Integration of Trigonometric Functions

13.6

Introduction

Integrals involving trigonometric functions are commonplace in engineering mathematics. This is especially true when modelling waves and alternating current circuits. When the root-mean-square (rms) value of a waveform, or signal is to be calculated, you will often find this results in an integral of the form

$$\int \sin^2 t \, dt$$

In this Section you will learn how such integrals can be evaluated.

Prerequisites

Before starting this Section you should ...

- be able to find a number of simple definite and indefinite integrals
- be able to use a table of integrals
- be familiar with standard trigonometric identities

Learning Outcomes

On completion you should be able to ...

- use trigonometric identities to write integrands in alternative forms to enable them to be integrated

1. Integration of trigonometric functions

Simple integrals involving trigonometric functions have already been dealt with in Section 13.1. See what you can remember:

Task

Write down the following integrals:

(a) $\displaystyle\int \sin x\, dx$, (b) $\displaystyle\int \cos x\, dx$, (c) $\displaystyle\int \sin 2x\, dx$, (d) $\displaystyle\int \cos 2x\, dx$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) $-\cos x + c$, (b) $\sin x + c$, (c) $-\dfrac{1}{2}\cos 2x + c$, (d) $\dfrac{1}{2}\sin 2x + c$.

The basic rules from which these results can be derived are summarised here:

Key Point 8

$$\int \sin kx\, dx = -\frac{\cos kx}{k} + c \qquad \int \cos kx\, dx = \frac{\sin kx}{k} + c$$

In engineering applications it is often necessary to integrate functions involving powers of the trigonometric functions such as

$$\int \sin^2 x\, dx \qquad \text{or} \qquad \int \cos^2 \omega t\, dt$$

Note that these integrals cannot be obtained directly from the formulas in Key Point 8 above. However, by making use of trigonometric identities, the integrands can be re-written in an alternative form. It is often not clear which identities are useful and each case needs to be considered individually. Experience and practice are essential. Work through the following Task.

Task

Use the trigonometric identity $\sin^2\theta \equiv \dfrac{1}{2}(1-\cos 2\theta)$ to express the integral $\displaystyle\int \sin^2 x\; dx$ in an alternative form and hence evaluate it.

(a) First use the identity:

Your solution

$$\int \sin^2 x\, dx = \int$$

Answer

The integral can be written $\displaystyle\int \frac{1}{2}(1-\cos 2x)dx.$

Note that the trigonometric identity is used to convert a power of $\sin x$ into a function involving $\cos 2x$ which can be integrated directly using Key Point 8.

(b) Now evaluate the integral:

Your solution

Answer

$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x + c\right) = \frac{1}{2}x - \frac{1}{4}\sin 2x + K$ where $K = c/2$.

Task

Use the trigonometric identity $\sin 2x \equiv 2\sin x\cos x$ to find $\displaystyle\int \sin x\cos x\, dx$

(a) First use the identity:

Your solution

$$\int \sin x\cos x\, dx = \int$$

Answer

The integrand can be written as $\frac{1}{2}\sin 2x$

(b) Now evaluate the integral:

Your solution

Answer

$$\int_0^{2\pi} \sin x\cos x\; dx = \int_0^{2\pi} \frac{1}{2}\sin 2x\, dx = \left[-\frac{1}{4}\cos 2x + c\right]_0^{2\pi} = -\frac{1}{4}\cos 4\pi + \frac{1}{4}\cos 0 = -\frac{1}{4} + \frac{1}{4} = 0$$

This result is one example of what are called **orthogonality relations**.

Engineering Example 3

Magnetic flux

Introduction

The magnitude of the magnetic flux density on the axis of a solenoid, as in Figure 13, can be found by the integral:

$$B = \int_{\beta_1}^{\beta_2} \frac{\mu_0 n I}{2} \sin\beta \; d\beta$$

where μ_0 is the permeability of free space ($\approx 4\pi \times 10^{-7}$ H m^{-1}), n is the number of turns and I is the current.

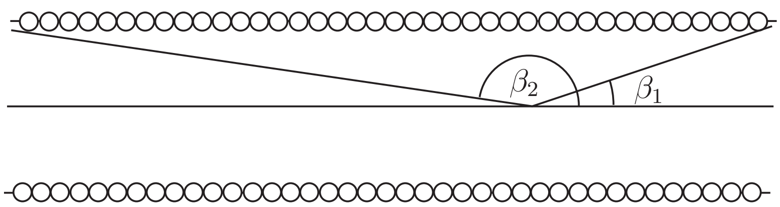

Figure 13: A solenoid and angles defining its extent

Problem in words

Predict the magnetic flux in the middle of a long solenoid.

