P_0$.

At $t = T,\ a \times T + b = -P_0$ implies that $a = -2P_0/T$.

Consequently, the sound pressure in the interval $[0, T]$ may be written $\quad p(t) = \dfrac{-2P_0}{T}\, t + P_0$.

(b) This expression for $p(t)$ may be used to compute the integral (2)

$$\begin{aligned}
\frac{1}{T}\int_0^T p(t)^2\, dt &= \frac{1}{T}\int_0^T \left(\frac{-2P_0}{T}\, t + P_0\right)^2 dt = \frac{1}{T}\int_0^T \left(\frac{4P_0^2}{T^2}\, t^2 - \frac{4P_0^2}{T}\, t + P_0^2\right) dt \\
&= \frac{1}{T}\left[\frac{4P_0^2}{3T^2}\, t^3 - \frac{2P_0^2}{T}\, t^2 + P_0^2 t\right]_0^T \\
&= \frac{P_0^2}{T}\left(\frac{4}{3T^2}\, T^3 - \frac{2}{T}\, T^2 + T\right) - 0 = P_0^2/3.
\end{aligned}$$

Hence, from Equation (1), the total acoustic energy E carried across unit area normal to the sonic-boom wave front during time T is $E = \dfrac{TP_0^2}{3\rho c}$.

Interpretation

The energy in an N-wave is given by a third of the sound intensity corresponding to the peak pressure multiplied by the duration.

Exercises

1. Calculate the mean value of the given functions across the specified interval.

   (a) $f(t) = 1 + t$ across $[0, 2]$
   (b) $f(x) = 2x - 1$ across $[-1, 1]$
   (c) $f(t) = t^2$ across $[0, 1]$
   (d) $f(t) = t^2$ across $[0, 2]$
   (e) $f(z) = z^2 + z$ across $[1, 3]$

2. Calculate the mean value of the given functions over the specified interval.

   (a) $f(x) = x^3$ across $[1, 3]$
   (b) $f(x) = \dfrac{1}{x}$ across $[1, 2]$
   (c) $f(t) = \sqrt{t}$ across $[0, 2]$
   (d) $f(z) = z^3 - 1$ across $[-1, 1]$

3. Calculate the mean value of the following:

   (a) $f(t) = \sin t$ across $\left[0, \frac{\pi}{2}\right]$
   (b) $f(t) = \sin t$ across $[0, \pi]$
   (c) $f(t) = \sin \omega t$ across $[0, \pi]$
   (d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$
   (e) $f(t) = \cos t$ across $[0, \pi]$
   (f) $f(t) = \cos \omega t$ across $[0, \pi]$
   (g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, 1]$

4. Calculate the mean value of the following functions:

   (a) $f(t) = \sqrt{t + 1}$ across $[0, 3]$
   (b) $f(t) = e^t$ across $[-1, 1]$
   (c) $f(t) = 1 + e^t$ across $[-1, 1]$

Answers

1. (a) 2 (b) -1 (c) $\dfrac{1}{3}$ (d) $\dfrac{4}{3}$ (e) $\dfrac{19}{3}$

2. (a) 10 (b) 0.6931 (c) 0.9428 (d) -1

3. (a) $\dfrac{2}{\pi}$ (b) $\dfrac{2}{\pi}$ (c) $\dfrac{1}{\pi\omega}[1 - \cos(\pi\omega)]$ (d) $\dfrac{2}{\pi}$ (e) 0 (f) $\dfrac{\sin(\pi\omega)}{\pi\omega}$

   (g) $\dfrac{1 + \sin\omega - \cos\omega}{\omega}$

4. (a) $\dfrac{14}{9}$ (b) 1.1752 (c) 2.1752

2. Root-mean-square value of a function

If $f(t)$ is defined on the interval $a \leq t \leq b$, the **mean-square** value is given by the expression:

$$\frac{1}{b-a}\int_a^b [f(t)]^2\,dt$$

This is simply the mean value of $[f(t)]^2$ over the given interval.

The related quantity: the **root-mean-square (r.m.s.)** value is given by the following formula.

Key Point 3

Root-Mean-Square Value

$$\text{r.m.s value } = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2\,dt}$$

The r.m.s. value depends upon the interval chosen. If the values of a or b are changed, then the r.m.s. value of the function across the interval from a to b will in general change as well. Note that when finding an r.m.s. value the function must be squared before it is integrated.

Example 3

Find the r.m.s. value of $f(t) = t^2$ across the interval from $t = 1$ to $t = 3$.

Solution

$$\text{r.m.s} = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2\,dt} = \sqrt{\frac{1}{3-1}\int_1^3 [t^2]^2\,dt} = \sqrt{\frac{1}{2}\int_1^3 t^4\,dt} = \sqrt{\frac{1}{2}\left[\frac{t^5}{5}\right]_1^3} \approx 4.92$$

Example 4

Calculate the r.m.s value of $f(t) = \sin t$ across the interval $0 \leq t \leq 2\pi$.

Solution

Here $a = 0$ and $b = 2\pi$ so r.m.s $= \sqrt{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2 t\, dt}.$

The integral of $\sin^2 t$ is performed by using trigonometrical identities to rewrite it in the alternative form $\frac{1}{2}(1 - \cos 2t)$. This technique was described in HELM 13.7.

$$\text{r.m.s. value} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\cos 2t)}{2} dt} = \sqrt{\frac{1}{4\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}} = \sqrt{\frac{1}{4\pi}(2\pi)} = \sqrt{\frac{1}{2}} = 0.707$$

Thus the r.m.s value is 0.707 to 3 d.p.

In the previous Example the amplitude of the sine wave was 1, and the r.m.s. value was 0.707. In general, if the amplitude of a sine wave is A, its r.m.s value is $0.707A$.

Key Point 4

The r.m.s value of any sinusoidal waveform taken across an interval of width equal to one period is $0.707 \times$ amplitude of the waveform.

Engineering Example 3

Electrodynamic meters

Introduction

A dynamometer or electrodynamic meter is an analogue instrument that can measure d.c. current or a.c. current up to a frequency of 2 kHz. A typical dynamometer is shown in Figure 7.

It consists of a circular dynamic coil positioned in a magnetic field produced by two wound circular stator coils connected in series with each other. The torque T on the moving coil depends upon the mutual inductance between the coils given by:

$$T = I_1 I_2 \frac{dM}{d\theta}$$

where I_1 is the current in the fixed coil, I_2 the current in the moving coil and θ is the angle between the coils. The torque is therefore proportional to the square of the current. If the current is alternating the moving coil is unable to follow the current and the pointer position is related to the mean value of the square of the current. The scale can be suitably graduated so that the pointer position shows the square root of this value, i.e. the r.m.s. current.

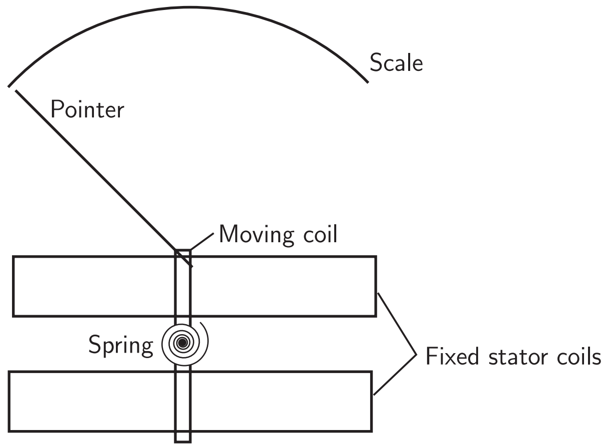


Figure 7: An electrodynamic meter

Problem in words

A dynamometer is in a circuit in series with a 400 Ω resistor, a rectifying device and a 240 V r.m.s alternating sinusoidal power supply. The rectifier resists current with a resistance of 200 Ω in one direction and a resistance of 1 kΩ in the opposite direction. Calculate the reading indicated on the meter.

Mathematical Statement of the problem

We know from Key Point 4 in the text that the r.m.s. value of any sinusoidal waveform taken across an interval equal to one period is 0.707 $\times$ amplitude of the waveform. Where 0.707 is an approximation of $\dfrac{1}{\sqrt{2}}$. This allows us to state that the amplitude of the sinusoidal power supply will be:

$$V_{\text{peak}} = \frac{V_{\text{rms}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}V_{\text{rms}}$$

In this case the r.m.s power supply is 240 V so we have

$$V_{\text{peak}} = 240 \times \sqrt{2} = 339.4\ V$$

During the part of the cycle where the voltage of the power supply is positive the rectifier behaves as a resistor with resistance of 200 Ω and this is combined with the 400 Ω resistance to give a resistance of 600 Ω in total. Using Ohm's law

$$V = IR \Rightarrow I = \frac{V}{R}$$

As $V = V_{\text{peak}}\sin(\theta)$ where $\theta = \omega t$ where ω is the angular frequency and t is time we find that during the positive part of the cycle

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta$$

During the part of the cycle where the voltage of the power supply is negative the rectifier behaves as a resistor with resistance of 1 kΩ and this is combined with the 400 Ω resistance to give 1400 Ω in total.