Mathematical statement of the problem

We assume that the solenoid is so long that $\beta_1 \approx 0$ and $\beta_2 \approx \pi$ so that

$$B = \int_{\beta_1}^{\beta_2} \frac{\mu_0 n I}{2} \sin\beta \; d\beta \approx \int_0^{\pi} \frac{\mu_0 n I}{2} \sin\beta \; d\beta$$

Mathematical analysis

The factor $\dfrac{\mu_0 n I}{2}$ can be taken outside the integral i.e.

$$B = \frac{\mu_0 n I}{2} \int_0^{\pi} \sin\beta \; d\beta = \frac{\mu_0 n I}{2} \Big[-\cos\beta \Big]_0^{\pi} = \frac{\mu_0 n I}{2}(-\cos\pi + \cos 0)$$

$$= \frac{\mu_0 n I}{2}(-(-1) + 1) = \mu_0 n I$$

Interpretation

The magnitude of the magnetic flux density at the midpoint of the axis of a long solenoid is predicted to be approximately $\mu_0 n I$ i.e. proportional to the number of turns and proportional to the current flowing in the solenoid.

2. Orthogonality relations

In general two functions $f(x), g(x)$ are said to be **orthogonal** to each other over an interval $a \leq x \leq b$ if

$$\int_a^b f(x)g(x)\ dx = 0$$

It follows from the previous Task that $\sin x$ and $\cos x$ are orthogonal to each other over the interval $0 \leq x \leq 2\pi$. This is also true over any interval $\alpha \leq x \leq \alpha + 2\pi$ (e.g. $\pi/2 \leq x \leq 5\pi$, or $-\pi \leq x \leq \pi$).

More generally there is a whole set of orthogonality relations involving these trigonometric functions on intervals of length 2π (i.e. over one period of both $\sin x$ and $\cos x$). These relations are useful in connection with a widely used technique in engineering, known as **Fourier analysis** where we represent periodic functions in terms of an infinite series of sines and cosines called a Fourier series. (This subject is covered in HELM 23.)

We shall demonstrate the orthogonality property

$$I_{mn} = \int_0^{2\pi} \sin mx \sin nx\ dx = 0$$

where m and n are integers such that $m \neq n$.

The secret is to use a trigonometric identity to convert the integrand into a form that can be readily integrated.

You may recall the identity

$$\sin A \sin B \equiv \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

It follows, putting $A = mx$ and $B = nx$ that provided $m \neq n$

$$\begin{aligned} I_{mn} &= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\ dx \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

because $(m-n)$ and $(m+n)$ will be integers and $\sin(\text{integer}\times 2\pi) = 0$. Of course $\sin 0 = 0$.

Why does the case $m = n$ have to be excluded from the analysis? (left to the reader to figure out!)

The corresponding orthogonality relation for cosines

$$J_{mn} = \int_0^{2\pi} \cos mx \cos nx\ dx = 0$$

follows by use of a similar identity to that just used. Here again m and n are integers such that $m \neq n$.

Example 23

Use the identity $\sin A\cos B \equiv \frac{1}{2}(\sin(A+B)+\sin(A-B))$ to show that

$$K_{mn} = \int_0^{2\pi} \sin mx \cos nx \; dx = 0 \qquad m \text{ and } n \text{ integers, } m \neq n.$$

Solution

$$\begin{aligned} K_{mn} &= \frac{1}{2}\int_0^{2\pi} [\sin(m+n)x + \sin(m-n)x] \; dx \\ &= \frac{1}{2}\left[-\frac{\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{(m-n)}\right]_0^{2\pi} \\ &= -\frac{1}{2}\left[\frac{\cos(m+n)2\pi - 1}{(m+n)} + \frac{\cos(m-n)2\pi - 1}{(m-n)}\right] = 0 \end{aligned}$$

(recalling that $\cos(\text{integer} \times 2\pi) = 1$)

Task

Derive the orthogonality relation

$$K_{mn} = \int_0^{2\pi} \sin mx \cos nx \; dx = 0 \qquad m \text{ and } n \text{ integers, } m = n$$

Hint: You will need to use a different trigonometric identity to that used in Example 23.