So we find that during the negative part of the cycle

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

Therefore over an entire cycle

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

We can calculate this value to find I_{rms}^2 and therefore I_{rms}.

Mathematical analysis

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

$$I_{\text{rms}}^2 = \frac{339.4^2}{2\pi\times 10000}\left(\int_{0}^{\pi}\frac{\sin^2(\theta)}{36}\,d\theta + \int_{\pi}^{2\pi}\frac{\sin^2(\theta)}{196}\,d\theta\right)$$

Substituting the trigonometric identity $\sin^2(\theta) \equiv \dfrac{1-\cos(2\theta)}{2}$ we get

$$\begin{aligned}
I_{\text{rms}}^2 &= \frac{339.4^2}{4\pi\times 10000}\left(\int_{0}^{\pi}\frac{1-\cos(2\theta)}{36}\,d\theta + \int_{\pi}^{2\pi}\frac{1-\cos(2\theta)}{196}\,d\theta\right)\\
&= \frac{339.4^2}{4\pi\times 10000}\left(\left[\frac{\theta}{36}-\frac{\sin(2\theta)}{72}\right]_0^{\pi} + \left[\frac{\theta}{196}-\frac{\sin(2\theta)}{392}\right]_{\pi}^{2\pi}\right)\\
&= \frac{339.4^2}{4\pi\times 10000}\left(\frac{\pi}{36}+\frac{\pi}{196}\right) = 0.0946875\ A^2\\
I_{\text{rms}} &= 0.31\ A \text{ to } 2 \text{ d.p.}
\end{aligned}$$

Interpretation

The reading on the meter would be 0.31 A.

Exercises

1. Calculate the r.m.s values of the given functions across the specified interval.

   (a) $f(t) = 1 + t$ across $[0, 2]$
   (b) $f(x) = 2x - 1$ across $[-1, 1]$
   (c) $f(t) = t^2$ across $[0, 1]$
   (d) $f(t) = t^2$ across $[0, 2]$
   (e) $f(z) = z^2 + z$ across $[1, 3]$

2. Calculate the r.m.s values of the given functions over the specified interval.

   (a) $f(x) = x^3$ across $[1, 3]$
   (b) $f(x) = \dfrac{1}{x}$ across $[1, 2]$
   (c) $f(t) = \sqrt{t}$ across $[0, 2]$
   (d) $f(z) = z^3 - 1$ across $[-1, 1]$

3. Calculate the r.m.s values of the following:

   (a) $f(t) = \sin t$ across $\left[0, \dfrac{\pi}{2}\right]$
   (b) $f(t) = \sin t$ across $[0, \pi]$
   (c) $f(t) = \sin \omega t$ across $[0, \pi]$
   (d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$
   (e) $f(t) = \cos t$ across $[0, \pi]$
   (f) $f(t) = \cos \omega t$ across $[0, \pi]$
   (g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, 1]$

4. Calculate the r.m.s values of the following functions:

   (a) $f(t) = \sqrt{t + 1}$ across $[0, 3]$
   (b) $f(t) = e^t$ across $[-1, 1]$
   (c) $f(t) = 1 + e^t$ across $[-1, 1]$

Answers

1. (a) 2.0817 (b) 1.5275 (c) 0.4472 (d) 1.7889 (e) 6.9666
2. (a) 12.4957 (b) 0.7071 (c) 1 (d) 1.0690
3. (a) 0.7071 (b) 0.7071 (c) $\sqrt{\dfrac{1}{2} - \dfrac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$

   (d) 0.7071 (e) 0.7071 (f) $\sqrt{\dfrac{1}{2} + \dfrac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$ (g) $\sqrt{1 + \dfrac{\sin^2 \omega}{\omega}}$
4. (a) 1.5811 (b) 1.3466 (c) 2.2724