Your solution

Answer

$K_{mn} = \int_0^{2\pi} \sin mx \cos mx \; dx$

Putting $m = n \neq 0$, and then using the identity $\sin 2A \equiv 2 \sin A \cos A$ we get

$$\begin{aligned} K_{mm} &= \int_0^{2\pi} \sin mx \cos mx \; dx \\ &= \frac{1}{2} \int_0^{2\pi} \sin 2mx \; dx \\ &= \frac{1}{2} \left[-\frac{\cos 2mx}{2m} \right]_0^{2\pi} = -\frac{1}{4m}(\cos 4m\pi - \cos 0) = -\frac{1}{4m}(1-1) = 0 \end{aligned}$$

Putting $m = n = 0$ gives $K_{00} = \frac{1}{2} \int_0^{2\pi} \sin 0 \cos 0 \; dx = 0.$

Note that the particular case $m = n = 1$ was considered earlier in this Section.

3. Reduction formulae

You have seen earlier in this Workbook how to integrate $\sin x$ and $\sin^2 x$ (which is $(\sin x)^2$). Applications sometimes arise which involve integrating higher powers of $\sin x$ or $\cos x$. It is possible, as we now show, to obtain a **reduction formula** to aid in this Task.

Task

Given $I_n = \int \sin^n(x) \; dx$ write down the integrals represented by I_2, I_3, I_{10}

Your solution

$I_2 =$ $\qquad$ $I_3 =$ $\qquad$ $I_{10} =$

Answer

$I_2 = \int \sin^2 x \; dx \qquad I_3 = \int \sin^3 x \; dx \qquad I_{10} = \int \sin^{10} x \; dx$

To obtain a reduction formula for I_n we write

$\sin^n x = \sin^{n-1}(x) \sin x$

and use integration by parts.

Task

In the notation used earlier in this Workbook for integration by parts (Key Point 5, page 31) put $f = \sin^{n-1} x$ and $g = \sin x$ and evaluate $\dfrac{df}{dx}$ and $\displaystyle\int g\ dx$.

Your solution

Answer

$\dfrac{df}{dx} = (n-1)\sin^{n-2}x\ \cos x$ (using the chain rule of differentiation),

$\displaystyle\int g\ dx = \int \sin x\ dx = -\cos x$

Now use the integration by parts formula on $\displaystyle\int \sin^{n-1} x \sin x\ dx$. [Do not attempt to evaluate the second integral that you obtain.]

Your solution

Answer

$$\begin{aligned}\int \sin^{n-1} x \sin x\ dx &= \sin^{n-1}(x)\int g\ dx - \int \frac{df}{dx}\int g\ dx \\ &= \sin^{n-1}(x)(-\cos x) + (n-1)\int \sin^{n-2} x \cos^2 x\ dx\end{aligned}$$

We now need to evaluate $\displaystyle\int \sin^{n-2} x \cos^2 x dx$. Putting $\cos^2 x = 1 - \sin^2 x$ this integral becomes:

$$\int \sin^{n-2}(x)\ dx - \int \sin^{n}(x)\ dx$$

But this is expressible as $I_{n-2} - I_n$ so finally, using this and the result from the last Task we have
$I_n = \displaystyle\int \sin^{n-1}(x) \sin x\ dx = \sin^{n-1}(x)(-\cos x) + (n-1)(I_{n-2} - I_n)$

from which we get Key Point 9:

Key Point 9

Reduction Formula

Given $I_n = \int \sin^n x dx$

$$I_n = -\frac{1}{n}\sin^{n-1}(x)\cos x + \frac{n-1}{n}I_{n-2}$$

This is our **reduction formula** for I_n. It enables us, for example, to evaluate I_6 in terms of I_4, then I_4 in terms of I_2 and I_2 in terms of I_0 where

$$I_0 = \int \sin^0 x\ dx = \int 1\ dx = x.$$

Task

Use the reduction formula in Key Point 9 with $n = 2$ to find I_2.

Your solution

Answer

$$\begin{aligned} I_2 &= -\frac{1}{2}[\sin x \cos x] + \frac{1}{2}I_0 \\ &= -\frac{1}{2}[\frac{1}{2}\sin 2x] + \frac{x}{2} + c \\ \text{i.e.} \qquad \int \sin^2 x\ dx &= -\frac{1}{4}\sin 2x + \frac{x}{2} + c \end{aligned}$$

as obtained earlier by a different technique.

Task

Use the reduction formula in Key Point 9 to obtain $I_6 = \int \sin^6 x\ dx$.

Firstly obtain I_6 in terms of I_4, then I_4 in terms of I_2:

Your solution

Answer

Using Key Point 9 with $n = 6$ gives $I_6 = -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6} I_4$.

Then, using Key Point 9 again with $n = 4$, gives $I_4 = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4} I_2$

Now substitute for I_2 from the previous Task to obtain I_4 and hence I_6.

Your solution

Answer

$$I_4 = -\frac{1}{4}\sin^3 x \cos x - \frac{3}{16}\sin 2x + \frac{3}{8}x + \text{ constant}$$

$$\therefore \qquad I_6 = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{32}\sin 2x + \frac{5}{16}x + \text{ constant}$$

Definite integrals can also be readily evaluated using the reduction formula in Key Point 9. For example,

$$I_n = \int_0^{\pi/2} \sin^n x\ dx \quad \text{so} \quad I_{n-2} = \int_0^{\pi/2} \sin^{n-2} x\ dx$$

We obtain, immediately

$$I_n = \frac{1}{n}\Big[-\sin^{n-1}(x)\cos x \Big]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

or, since $\cos\frac{\pi}{2} = \sin 0 = 0$, $\qquad I_n = \frac{(n-1)}{n} I_{n-2}$

This simple easy-to-use formula is well known and is called **Wallis' formula**.

Key Point 10

Reduction Formula - Wallis' Formula

Given $I_n = \int_0^{\pi/2} \sin^n x\,dx$ or $I_n = \int_0^{\pi/2} \cos^n x\,dx$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Task

If $I_n = \int_0^{\pi/2} \sin^n x\ dx$ calculate I_1 and then use Wallis' formula, without further integration, to obtain I_3 and I_5.

Your solution

Answer

$$I_1 = \int_0^{\pi/2} \sin x\ dx = \Big[-\cos x \Big]_0^{\pi/2} = 1$$

Then using Wallis' formula with $n = 3$ and $n = 5$ respectively

$$I_3 = \int_0^{\pi/2} \sin^3 x\ dx = \frac{2}{3} I_1 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I_5 = \int_0^{\pi/2} \sin^5 x\ dx = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

Task

The total power P of an antenna is given by

$$P = \int_0^\pi \frac{\eta L^2 I^2 \pi}{4\lambda^2} \sin^3\theta \, d\theta$$

where η, λ, I are constants as is the length L of antenna. Using the reduction formula for $\int \sin^n x \, dx$ in Key Point 9, obtain P.

Your solution

Answer

Ignoring the constants for the moment, consider

$I_3 = \int_0^\pi \sin^3\theta \, d\theta$ which we will reduce to I_1 and evaluate.

$$I_1 = \int_0^\pi \sin\theta \, d\theta = \Big[-\cos\theta \Big]_0^\pi = 2$$

so by the reduction formula with $n = 3$

$$I_3 = \frac{1}{3}\Big[-\sin^2 x \cos x \Big]_0^\pi + \frac{2}{3} I_1 = 0 + \frac{2}{3} \times 2 = \frac{4}{3}$$

We now consider the actual integral with all the constants.

Hence $P = \dfrac{\eta L^2 I^2 \pi}{4\lambda^2} \displaystyle\int_0^\pi \sin^3\theta \, d\theta = \dfrac{\eta L^2 I^2 \pi}{4\lambda^2} \times \dfrac{4}{3}$, so $P = \eta \dfrac{L^2 I^2 \pi}{3\lambda^2}$.

A similar reduction formula to that in Key Point 9 can be obtained for $\int \cos^n x \, dx$ (see Exercise 5 at the end of this Workbook). In particular if

$$J_n = \int_0^{\pi/2} \cos^n x \, dx \quad \text{then} \quad J_n = \frac{(n-1)}{n} J_{n-2}$$

i.e. Wallis' formula is the same for $\cos^n x$ as for $\sin^n x$.

4. Harder trigonometric integrals

The following seemingly innocent integrals are examples, important in engineering, of trigonometric integrals that **cannot** be evaluated as **indefinite** integrals:

(a) $\displaystyle\int \sin(x^2)\ dx$ and $\displaystyle\int \cos(x^2)\ dx$ These are called Fresnel integrals.

(b) $\displaystyle\int \frac{\sin x}{x}\ dx$ This is called the Sine integral.

Definite integrals of this type, which are what normally arise in applications, have to be evaluated by **approximate numerical methods**.

Fresnel integrals with limits arise in wave and antenna theory and the Sine integral with limits in filter theory.

It is useful sometimes to be able to visualize the definite integral. For example consider

$$F(t) = \int_0^t \frac{\sin x}{x}\ dx \qquad t > 0$$

Clearly, $F(0) = \displaystyle\int_0^0 \frac{\sin x}{x}\ dx = 0$. Recall the graph of $\dfrac{\sin x}{x}$ against x, $x > 0$:

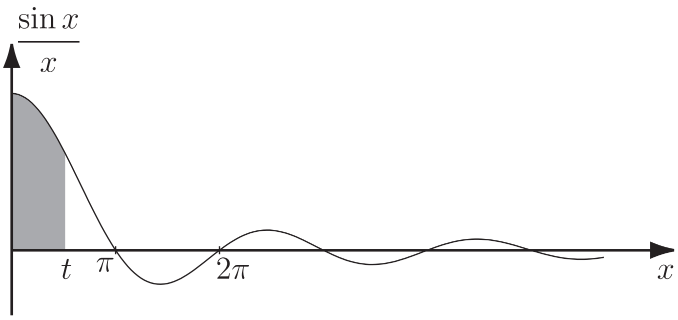

Figure 14

For any positive value of t, $F(t)$ is the shaded area shown (the area interpretation of a definite integral was covered earlier in this Workbook). As t increases from 0 to π, it follows that $F(t)$ increases from 0 to a maximum value

$$F(\pi) = \int_0^\pi \frac{\sin x}{x}\ dx$$

whose value could be determined numerically (it is actually about 1.85). As t further increases from π to 2π the value of $F(t)$ will decrease to a local minimum at 2π because the $\dfrac{\sin x}{x}$ curve is below the x-axis between π and 2π. Note that the area below the curve **is** considered to be negative in this application.

Continuing to argue in this way we can obtain the shape of the $F(t)$ graph in Figure 15: (can you see why the oscillations decrease in amplitude?)

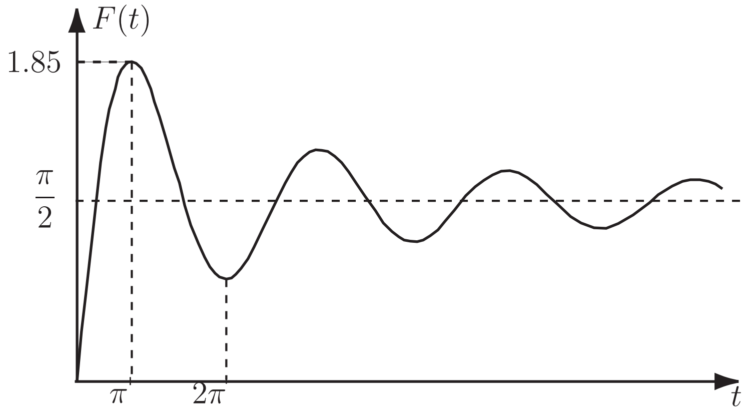

Figure 15

The result $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ is clearly illustrated in the graph (you are not expected to know how this result is obtained). Methods for solving such problems are dealt with in HELM 31.

Exercises

You will need to refer to a Table of Trigonometric Identities to answer these questions.

1. Find (a) $\int \cos^2 x dx$ (b) $\int_0^{\pi/2} \cos^2 t dt$ (c) $\int (\cos^2\theta + \sin^2\theta) d\theta$

2. Use the identity $\sin(A+B) + \sin(A-B) \equiv 2\sin A\cos B$ to find $\int \sin 3x \cos 2x dx$

3. Find $\int (1 + \tan^2 x) dx$.

4. The mean square value of a function $f(t)$ over the interval $t = a$ to $t = b$ is defined to be

$$\frac{1}{b-a}\int_a^b (f(t))^2 dt$$

Find the mean square value of $f(t) = \sin t$ over the interval $t = 0$ to $t = 2\pi$.

5. (a) Show that the reduction formula for $J_n = \int \cos^n x\ dx$ is

$$J_n = \frac{1}{n}\cos^{n-1}(x)\sin x + \frac{(n-1)}{n}J_{n-2}$$

(b) Using the reduction formula in (a) show that

$$\int \cos^5 x\ dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x$$

(c) Show that if $J_n = \int_0^{\pi/2} \cos^n x\ dx$, then $J_n = \left(\frac{n-1}{n}\right) J_{n-2}$ (Wallis' formula).

(d) Using Wallis' formula show that $\int_0^{\pi/2} \cos^6 x\ dx = \frac{5}{32}\pi$.

Answers

1. (a) $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$ (b) $\pi/4$ (c) $\theta + c$.
2. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$.
3. $\tan x + c$.
4. $\frac{1}{2}$.